THEOREM 5.

$$(V_q^{(n)}1,\ 1) = g^{-n\,\mathrm{rg}(g-I)} = m\{x:\ gx = x\},$$

and, consequently, the representations $V^{(n)}$, $n = 0, 1, \ldots$, *and the regular representation together with their tensor products with one-dimensional representations realize all* (*see* [13]) *factor representations of the group* $\mathrm{GL}(\infty, \mathrm{F}_q)$.

Thus, all the realizations of representations we have constructed are based on some (nonfree) action of the group itself in a measure space and its orbit partition. This distinguishes them from the direct GNS realization and from the mixed product realization, whose structure we have in any algebra that is locally semisimple or very nearly so (see [4]). Obviously, this is a general fact connected with the asymptotic coincidence of the classes of the induced and primary representations of the classical series (the law of large numbers).

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AN ANALOGUE OF THE BRUHAT DECOMPOSITION FOR THE CLOSURE OF THE CONE OF A CHEVALLEY GROUP OF THE CLASSICAL SERIES

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If $\mathcal{L}$, a simple Lie algebra over the infinite field F, is in one of the four classical series A, B, C or D and G is some Chevalley group considered as a group of automorphisms of $\mathcal{L}$, then by $c(G)$ we denote the subset $\{\lambda g:\ g \in G,\ 0 \neq \lambda \in F\}$ (cone without zero) of the algebra of F-linear operators of the additive group of $\mathcal{L}$. In the sequel, we will denote the closure of $c(G)$ in the Zariski topology by Γ. The principal results of this note are that an analogue of the Bruhat decomposition (Theorem 2) holds for Γ as well as an analogue of the Chevalley theorem on the structure of the closure of a double coset of a Borel subgroup (Corollary 1); cf. [1], §3,8. Originally these results were obtained for the series A and were used by the author as tools in the theory of complexity of computation (besides Theorem 2 and Corollaries 1 and 2, Proposition 1 played an important role there).

The following notation will be used: B denotes a Borel subgroup of G; $c(B) = \{\lambda b:\ b \in B,\ 0 \neq \lambda \in F\}$; $\mathrm{T} = \overline{c(B)}$; a bar over a subset of Γ denotes here, and elsewhere, its closure in the Zariski topology. W denotes the Weyl group of G; $l(w)$ the length of an element $w \in W$, (cf. [2], Chapter VI, §1); m the size of the matrices under consideration in Γ; E the identity matrix; S_m the permutation group on a set of m elements; and

$$s = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$$

the permutation matrix with ones on the collateral diagonal. Given any nondegenerate bilinear form ψ on the vector space V, we denote by Q^* the adjoint operator of Q with respect to ψ; $\mathrm{spin}(a)$ is the spinor norm of the orthogonal matrix a ([4], Chapter 5, §5). K denotes a set of indices of basis vectors of the vector space V, where

$$K = \begin{cases} \{m,\ m-1, \ldots, 1\} & \text{for series } A, \\ \{n,\ n-1, \ldots, 1, 0, -1, \ldots, -n\} & \text{for series } B, \\ \{n, \ldots, 1, -1, \ldots, -n\} & \text{for series } C \text{ and } D; \end{cases}$$

σ_i denotes the transposition of coordinates i and $(i+1)$ for $i \neq -1$ and σ_{-1} the transposition of the coordinates 1 and -1; $|I|$ denotes the cardinality of the subset $I \subset K$, and $|I|^+$ the number of positive indices in I. By $a_{I,J}$ for a matrix a and $I, J \subset K$ we denote the $|I| \times |J|$ submatrix of a consisting of the matrix elements of a whose row indices are in I and whose column indices are in J.

Now we describe $c(G)$, G, W and its generators as a Coxeter group (see [2], Chapter IV, §1) in terms of standard matrix realizations (see [3], Chapter VIII, §13, and [5], §11) for

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each of the series individually. The formulation of the principal results (Theorem 2 and Corollary 1) is not dependent on these realizations, in contrast to the auxiliary results and constructions in the proof of Theorem 2. It would be interesting to obtain an invariant proof of Theorem 2. In some places below we omit, for simplicity, the passage to the factor group by the center: this does not influence the formulation of the results.

SERIES *A*. $G = SL_m(F)$ and $c(G) = \{a: \det a \in (F^*)^m\}$; $\Gamma = M_m$ is the manifold of all $m \times m$ matrices; $T = R_m$ is the manifold of all upper triangular matrices (i.e. with zeroes below the main diagonal); and $W = S_m$ with generators $\sigma_1, \ldots, \sigma_{m-1}$.

SERIES *B*. Let

$$\psi = \begin{pmatrix} & & s \\ & -2 & \\ s & & \end{pmatrix}$$

be a $(2n+1)$-dimensional form. Then

$$G = \Omega\, O_{2n+1}(F) = \{a:\ aa^* = E,\ \det a = \operatorname{spin}(a) = 1\}$$

(see [4], Chapter 5, §5), and

$$c(G) = \{a:\ aa^* = \mu E,\quad \operatorname{spin}(a\mu^n/\det a) = 1,\quad \mu \in F^*\}.$$

The manifold Γ is given by

$$\Gamma = \{a \in M_{2n+1}:\ aa^* = a^*a = \mu E,\quad \mu \in F\};\qquad T = \Gamma \cap R_{2n+1}.$$

The Weyl group $W \subset S_{2n+1}$ consists of those permutation matrices whose nonzero entries are symmetric with respect to the center of the matrix. Generators of W are $\sigma_{n-1}\sigma_{-n}, \ldots, \sigma_1\sigma_{-2}, \sigma_{-1}$. For each $w \in W$, it is possible to pick a monomial matrix $d \in G$, representing w and satisfying $\det d = \operatorname{spin}(d) = 1$, from which w may be obtained by changing some of the signs of d.

SERIES *C*. Let $\psi = (_{-s}{}^{s})$ be a skew-symmetric $2n$-dimensional form. The group G is given by

$$G = \operatorname{Sp}_{2n}(F) = \{a:\ aa^* = E\};\qquad c(G) = \{a:\ aa^* = \mu E,\ \mu \in (F^*)^2\}.$$

The manifold Γ is given by

$$\Gamma = \{a \in M_{2n}:\ aa^* = a^*a = \mu E,\quad \mu \in F\};\qquad T = \Gamma \cap R_{2n}.$$

The Weyl group $W \subset S_m$ consists of those permutation matrices in which the disposition of the nonzero entries is symmetric with respect to the center of the matrix. Generators of W are $\sigma_{n-1}\sigma_{-n}, \ldots, \sigma_1\sigma_{-2}, \sigma_{-1}$. For each element $w \in W$, we can pick a monomial matrix in G, representing w, from which w can be obtained by changing the signs of the nonzero entries situated in the lower left quadrant.

SERIES *D*. Let $\psi = (_{s}{}^{s})$ be a $2n$-dimensional symmetric form. The group G is given by

$$G = \Omega\, O_{2n}(F) = \{a:\ aa^* = E,\quad \det a = \operatorname{spin}(a) = 1\},$$

$$c(G) = \begin{cases} \{a:\ aa^* = \mu E,\ \det a = \mu^n,\ \mu \in (F^*)^2\} & \text{for odd } n, \\ \{a:\ aa^* = \mu E,\ \det a = \mu^n,\ \operatorname{spin}(a/\sqrt{\mu}) = 1,\ \mu \in (F^*)^2\} & \text{for even } n. \end{cases}$$

The manifold Γ consists of all $a \in M_{2n}$ satisfying $aa^* = a^*a = \mu E$, $\mu \in F$, and the following system of equations (henceforth we will call them the parity equations): $\{\det(a_{I,J}) = 0\}$ for all $I, J \subseteq K$ such that $|I| = |J| = n$, $I \cap (-I) = \varnothing$, $J \cap (-J) = \varnothing$ and $|I|^+ + |J|^+$ is odd. Set $K_i = \{i, (i-1), \ldots, 1, -1, \ldots, -(i-1), -i\}$. The manifold T consists of all $a \in \Gamma \cap R_{2n}$ such that for all $1 \le i \le n$ the matrix a_{K_i,K_i} satisfies the parity equations. The Weyl group $W \subset S_{2n}$ consists of all even permutations whose nonzero entries are symmetric with respect to the center of the matrix. Generators for W are $\sigma_{n-1}\sigma_{-n}, \ldots, \sigma_1\sigma_{-2}, \sigma_{-1}\sigma_1\sigma_{-2}\sigma_{-1}$. Each element $w \in W$ may be obtained from a monomial matrix $d \in G$ satisfying $\operatorname{spin}(d) = 1$ by changing signs of pairs of symmetric nonzero entries.

For the proof of Theorem 2 we must effectively describe the well-known partial ordering $\le$ in the Weyl group ([1], §8); this is done in Theorem 1 (see below). It has an independent interest. The basic tool which will be used for this is the integer-valued function $r_a(i, j)$ of two integer arguments, where $0 \le i, j \le m$, defined below for any $m \times m$ matrix a. Namely, for $i, j \ge 1$ we define $r_a(i, j)$ to be equal to the rank of the $i \times j$ submatrix of a situated in the lower left corner, and we define $r_a(0, j) = r_a(i, 0) = 0$. We note that $r_a = r_{b_1ab_2}$ for $b_1, b_2 \in GL_m \cap R_m$. The function $r = r_a$ satisfies the following conditions:

α) $0 \le r(i, j) \le \min\{i, j\}$;

β) $r(i, j) \le r(i, j+1) \le r(i, j) + 1$, $r(i, j) \le r(i+1, j) \le r(i, j) + 1$;

γ) if $r(i, j+1) = r(i+1, j) = r(i, j) + 1$, then $r(i+1, j+1) = r(i, j) + 2$.

For any function $f(i, j)$, defined for $0 \le i, j \le m$, we denote $f^s(i, j) = f(m-i, m-j)$.

LEMMA 1. *Let the function r satisfy properties α), β) and γ), and let $r(m, m) = m$. Then the following obtain:*

Series A. There is a unique permutation matrix $w \in W$ such that $r = r_w$.

Series B and C. The constructed permutation matrix (see Series A above) lies in W if and only if $\chi_w(i, j) = i + j - m + r^s(i, j) - r(i, j) = 0$ for all $0 \le i, j \le m$.

Series D. The constructed permutation matrix (Series A) lies in W if and only if $\chi_w = 0$ and $r(n, n)$ is even ($m = 2n$).

We note that the upper right matrix element of the $i \times j$ submatrix situated in the lower left corner of the matrix w is nonzero if and only if

$$r(i, j) - 1 = r(i-1, j) = r(i, j-1) = r(i-1, j-1).$$

THEOREM 1. 1) *For the series A, B and C the partial order relation $w_1 \le w_2$, $w_1, w_2 \in W$, is equivalent to the functional inequality $r_{w_1} \le r_{w_2}$.*

2) *For series D the relation $w_1 \le w_2$ is valid if and only $r_{w_1} \le r_{w_2}$ and, simultaneously, the following condition holds.*

Suppose $r_{w_1}(i, j) = r_{w_2}(i, j)$ for some $m \ge i > n \ge j \ge 1$ ($m = 2n$) and

$$r_{w_1}(i, j) - r_{w_1}(m-i, j) = r_{w_2}(i, j) - r_{w_2}(m-i, j) = i - n.$$

Set $I = K_{i-n}$ and $J = \{n, n-1, \ldots, n-j+1\} \subset K$. Suppose that the nonzero entries of the $2(i-n) \times j$ matrix $(w_p)_{I,J}$ belong to rows with indices in $I_p \subset K$, $p = 1, 2$. Then $|I_1|^+ + |I_2|^+$ is even (according to the assumptions made above, $|I_1| = |I_2| = i-n$, $I_1 \cap (-I_1) = \varnothing$ and $I_2 \cap (-I_2) = \varnothing$).

Although the rows and columns enter into the condition asymetrically, it can be shown that this condition is equivalent to a certain symmetric condition.

THEOREM 2. *For every* $a \in \Gamma$ *there is a unique* $w_a \in W$ *such that*

1) $a \in \mathrm{T}w_aB \cap Bw_a\mathrm{T} \subset \mathrm{T}w_a\mathrm{T}$;
2) $w_a \leqslant w$ *for every* $w \in W$ *such that* $a \in \mathrm{T}w\mathrm{T}$.

One of the forms of the Bruhat decomposition is the following proposition: for every $a \in G$ there is a unique $w_a \in W$ such that $a \in Bw_aB$. Theorem 2 is an analogue of the proposition for Γ: each element in Γ lies in some class $\mathrm{T}w\mathrm{T}$; w is not necessarily unique, but among such w there is a unique canonical element smaller than all the rest (in the sense of the partial ordering $\leqslant$).

If $a \in \Gamma$ is nonsingular then for some diagonal matrix $h \in \mathrm{T}$ we have $ha \in G$, and the element $w_a = w_{ha}$ is found from the Bruhat decomposition.

The construction of w_a only remains for singular $a \in \Gamma$ (i.e. $aa^* = a^*a = 0$). By virtue of Lemma 1 it suffices to construct $r = r_{w_a}$.

Series A. Set $r(i, j) = \max\{i + j - m, r_a(i, j)\}$.

Series B and C. Set $r(i, j) = \max\{i + j - m + r_a^s(i, j), r_a(i, j)\}$.

Series D. Define the auxiliary function $\Delta r_a(i, j)$ for $0 \leqslant i, j \leqslant m$, taking on the value zero or one. We put $\Delta r_a(i, j) = 1$ for any of the following cases (otherwise $\Delta r_a(i, j) = 0$):

(I) if $1 \leqslant i, j \leqslant n$, $r_a(n, n)$ is odd and

$$r_a(m - i, j) - r_a(i, j) = n - i, \qquad r_a(i, m - j) - r_a(i, j) = n - j;$$

(II) if $1 \leqslant j \leqslant n < i \leqslant m$, $r_a(i, m) - r_a(n, m) - r_a(i, n)$ is odd and

$$r_a(i, m) - r_a(m - i, m) = i - n, \quad r_a(i, m - j) - r_a(i, j) = n - j;$$

(III) if $1 \leqslant i \leqslant n < j \leqslant m$, $r_a(m, j) - r_a(m, n) - r_a(n, j)$ is odd and

$$r_a(m, j) - r_a(m, m - j) = i - n, \quad r_a(m - i, j) - r_a(i, j) = n - i;$$

(IV) if $n < i, j \leqslant m$ and $\Delta r_a(m - i, m - j) = 1$.

Finally, we define

$$r(i, j) = \max\{i + j - m + r_a^s(i, j) + (\Delta r_a)^s(i, j), \; r_a(i, j) + \Delta r_a(i, j)\}.$$

To check the hypotheses of Lemma 1 it is convenient to reformulate them in terms of the function $i + j - m - r_a(i, j)$ in the case of series A and the function χ_a (see above) for the series B, C and D. The proof of statement 1) in Theorem 2 is based on the above-mentioned description of T. Statement 2) of Theorem 2 is proved by using Theorem 1.

Now the analogue for Γ is reduced to the following theorem of Chevalley:

$$cl_G(BwB) = \bigcup_{w' \leqslant w} Bw'B,$$

where cl_G means closure in G.

COROLLARY 1. $\overline{\mathrm{T}w\mathrm{T}} = \bigcup_{w' \leqslant w} \mathrm{T}w'\mathrm{T}$.

Theorem 2 also shows that it is possible to extend the function l to Γ by setting $l(a) = l(w_a)$.

COROLLARY 2. *The function* l *is lower semicontinuous, i.e., for every set* $Y \subseteq \Gamma$ *and any* $a \in \bar{Y}$ *we have* $l(a) \leqslant \max_{y \in Y} l(y)$.

The last result of this note relates to the standard matrix realizations of Chevalley groups (see above). The principal submatrices of a matrix $a \in \Gamma$ are those matrices of the form $a_{I,I}$ where $I \subset K$ satisfies $I = -I$ in the case of series B, C, D and, also, $0 \in I$ in the case of series B.

PROPOSITION 1. *Let* d *be a principal submatrix of the matrix* a, *where* $a, d \in G$. *Then*:

1) $l(d) \leqslant l(a)$ *for series A, B and C.*
2) *For series D this inequality holds, in general, only for nonsingular d.*

REMARK. The results of this article are valid for a finite field as well, if by Γ and T we mean their matrix realizations, as considered above, and in Theorem 2 we replace $\overline{\mathrm{T}w\mathrm{T}}$ by $\overline{\mathrm{T}_1w\mathrm{T}_1} \cap \Gamma$, where T_1 is the set T for the group $G \otimes \bar{F}$, defined over the algebraic closure of F.

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