# A tropical version of Hilbert polynomial (in dimension one)

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**Abstract**

For a tropical univariate polynomial $f$ we define its tropical Hilbert function as the dimension of a tropical linear prevariety of solutions of the tropical Macauley matrix of the polynomial up to a (growing) degree. We show that the tropical Hilbert function equals (for sufficiently large degrees) a sum of a linear function and a periodic function with an integer period. The leading coefficient of the linear function coincides with the tropical entropy of $f$. Also we establish sharp bounds on the tropical entropy.

**keywords:** tropical Hilbert function, tropical Macauley matrix, tropical entropy

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## Introduction

One can find the basic concepts of tropical algebra in [8].

Consider a tropical univariate polynomial $f := \min_{0\le i\le n}\{iX + a_i\}$ where $a_i \in \mathbb{Z}$, $0 \le i \le n$. We call $y := (y_1, y_2, \ldots)$, $y_j \in \mathbb{R}$, $j \ge 1$ a *tropical recurrent sequence satisfying the vector* $a := (a_0, \ldots, a_n)$ [3] if for any $j \ge 1$ the following tropical (linear) polynomial is satisfied:

$$\min_{0\le i\le n}\{y_{j+i} + a_i\}, \qquad (1)$$

i. e. the minimum in (1) is attained at least for two different values among $0 \le i \le n$.

When one considers classical recurrent sequences $(x_1, x_2, \ldots)$ satisfying relations $\sum_{0\le i\le n} a_i x_{i+j} = 0$ similar to (1), the first $n$ values $x_1, \ldots, x_n$ determine the rest of the sequences uniquely. This is not the case for tropical recurrent sequences.

Denote by $D(s) \subset \mathbb{R}^s$ a polyhedral complex of all the sequences $(y_1, \ldots, y_s)$ satisfying (1) for $1 \le j \le s-n$. The function $d(s) := d_a(s) := \dim(D(s))$ we call *the tropical Hilbert function* of the tropical polynomial $f$ (or equivalently, of the vector $a$ of its coefficients). Obviously, $d(s) \le d(s+1) \le d(s)+1$. It is observed in [3] that $d(s+t) \le d(s)+d(t)$. Therefore, due to Fekete's subadditivity lemma [10] there exists the limit

$$H := H(a) = \lim_{s\to\infty} d(s)/s \qquad (2)$$

which is called [3] *the tropical entropy of the tropical polynomial $f$ or of the vector $a$*. Evidently, $0 \le H \le 1$.

In the classical commutative algebra Hilbert function of a polynomial $g = \sum_I g_I X^I \in F[X_1, \ldots, X_m]$ is defined as the growth function of the quetient ring $F[X_1, \ldots, X_m]/(g)$ in the filtration with respect to degree. For a given degree $e$ this function coincides with the dimension of the space of solutions of a linear system

$$\sum_I g_I Y_{I+J} = 0 \qquad (3)$$

for all vectors $J := (j_1, \ldots, j_m) \in \mathbb{Z}^m$, $0 \le j_1, \ldots, j_m$ such that for every vector $I = (i_1, \ldots, i_m)$ from the support of $g$ we have $i_1 + j_1 + \cdots + i_m + j_m \le e$. Note that a linear system (3) forms the rows of Macauley matrix.

In the tropical algebra there is no concept of a quotient linear space, that is why we stick with an alternative definition of Hilbert function based on the dimension of the space of solutions of a linear system (1) which is equivalent to tropical recurrent sequences.

Note that multidimensional tropical recurrent sequences appear also as the solutions of the tropical Macauley matrix [3] (generalizing tropical equations (1)). Macauley matrix emerges in a tropical version of the weak Hilbert Nullstellensatz (see [1], [2], [5], [6], [7]).

The main result of the paper (see Theorem 5.4 and Corollary 5.5) states that the tropical Hilbert function $d(s)$ is *quasi-linear*, i. e. coincides with a sum $Hs + r(s)$ (for sufficiently large $s$) of a linear function $Hs$ (see (2)) and a periodic function $r(s)$ with an integer period.

Recall that in the classical commutative algebra Hilbert function of an ideal in $F[X_1, \ldots, X_m]$ is a polynomial (for sufficiently large degrees filtrations). In its turn, the degree of this polynomial is less than $m$ (in particular, in case $m = 1$ Hilbert polynomial is a constant). In the tropical setting which we study, the degree of the tropical Hilbert function can be less or equal to $m$, and the coefficient at the power of $m$ equals the entropy $H$ (provided that the function is approximated by a polynomial). Thus, in case of dimension $m = 1$ which we study in the present paper, the tropical Hilbert function $d(s)$ coincides with the linear function $Hs$ up to a periodic function (for sufficiently large $s$).

We mention that in [2], [3] it is proved that $H = 0$ iff each point $(i, a_i) \in \mathbb{R}^2$, $0 \le i \le n$ is a vertex of Newton polygon being the convex hull of the rays $\{(i, x \ge a_i)\}$, $0 \le i \le n$.

It would be interesting to extend the results of the paper to vectors $(a_0, \dots, a_n)$ with $a_i \in \mathbb{Z} \cup \{\infty\}$. Another problem is to improve the bound on the period in the function $r(s)$ and the bound on $s$ starting with which the tropical Hilbert polynomial coincides with $Hs + r(s)$ (sometimes, a bound on $s$ is called the *regularity* of $f$).

In section 1 we prove some auxiliary bounds on tropical recurent sequences. In section 2 we describe a directed graph $G := G_a$ and provide a construction, how following paths of $G$ to yield tropical recurrent sequences (satisfying the vector $a$). In section 3 we show that this construction is complete, so that any tropical recurrent sequence (satisfying the vector $a$) can be yielded following an appropriate path of $G$. In section 4 we prove that the tropical Hilbert function fulfils inequalities $Hs + b \le d(s) \le Hs + e$ for explicitly given constants $b$, $e$. Also, an explicit calculation (and an algorithm as a by-product) of $H$ is provided in terms of $G$, thereby in terms of the vector $a$. In particular, we obtain that $H$ is a rational number. In section 5 the main result of the paper is established. An explicit bound on $s$ starting with which $d(s) = Hs + r(s)$ holds (so, the regularity) is provided. Also, an explicit bound on the period of $r(s)$ is exhibited. In section 6 we consider tropical recurrent sequences $y := (y_1, y_2, \dots)$, $y_j \in \mathbb{R}$, $j \ge 1$ satisfying a *tropical boolean vector* $a := (a_0, \dots, a_n)$ where $a_0 = a_n = 0$ and each $a_i$, $0 \le i \le n$ equals either 0 or $\infty$ and prove the similar to the previous sections results in this case, in particular, the quasi-linearity of the tropical Hilbert function. Finally, in section 7 we establish the sharp lower bound $H(a) \ge 1/4$ on the tropical entropy when $H(a)$ is positive. Also we show the sharp upper bound $H(a) \le 1 - 2/(n+1)$ in case when Newton polygon of $a$ has a single bounded edge. We conjecture that the latter bound holds for an arbitrary vector $a$.

# 1 Bounds on connected coordinates

Let $a := (a_0, \dots, a_n) \in \mathbb{Z}^{n+1}$ be a vector, assume that for its *amplitude* an inequality holds

$$\max_{0 \le i \le n} \{a_i\} - \min_{0 \le i \le n} \{a_i\} \le M. \tag{4}$$

Consider a tropical recurrent sequence $z := (z_0, z_1, \ldots)$, $z_j \in \mathbb{R}$ satisfying vector $a$. We call a coordinate $z_{j_0}$ (or, more precisely, $j_0$) *connected* if there exists $0 \le k_0 \le n$ such that $z_{j_0} + a_{k_0} = \min_{0 \le k \le n}\{z_{k+j_0-k_0} + a_k\}$. In other words, one can't diminish the value of $z_{j_0}$ without changing all other $z_j$, $j \ne j_0$ and keeping the property of being a tropical recurrent sequence satisfying $a$. Otherwise, we call $z_{j_0}$ *disconnected*. We say that connected coordinates $j_0 < j_1$ are *neighbouring* if any intermediate coordinate $j_0 < j < j_1$ is disconnected.

**Lemma 1.1** *Assume that a vector $a \in \mathbb{Z}^{n+1}$ fulfils (4) and a tropical recurrent sequence $z$ satisfies $a$. Let $j_0 < j_1$ be a pair of neighbouring connected coordinates. Then*

*i) $j_1 - j_0 \le n$;*

*ii) $|z_{j_0} - z_{j_1}| \le 2M$.*

**Proof**. To prove i) suppose the contrary, then the minimum $\min_{0 \le k \le n}\{z_{j_0+k} + a_k\}$ is attained only for $k = 0$ which contradicts to that $z$ satisfies $a$.

To prove ii) suppose the contrary. First, for definiteness assume that $z_{j_1} \ge z_{j_0}$, hence $z_{j_1} - z_{j_0} > 2M$. There exists $0 \le k_1 \le n$ such that

$$z_{j_1} + a_{k_1} = \min_{0 \le k \le n}\{z_{j_1+k-k_1} + a_k\}. \tag{5}$$

If $j_1 - k_1 \le j_0$ then $z_{j_0} + a_{j_0-j_1+k_1} < z_{j_1} - 2M + a_{j_0-j_1+k_1} \le z_{j_1} - M + a_{k_1}$. and we get a contradiction with (5), thus $j_1 - k_1 > j_0$.

We claim that the minimum $\min_{0 \le k \le n}\{z_{j_0+k} + a_k\}$ attains only for $k = 0$. Indeed, for any connected $j_2 \le j_0 + n$ we have

$$z_{j_2} + a_{j_2-j_0} \ge z_{j_1} + a_{k_1} - a_{j_2-j_1+k_1} + a_{j_2-j_0} > z_{j_0} + a_0,$$

where the first inequality is due to (5), while the second inequality follows from $z_{j_1} - z_{j_0} > 2M$ and from (4). This proves the claim. We come to a contradiction with that $z$ satisfies $a$, which completes the proof of ii) in case $z_{j_1} \ge z_{j_0}$.

The case $z_{j_1} \le z_{j_0}$ is handled in a similar way. The lemma is proved. $\Box$

**Corollary 1.2** *For a connected coordinate $j$ of a tropical recurrent sequence satisfying $a$*

*i) and any connected coordinate $s$ an inequality holds $z_s \le z_j + 2M|s - j|$:*

*ii) and any coordinate $s$ an inequality holds $z_s \ge z_j - 2M \cdot \max\{|s - j|, n\}$:*

*iii) if $z_{s+n} > \min_{s_0 \le k < s+n}\{z_k\} + 2Mn$ for some $s_0 \ge s$, $s_0 \ge 0$ then the coordinate $s + n$ is disconnected.*

**Proof**. i) follows immediately from Lemma 1.1 ii).

ii) follows from i) when a coordinate $s$ is connected, moreover, in this case

$$z_s \geq z_j - 2M|s-j|. \tag{6}$$

For a disconnected coordinate $s$ one can assume w.l.o.g. that $s > j$. Take the maximal connected coordinate $s_0 < s$. Lemma 1.1 i) implies that $s-s_0 < n$. The minimum $\min_{0\leq k\leq n}\{z_{k+s-n}+a_k\}$ is attained for some connected coordinate $k_0+s-n \leq s_0$. Therefore, when $k_0+s-n \geq j$, we obtain

$$z_s + a_n \geq z_{k_0+s-n} + a_{k_0} \geq z_j - 2M(k_0+s-n-j) + a_{k_0}$$

due to (6) which proves ii) in this case.

When $k_0+s-n < j$, we obtain

$$z_s + a_n \geq z_{k_0+s-n} + a_{k_0} \geq z_j - 2M(j-k_0-s+n) + a_{k_0} \geq z_j - 2M(n-1) + a_{k_0}$$

again due to (6). This completes the proof of ii).

iii) follows from ii). □

# 2 Construction of a graph of tropical recurrent sequences

Assume that we are producing by recursion a tropical recurrent sequence satisfying vector $a = (a_0, \dots, a_n)$, and that a finite fragment (a prefix) of the sequence is already produced. Denote by $(y_1, \dots, y_n) \in \mathbb{R}^n$ the last $n$ coordinates (a suffix) of the fragment. The continuations $y_{n+1} \in \mathbb{R}$ of the fragment depend just on signs of certain linear inequalities on the differences between the pairs of the coordinates $y_i - y_j$, $1 \leq i < j \leq n$. These linear inequalities define a polyhedron $P := P_v \subset \mathbb{R}^n$ which corresponds to a vertex $v$ of a directed finite graph $G := G_a$ which we construct in the present section.

The edges of $G$ outcoming from $v$ (to certain vertices $w$) are determined by possible signs of (linear) inequalities on the differences $y_i - y_{n+1}$, $1 \leq i \leq n$. The latter inequalities together with the (linear) inequalities on the differences $y_i - y_j$, $2 \leq i < j \leq n$ inherited from $P_v$, define a polyhedron $P_w \subset \mathbb{R}^n$ (with the coordinates $y_2, \dots, y_{n+1}$), a vertex $w$ of $G$ and an edge $(v, w)$.

In section 3 we show that the tropical recurrent sequences satisfying $a$ are encoded by paths in $G$ (and vice versa).

## 2.1 Vertices of graph $G_a$

**Definition 2.1** *We define a vertex $v$ of the graph $G$ and a corresponding polyhedron $P := P_v$. The coordinates $y_1, \dots, y_n$ of $P$ are partitioned into* *two parts: we call them bounded and unbounded (we require that at least one coordinate is bounded).*

- *A bounded coordinate $y_{s_0}$ is distinguished and the inequalities*

$$y_{s_0} - y_j \leq 0,\ 1 \leq j \leq n \tag{7}$$

*belong to defining inequalities of $P_v$.*

- *For each pair of bounded coordinates $y_l, y_k, n \geq l > k \geq 1$ an integer*

$$(k-2n)M \leq m := m(k,l) \leq (2n-k)M \tag{8}$$

*is determined such that either an equality*

$$y_k = y_l + m \tag{9}$$

*or inequalities*

$$y_k < y_l + m < y_k + 1 \tag{10}$$

*belong to the defining inequalities of $P_v$.*

- *For each unbounded coordinate $y_j$ the inequality*

$$y_j - y_{s_0} > jM \tag{11}$$

*belongs to defining inequalities of $P_v$.*

*Possible choices of bounded coordinates, of $s_0, m$, of either equations or inequalities provide all the polyhedra $P_v$ and thereby, the vertices $v$ of the graph $G$ under construction.*

**Remark 2.2** *i) If a polyhedron $P_v$ is empty we ignore a vertex $v$.*

*ii) If for bounded coordinates $y_r$, $y_t, 1 \leq r, t \leq n$ holds $y_r - y_t = m$ (see (9)), we say that $r$, $t$ belong to the same (equivalence) class.*

*iii) Informally, the difference (repectively, the ceiling function of the difference) of each pair of bounded coordinates is given in (9) (respectively, in (10)), while for unbounded coordinates just lower bounds (11) via the minimal coordinate, which is always a bounded one, are given.*

## 2.2 An edge of the graph $G$ in case of a unique continuation of a prefix of a tropical recurrent sequence

Now we describe when $G$ has an edge from a vertex $v$ to a vertex $w$. Denote by $x_1, \dots, x_n$ the coordinates of the polyhedron $P_w$ under construction. The polyhedron $P_w$ relates to $P_v$ informally as follows. For any point $(y_1, \dots, y_n) \in P_v$

there exists a point $(y_2, \dots, y_n, x_n) \in P_w$, and $x_n$ fulfills the conditions described below in Definitions 2.5, 2.7, 2.8 (see Theorem 3.1 below). A value of $x_n$ is either unique or varies in an open interval. Formally, in Definitions 2.3, 2.5, 2.7, 2.8 we describe linear inequalities determining $P_w$.

In the following definition we provide a part of equalities and inequalities of the forms (9), (10) describing $P_w$ and complete the description in Definitions 2.5, 2.7, 2.8.

**Definition 2.3** *Denote the coordinates of $P_w$ by $x_1, \dots, x_n$. First, we impose that a coordinate $x_{r-1}, 2 \le r \le n$ is bounded iff the coordinate $y_r$ is bounded. The status of boundness of the coordinate $x_n$ is specified in Definitions 2.5, 2.7, 2.8. Also we require that the inequalities of the forms (9), (10) determining $P_v$ on the pairs of bounded coordinates among $y_k, y_l, 2 \le k < l \le n$ are imposed on the coordinates $x_{k-1}, x_{l-1}$ of $P_w$. In particular, for each equality (respectively, inequalities) of the form (9) (respectively, (10)) the equality $x_{k-1} = x_{l-1} + m$ (respectively, inequalities $x_{k-1} < x_{l-1} + m < x_{k-1} + 1$) belong to determining equalities of the form (9) (respectively, inequalities of the form (10)) of $P_w$. Respectively, we impose the inequalities $(k - 1 - 2n)M \le m \le (2n - k + 1)M$ of the form (8).*

In Definitions 2.5, 2.7, 2.8 we impose equalities and inequalities of the forms (9), (10) which involve the coordinate $x_n$, and also inequalities of the forms (7), (11) for $P_w$.

**Lemma 2.4** *For points $(y_1, \dots, y_n) \in P_v$ the minimum*

$$\min_{1 \le r \le n} \{y_r + a_{r-1}\} \tag{12}$$

*is attained on a suitable subset of the bounded coordinates independent from a point $(y_1, \dots, y_n) \in P_v$.*

**Proof**. Due to (11) the minimum in (12) is attained only on bounded coordinates $y_r$. Let two points $(y_1^{(1)}, \dots, y_n^{(1)})$, $(y_1^{(2)}, \dots, y_n^{(2)}) \in P_v$. Assume that for a pair of bounded coordinates $y_r$, $y_t$ an inequality holds $y_r^{(1)} + a_{r-1} \le y_t^{(1)} + a_{t-1}$. Then $y_r^{(2)} + a_{r-1} \le y_t^{(2)} + a_{t-1}$ because of inequalities (9), (10), taking into the account that $a_{r-1}$, $a_{t-1}$ are integers. $\square$

Denote by $S := S_v$ the set of $r, 1 \le r \le n$ on which the minimum in (12) is attained. In particular, all the elements from $S$ belong to the same class (see Remark 2.2). First consider the case when $S$ consists of a single element $t$.

**Definition 2.5** *Let the set $S = \{t\}$ be a singleton. We define a unique edge in $G$ outcoming from the vertex $v$ (to a vertex $w$) and describe a system* *of equations and inequalities defining a polyhedron $P_w$. Recall that the inequalities on the pairs of bounded coordinates among $x_1, \dots, x_{n-1}$ correspond to the inequalities of the forms (9), (10) on the coordinates $y_2, \dots, y_n$, respectively, defining $P_v$ (see Definition 2.3).*

*Declare the coordinate $x_n$ to be bounded. When $t > 1$ we add the equality $x_n - a_{t-1} + a_n = x_{t-1}$ to the description of $P_w$. For each bounded coordinate $y_r, 2 \leq r \leq n$ if there is an equality $y_t = y_r + m$ of the form (9) in the description of $P_v$, then we add the equality $x_n - a_{t-1} + a_n = x_{r-1} + m$ of the form (9) to the description of $P_w$. In a similar way if there are inequalities $y_r < y_t + m < y_r + 1$ of the form (10) in the description of $P_v$, then we add inequalities $y_r < x_n - a_{t-1} + a_n + m < y_r + 1$ of the form (10) to the description of $P_w$. We impose the inequalities of the form (8) on the integers in the produced inequalities of the forms (9), (10).*

*Based on the produced so far equalities and inequalities of the forms (9), (10) from the description of $P_w$ one can compute a minimal possible $1 \leq s \leq n$ such that $x_s$ is a bounded coordinate and $x_s \leq x_i$ for every bounded coordinate $x_i, 1 \leq i \leq n$. We add the inequality $x_s \leq x_i$ for every $1 \leq i \leq n$ (cf. (7)) and the inequality $x_j - x_s > jM$ for every unbounded coordinate $x_j, 1 \leq j \leq n-1$ (cf. (11)) to the description of $P_w$.*

**Remark 2.6** *In a particular case $t = 1$ and all the coordinates $y_2, \dots, y_n$ are unbounded, the only bounded coordinate of $P_w$ is $x_n$. The description of $P_w$ consists of inequalities $x_n \leq x_i$ (cf. (7)) and of inequalities $x_i - x_n > iM, 1 \leq i \leq n-1$ (cf. (11)).*

## 2.3 Edges of $G$ in case of non-uniqueness of continuations of a prefix of a tropical recurrent sequence

Now we study the case when the set $S$ consists of more than one elements. Take a minimal $t > 1$ such that $t \in S$. There can be several edges in the graph $G$ outcoming from the vertex $v$.

**Definition 2.7** *First define a single edge from the vertex $v$ to a vertex $w$ such that the coordinate $x_n$ is unbounded in $P_w$. Recall that the description of $P_w$ already contains the equalities and inequalities of the forms (9), (10) introduced in Definition 2.3. Based on the latter equalities and inequalities one can compute a minimal possible $1 \leq s \leq n-1$ such that $x_s$ is a bounded coordinate and $x_s \leq x_i$ for every bounded coordinate $x_i, 1 \leq i \leq n-1$. Then we add inequalities $x_s \leq x_i, 1 \leq s \leq n$ to the description of $P_w$ (cf. (7)). Finally, we add the inequality*

$$x_n - x_s > nM \tag{13}$$

*and the inequalities $x_i - x_s > iM$ for all the unbounded coordinates $x_i, 1 \le i \le n$ (cf. (11)).*

We distinguish (13) among the latter inequalities of the form (11) for the sake of easier references below.

The constructed vertex $w$ is the unique one to which there is an edge in the graph $G$ from the vertex $v$ such that the coordinate $x_n$ is unbounded. Still we assume that $|S| \ge 2, t \in S$ with a minimal possible $t > 1$. Now we construct vertices $w$ with a bounded coordinate $x_n$ to which there are edges from $v$.

**Definition 2.8** *We declare the coordinate $x_n$ to be bounded. Recall that the description of $P_w$ already contains the equalities and inequalities of the forms (9), (10) introduced in Definition 2.3. We add to the description of $P_w$ either the equality*

$$x_n - x_{t-1} = a_{t-1} - a_n \tag{14}$$

*or inequalities*

$$nM \ge x_n - x_{t-1} > a_{t-1} - a_n. \tag{15}$$

*For every bounded coordinate $x_l, 1 \le l \le n$ we consider all possible consistent (with the equalities and inequalities of the forms (9), (10) introduced in Definition 2.3 and with either (14) or with (15), respectively) either equalities of the form $x_l = x_n + m^{(l)}$ or inequalities of the form $x_l < x_n + m^{(l)} < x_l + 1$ for some integers $m^{(l)}$. Due to either (14) or (15), respectively, there is a finite numbers of possible integers $m^{(l)}$. The produced equations and inequalities constitute the ones of the forms (9), (10) for $P_w$.*

*Based on the latter equalities and inequalities and the ones introduced in Definition 2.3 we compute a minimal possible $1 \le s \le n$ such that $x_s \le x_i$ for every bounded coordinate $x_i, 1 \le i \le n$. We add inequalities $x_s \le x_i, 1 \le i \le n$ (cf. (7)) to the description of $P_w$. Also we impose the inequality*

$$x_n - x_s \le nM \tag{16}$$

*which in its turn imposes inequalities on the integers $m^{(l)}$. We impose the inequalities of the form (8) on the integers in the produced inequalities of the forms (9), (10). Finally, we add to the description of $P_w$ inequalities $x_i - x_s > iM$ for every unbounded coordinate $x_i, 1 \le i \le n-1$ (cf. (11)).*

**Remark 2.9** *i) Choosing all possible integers $m^{(l)}$ and either equalities or inequalities, we obtain all the vertices $w$ to which there are edges from $v$ with a bounded coordinate $x_n$.*

*ii) The inequalities either (14) or (15) and the inequality (16) follow from the produced ones of the forms (9), (10) defining $P_w$. In particular, the lefthand* *inequality of (15) and (16) follow from either an equality $x_n - x_s = m$ of the form (9) or an inequality $x_n - x_s < m$ of the form (10) in the description of $P_w$ for some integer $m \leq nM$ (cf. (8)). The similar concerns the equality (14) and the righthand inequality of (15). Thus, (14), (15) and (16) do not formally occur in the description of $P_w$ (to be compatible with Definition 2.1). We have distinguished (14), (15) and (16) just for the sake of easier references below.*

*iii) One can (equivalently) construct the edges outcoming from the vertex $v$ with a bounded coordinate $x_n$ in a more explicit manner. The equations of the form $x_i - x_j = m$ and the inequalities of the form $x_i - x_j > m$, $1 \leq i, j \leq n-1$ defining $P_w$ produced in Definition 2.3, provide uniquely the linear ordering between $x_i + k$, $-N \leq k \leq N$ for all bounded coordinates among $x_1, \dots, x_{n-1}$ where $N$ bounds from above all integers $m^{(l)}$ occurred in Definition 2.8. Then a vertex $w$ is uniquely determined by a choice of either an equation $x_n = x_i + k$ or inequalities $x_i + k_1 < x_n < x_j + k_2$ for suitable bounded coordinates $x_i$, $x_j$ and $-N \leq k$, $k_1$, $k_2 \leq N$ such that for no bounded coordinate $x_l$, $1 \leq l \leq n-1$ and an integer $k_0$ holds $x_i + k_1 < x_l + k_0 < x_j + k_2$.*

This completes the description of all the edges outcoming from the vertex $v$ in the graph $G$.

# 3 Description of tropical recurrent sequences via paths in the graph

## 3.1 Yielding a short tropical recurrent sequence along an edge of the graph

In this subsection for any point $(y_1, \dots, y_n) \in P_v$ we prove the following claim. If a sequence $(y_1, \dots, y_n, x) \in \mathbb{R}^{n+1}$ satisfies the vector $a$ then for exactly one of the edges $(v, w)$ of the graph $G$ it holds that $(y_2, \dots, y_n, x) \in P_w$. Conversely, for every edge $(v, w)$ of $G$ constructed according to one of Definitions 2.5, 2.7, 2.8 there exists a point $(y_2, \dots, y_n, x_n) \in P_w$ such that the point $(y_1, \dots, y_n, x_n) \in \mathbb{R}^{n+1}$ satisfies the vector $a$ (for more precise statements see Theorem 3.1).

We assume that a point $(y_1, \dots, y_n, x_n) \in \mathbb{R}^{n+1}$ satisfies the vector $a$. Denote $x_i := y_{i+1}, 1 \leq i \leq n-1$. We declare that a coordinate $y_{i+1}$ is bounded in $P_v$ iff the coordinate $x_i$ is bounded in $P_w$. Then the bounded coordinates among $x_1, \dots, x_{n-1}$ fulfill the inequalities of the forms (9), (10) introduced in Definition 2.3, and moreover, they fulfill the inequalities of the form (8) for $P_w$ since the inequalities of the form (8) for the bounded coordinates $x_1, \dots, x_{n-1}$

in $P_w$ are weaker than the corresponding inequalities of the form (8) for the same coordinates $(y_2, \dots, y_n) = (x_1, \dots, x_{n-1})$ for $P_v$.

Consider the case of a singleton $S = \{t\}$. Then $x_n = y_t + a_{t-1} - a_n$. We claim that $(y_2, \dots, y_n, x_n) \in P_w$ where the edge $(v, w)$ of $G$ is constructed according to Definition 2.5. In particular, we declare the coordinate $x_n$ to be bounded. For every unbounded coordinate $y_i, 1 \le i \le n$ the inequalities (11) imply that $y_i - y_{s_0} > iM$. Recall (see Definitions 2.5, 2.7, 2.8) that for a bounded coordinate $x_s$ it holds $x_s \le x_l$ for each bounded coordinate $x_l, 1 \le l \le n$. If $s_0 = 1$ then $x_s \le x_n \le y_{s_0} + M$. Otherwise, if $1 < s_0 \le n$ then $x_s \le x_{s_0-1} = y_{s_0}$. Therefore, for $1 < i \le n$ it holds that

$$x_{i-1} - x_s = y_i - x_s \ge y_i - y_{s_0} - M > (i-1)M. \tag{17}$$

In particular, the point $(x_1, \dots, x_n)$ fulfills the inequalities of the form (7). Also, the point $(x_1, \dots, x_n)$ fulfills the inequalities of the form (11).

The inequality $|y_t - x_n| \le M$ implies that the point $(x_1, \dots, x_n)$ fulfills the inequalities of the forms (8), (9), (10) on the differences $x_n - x_j$ where $x_j, 1 \le j < n$ is a bounded coordinate. Thus, the point $(x_1, \dots, x_n) \in P_w$ for an edge $(v, w)$ of $G$ constructed according to Definition 2.5. This proves the claim in case $S = \{t\}$.

Now we study the case when $|S| \ge 2$ and the inequality (13) is true. We claim that in this case $(x_1, \dots, x_n) \in P_w$ where the edge $(v, w)$ of $G$ is constructed according to Definition 2.7. We declare the coordinate $x_n$ to be unbounded. There exists $2 \le t \le n$ for which $t \in S$. If $s_0 = 1$ then taking into the account that $y_t + a_{t-1} \le y_1 + a_0$, we get that $x_s \le x_{t-1} = y_t \le y_1 + M$. Otherwise, if $2 \le s_0 \le n$ then $y_{s_0} = x_s$. In any case it holds $x_s \le y_{s_0} + M$. As in case $S = \{y_t\}$ considered above, we obtain (17). This implies the inequalities of the forms (7) and (11) making use of (13). This completes the proof of the claim in case of $|S| \ge 2$ and (13).

Now we assume that $|S| \ge 2$ and (16) (so, (13) is not true). We claim that $(x_1, \dots, x_n) \in P_w$ where the edge $(v, w)$ of $G$ is constructed according to Definition 2.8. We declare the coordinate $x_n$ to be bounded. There exists $2 \le t \le n$ for which $t \in S$. It holds $x_n \ge x_{t-1} + a_{t-1} - a_n$. The inequality $y_t + a_{t-1} \le y_1 + a_0$ implies that $x_s \le y_{s_0} + M$ and (17) (as above). Again this implies the inequalities of the forms (7) and (11).

For each bounded coordinate $x_l, 1 \le l < n$ there exists a unique integer $m^{(l)}$ such that either $x_n = x_l + m^{(l)}$ or $x_n < x_l + m^{(l)} < x_n + 1$ holds. They constitute the inequalities of the form either (9) or (10), respectively, in the description of $P_w$ according to Definition 2.8. The inequality $x_n \ge x_{t-1} + a_{t-1} - a_n$ entails that $x_s \ge y_{s_0} - M$. Together with the inequalities (8) for $P_v$ and (16) this implies the inequalities of the form (8) for $P_w$ on the integers $m^{(l)}$. This completes the proof of the claim.

Conversely, assume that it holds $(x_1, \ldots, x_n) := (y_2, \ldots, y_n, x_n) \in P_w$ for an edge $(v, w)$ of $G$ constructed according to one of Definitions 2.5, 2.7, 2.8. First, we study the case

i) there exists $t \in S$, $2 \leq t \leq n$. If $S = \{t\}$ then $x_n = x_{t-1} + a_{t-1} - a_n$. Otherwise, if $|S| \geq 2$ then $x_n \geq x_{t-1} + a_{t-1} - a_n$. Therefore the point $(y_1, \ldots, y_n, x_n) \in \mathbb{R}^{n+1}$ satisfies the vector $a$.

Observe that if the edge $(v, w)$ is constructed according to Definition 2.7 then the values of the coordinate $x_n$ vary in an open infinite interval bounded from below. If the edge $(v, w)$ is constructed according to Definition 2.8 and the description of $P_w$ contains an equality $x_n = x_l + m^{(l)}$ of the form (9) for some $1 \leq l \leq n-1$ then the value of the coordinate $x_n$ is unique. Otherwise, the values of the coordinate $x_n$ vary in an open finite interval.

ii) Now assume that $S = \{1\}$, then the point $(x_1, \ldots, x_n) \in P_w$ for an edge $(v, w)$ constructed according to Definition 2.5. If the description of $P_v$ contains an equality $y_1 = y_l + m$ of the form (9) for some $2 \leq l \leq n$ then the description of $P_w$ contains the equality $x_n + a_n - a_0 = x_{l-1} + m$. Hence in this case for any point $(y_2, \ldots, y_n, x_n) \in P_w$ the point $(y_1, \ldots, y_n, x_n) \in \mathbb{R}^{n+1}$ satisfies the vector $a$ (in fact, $(y_2, \ldots, y_n, x_n) \in P_w$ implies that $x_n = y_1 + a_0 - a_n$). Otherwise, the values of the coordinate $x_n$ such that $(y_2, \ldots, y_n, x_n) \in P_w$ vary in an open interval (perhaps, an infinite one), while only for the value $x_n = y_1 + a_0 - a_n$ the point $(y_1, \ldots, y_n, x_n) \in \mathbb{R}^{n+1}$ satisfies the vector $a$.

Summarizing, we have proved the following theorem.

**Theorem 3.1** *Let a point $(y_1, \ldots, y_n) \in P_v$.*

*If a point $(y_1, \ldots, y_n, x_n) \in \mathbb{R}^{n+1}$ satisfies the vector $a$ then $(y_2, \ldots, y_n, x_n) \in P_w$ holds for exactly one edge $(v, w)$ of the graph $G$ constructed according to Definitions 2.5, 2.7, 2.8.*

*Conversely, let $(y_2, \ldots, y_n, x_n) \in P_w$ for an edge $(v, w)$ of $G$ constructed according to one of Definitions 2.5, 2.7, 2.8.*

*i) In case when there exists $t \in S$, $2 \leq t \leq n$ (see subsection 2.2) the point $(y_1, \ldots, y_n, x_n) \in \mathbb{R}^{n+1}$ satisfies the vector $a$. In case of an edge constructed according to*

- *Definition 2.5, the value of $x_n$ is unique;*
- *Definition 2.7, the values of $x_n$ vary in an open infinite interval bounded from below;*
- *Definition 2.8, the values of $x_n$ depending on the edge $(v, w)$, can be either unique or vary in an open finite interval.*

*ii) If $S = \{1\}$ then only for the value $x_n = y_1 + a_0 - a_n$ the point $(y_1, \ldots, y_n, x_n) \in \mathbb{R}^{n+1}$ satisfies the vector $a$.*

## 3.2 The polyhedron of tropical recurrent sequences produced along a path of the graph

We consider paths in the graph $G$ and describe how they correspond to the tropical recurrent sequences satisfying the vector $a$. Take an arbitrary vertex $v_0$ as the first vertex in a path and any sequence $y^{(0)} := (y_1^{(0)}, \dots, y_n^{(0)}) \in P_{v_0}$. As in subsection 2.2 consider a subset $S$. If $|S| = 1$ then there is a unique edge $(v_0, w_0)$ in $G$ outcoming from $v_0$. In this case one applies Definition 2.5 and obtains a unique $y_{n+1}^{(0)} := x_n^{(0)} \in \mathbb{R}$ such that $(y_2^{(0)}, \dots, y_{n+1}^{(0)}) \in P_{w_0}$ and $(y_1^{(0)}, \dots, y_{n+1}^{(0)})$ satisfies vector $a$ (see Theorem 3.1).

Otherwise, if $|S| > 1$ then there are several edges outcoming from $v_0$. For each edge $(v_0, v_1)$ one applies either Definition 2.7 or Definition 2.8, respectively, and produces $y_{n+1}^{(0)} := x_n^{(0)} \in \mathbb{R}$ such that $(y_2^{(0)}, \dots, y_{n+1}^{(0)}) \in P_{v_1}$ and $(y_1^{(0)}, \dots, y_{n+1}^{(0)})$ satisfies vector $a$ (see Theorem 3.1). Recall (see Theorem 3.1 i)) that for certain edges $(v_0, v_1)$ the value $y_{n+1}^{(0)}$ is unique, while for other edges $y_{n+1}^{(0)}$ runs over an open interval.

An edge $(v_0, v_1)$ for which the value $y_{n+1}^{(0)}$ is unique we call *rigid*, otherwise if the values run over an open interval we call an edge *augmenting*. Due to Theorem 3.1 i) the property of an edge to be rigid or augmenting does not depend on a point $y^{(0)}$. Note that in case of $S$ being a singleton, the edge is rigid (cf. also Theorem 3.1 ii)).

So far, we have yielded a short tropical recurrent sequence $(y_1^{(0)}, \dots, y_{n+1}^{(0)})$ corresponding to an edge of $G$. We treat this as a base of recursion. Suppose that we have yielded by recursion a tropical recurrent sequence $(y_1^{(0)}, \dots, y_{n+k}^{(0)})$ satisfying the vector $a$ corresponding to a path $T$ of the length $k$ in $G$ (the length of a path is defined as the number of its edges). Let $v$ be the last vertex of $T$. Then we apply to $v$ and to the suffix $(y_{k+1}^{(0)}, \dots, y_{n+k}^{(0)})$ of the yielded sequence one of Definitions 2.5, 2.7, 2.8 as above in the base of recursion, choosing an edge $(v, w)$ of $G$ and yielding $y_{n+k+1}^{(0)}$. Thereby, we get a tropical recurrent sequence $(y_1^{(0)}, \dots, y_{n+k+1}^{(0)})$ satisfying the vector $a$ and corresponding to the path $T_w$ obtained by extending $T$ by an edge $(v, w)$. This completes the recursive step.

Summarizing, we have established in this subsection the following proposition.

**Proposition 3.2** *For any path in the graph $G$ any yielded (by the described recursive process) sequence following this path is a tropical recurrent sequence satisfying vector $a$.*

Denote by $Q_T \subset \mathbb{R}^{k+n}$ a polyhedron of all the tropical recurrent sequences which are yielded following the path $T$ as described above (see Proposition 3.2).

Thus, any yielded tropical recurrent sequence satisfies the vector $a$. The polyhedron $Q_T$ is presented by the systems of linear equations and linear inequalities produced in Definitions 2.5, 2.7, 2.8, respectively, applied to the edges of the path $T$ (see Theorem 3.1). Observe that when $S \neq \{1\}$ Theorem 3.1 i) implies that for the inequalities describing $Q_T$ just the inequalities describing $P_v$ and $P_w$ suffice, while when $S = \{1\}$ one has to add to the latter inequalities also the equality $x_n = y_1 + a_0 - a_n$ (see Theorem 3.1 ii)).

Observe that for a rigid edge $(v, w)$ the polyhedron $Q_{T_w} \subset \mathbb{R}^{k+n+1}$ is homeomorphic to $Q_T$, and the homeomorphism is provided by the projection along the last coordinate. For an augmenting edge $(v, w)$ the polyhedron $Q_{T_w}$ is homeomorphic to the cylinder $Q_T \times \mathbb{R}$. In particular, in the latter case $\dim(Q_{T_w}) = \dim(Q_T) + 1$. Summarizing, we have established the following proposition.

**Proposition 3.3** *Let $T$ be a finite path of the graph $G$ with an ending vertex $v$, and $T_w$ be an extension of $T$ by an edge $(v, w)$. If the edge $(v, w)$ is rigid then the polyhedron $Q_{T_w}$ of all the finite tropical recurrent sequences yielded following $T_w$ (see Proposition 3.2) is homeomorphic to the polyhedron $Q_T$, while if $(v, w)$ is augmenting then $Q_{T_w}$ is homeomorphic to the cylinder $Q_T \times \mathbb{R}$.*

## 3.3 Completeness of the construction of tropical recurrent sequences

Now, conversely to Proposition 3.2, we claim that every tropical recurrent sequence $y := (y_1, y_2, \dots)$ satisfying the vector $a$ emerges while following an appropriate path of the graph $G$ (see subsection 3.2). For the base of recursion denote $q := \min_{1 \leq j \leq n}\{y_j\}$. If for some $1 \leq j \leq n$ it holds $y_j - q > jM$ then the coordinate $y_j$ we declare unbounded (cf. (11)), otherwise - bounded. There exists an initial vertex $v_0$ of $G$ with the bounded and unbounded coordinates specified as in the previous sentence (see Definition 2.1) such that $(y_1, \dots, y_n) \in P_{v_0}$.

For the recursive step suppose that a path $T$ of $G$ of a length $k$ is already produced such that the sequence $(y_1, \dots, y_{n+k})$ is yielded following $T$ as in subsection 3.2 (see Proposition 3.2). Let $v$ be the last vertex of $T$, then $y^{(k)} := (y_{k+1}, \dots, y_{k+n}) \in P_v$. Apply Theorem 3.1 to $y^{(k)}$, this provides a unique edge $(v, w)$ of $G$ such that $(y_{k+2}, \dots, y_{k+n+1}) \in P_w$, thus the sequence $(y_1, \dots, y_{n+k+1})$ is yielded following the extended path $T_w$. This completes the proof (by recursion) of the claim.

Observe that one could choose, perhaps, another initial vertex $v'$ of $G$ such that $(y_1, \dots, y_n) \in P_{v'}$ (the latter inclusion is the only property of $v'$ we require). In fact, one could declare (in an arbitrary way) any coordinate $y_j, 1 \leq j \leq n$ either bounded or unbounded if it fulfills the inequalities $jM <$

$y_j - q \le (2n-j)M$ (see (8)). If $y_j - q \le jM$ then $y_j$ should be bounded (cf. (11)), while if $y_j - q > (2n-j)M$ then $y_j$ should be unbounded. After choosing an initial vertex $v_0$, the rest of a path $T$ in $G$ is produced uniquely (see Theorem 3.1 and subsection 3.2). Therefore, each tropical recurrent sequence satisfying the vector $a$ corresponds to just a finite number of paths in $G$ as in subsection 3.2 (see Proposition 3.2). Moreover, this number does not exceed the number of vertices in $G$. Thus, a tropical prevariety of all the tropical recurrent sequences of a length $n+k$ satisfying the vector $a$ has the same dimension as the union of polyhedra $Q_T$ over all the paths $T$ of the length $k$ in $G$. Thus, the following proposition is established.

**Proposition 3.4** *i) The union of the polyhedra $P_v$ over all the vertices $v$ of the graph $G$ coincides with $\mathbb{R}^n$.*

*ii) For any tropical recurrent sequence $y := (y_1, y_2, \dots)$ satisfying the vector $a$ and a vertex $v$ of $G$ such that $(y_1, \dots, y_n) \in P_v$ there exists a unique path $T$ of $G$ starting with $v$ such that $y$ is yielded along $T$ as described in subsection 3.2 (see Proposition 3.2).*

For a path $T$ in the graph $G$ denote by $d(T)$ the number of augmenting edges in $T$. By $n(T) \le n$ denote the number of (equivalence) classes of the coordinates in the first vertex of $T$ (see Remark 2.2). We summarize the proved above in the following theorem taking into account Propositions 3.2, 3.3, 3.4.

**Theorem 3.5** *For any vector $a := (a_0, \dots, a_n) \in \mathbb{Z}^{n+1}$ with an amplitude $M$ (4) the constructed finite directed graph $G := G_a$ satisfies the following properties. For a path $T$ of a length $k$ in $G$ denote by $Q_T \subset \mathbb{R}^{k+n}$ the polyhedron of all the tropical recurrent sequences satisfying the vector $a$ and being yielded while following the path $T$ in $G$. Then $\dim(Q_T) = d(T) + n(T)$. Moreover, the union of polyhedra $Q_T$ over all the paths $T$ of the length $k$ coincides with the tropical prevariety of all the tropical recurrent sequences of the length $k+n$ satisfying the vector $a$.*

# 4 Calculating the entropy via the graph of tropical recurrent sequences

In this section we study the tropical Hilbert function $d(s) := d_a(s)$ (see the Introduction). Due to Theorem 3.5 $d(s)$ equals the maximum of $n(T) + d(T)$ over all the paths $T$ of the length $s - n$ in the graph $G$.

We call a simple cycle in $G$ *optimal* if the quotient of the number of augmenting edges in the cycle to the length of the cycle is the maximal among the simple cycles. This maximal quotient we denote by $\mathcal{H} := \mathcal{H}_a$. Later we show that $\mathcal{H}$ equals the entropy $H := H_a$ (Corollary 4.3). Clearly, $\mathcal{H}$ equals the maximum of the same quotient over all the cycles in $G$ (not necessary, simple).

First, we prove a lower bound on the tropical Hilbert function $d(s)$.

**Lemma 4.1** $d(s) \geq \mathcal{H}(s-n)$.

**Proof**. Take an optimal simple cycle $U$ in $G$. Denote the length of $U$ by $L$ and the number of augmenting edges in $U$ by $m$, then $\mathcal{H} = m/L$. Assign to each augmenting edge of $U$ the number $1-\mathcal{H}$ and to each rigid edge the number $-\mathcal{H}$. Then the sum of all these numbers equals 0. Due to the lemma about leaders [9] there exists a vertex $u$ of $U$ such that the sum of the assigned numbers along any subpath of $U$ starting with $u$ is non-negative.

Consider a path $T$ of a length $s-n$ starting with the vertex previous to $u$ in $U$ and following the cycle $U$ (i. e. $T$ can wind the cycle $U$ several times). According to Theorem 3.5 $\dim(Q_T) \geq \mathcal{H}(s-n)$. □

Denote by $V$ the number of vertices in $G$. Now we proceed to an upper bound on the tropical Hilbert function.

**Lemma 4.2** $d(s) \leq \mathcal{H}s + (1-\mathcal{H})(V+n)$.

**Proof**. Consider a path $T$ of a length $L$ in $G$. Take a vertex $v_1$ of $G$ which occurs in $T$ at least twice (provided that it does exist). Then the subpath of $T$ between these two occurrings constitues a cycle of a length $L_1$. Remove this cycle from $T$, and continue removing cycles from the resulting paths, while it is possible. Let $L_2, L_3, \dots, L_q$ be the lengths of the consecutively removed cycles. Then

$$d_T \leq \mathcal{H}(L_1 + \cdots + L_q) + (L - L_1 - \cdots - L_q) \leq \mathcal{H}(L_1 + \cdots + L_q) + V$$

(cf. Theorem 3.5). Therefore, $d(s) \leq \mathcal{H}(s-n) + (1-\mathcal{H})V + n$ taking into the account that $L - L_1 - \cdots - L_q \leq V$. □

Lemmata 4.1, 4.2 imply the following corollary (see (2)).

**Corollary 4.3** $\mathcal{H} = H$.

**Remark 4.4** *The entropy $H$ is a rational number.*

# 5 Quasi-linearity of the tropical Hilbert function

**Lemma 5.1** *Any path $T$ of a length $s-n$ greater than $V^2(V+n)+V$ in the graph $G$ such that $n(T)+d(T)=d(s)$, contains a vertex from an optimal cycle.*

**Proof**. First consider the case when $H = 0$. Then any simple cycle in $G$ is optimal, and the statement of the lemma is true even with a better bound $s - n > V$. Thus, from now on in the proof of the lemma we assume that $H > 0$.

Recall that according to Theorem 3.5 $\dim(Q_T) = n(T) + d(T)$. Slightly modifying the construction from the proof of Lemma 4.2 take the first repetition of some vertex $v$ in $T$ (provided that it is possible). Then the subpath of $T$ between these two occurrences of $v$ constitutes a simple cycle of a length $L_1$ in $T$. Remove this cycle from $T$ and continue removing cycles from the resulting paths in a similar way, while it is possible. Denote by $L_2, L_3, \dots, L_q$ the lengths of the consecutively removed cycles. Denote by $B$ the denominator of $H$ (cf. Remark 4.4), obviously $B \le V$ (see section 4).

Assume the contrary to the claim of the lemma. Then

$$d(T) \le H(L_1 + \cdots + L_q) - q/B + (s - n - L_1 - \cdots - L_q) \le H(s-n) - q/B + V.$$

The first inequality follows from the statement that the amount of augmenting edges in the cycle with the length $L_i$, $1 \le i \le q$ is not greater than $H \cdot L_i - \frac{1}{B}$. Making use of Lemma 4.1 we obtain an inequality $q/B \le V + n$, hence $q \le V(V+n)$. The path $T$ consists of $q$ cycles and a path without cycles. Each cycle has length not more than $V$ as well as the path without cycles. Therefore, $s - n \le V^2(V+n) + V$ since $L_1, \dots, L_q \le V$. $\square$

Denote by $R$ the least common multiple of the lengths of all the optimal cycles.

**Lemma 5.2** *For any $s > (V^2+1)(V+n)$ we have $d(s+R) \ge d(s) + HR$.*

**Proof**. Take a path $T$ of the length $s-n$ in $G$ such that $n(T)+d(T) = d(s)$ (cf. Theorem 3.5). Due to Lemma 5.1 $T$ contains a vertex $v$ which belongs to an optimal cycle $C$ of a length $c$. Glue in the path $T$ at the vertex $v$ the number $R/c$ of copies of the cycle $C$, the resulting path of the length $s-n+R$ denote by $T_1$. In other words, in $T_1$ one follows first $T$ till the vertex $v$, then there are $R/c$ windings of the cycle $C$ (finishing at $v$), finally after that one again follows path $T$ (starting at $v$). Clearly, $d(T_1) = d(T) + (R/c)Hc$. $\square$

**Lemma 5.3** *If for some $s > (V^2+1)(V+n)$ we have*

$$d(s+iR) = d(s) + HiR,\ 0 \le i \le V((1-H)V + n + 1)$$

*then $d(s+jr) = d(s) + HjR$ for any $j \ge 0$.*

**Proof**. Due to Lemma 5.2 it holds $d(s+jR) \ge d(s) + HjR$. Suppose that

$$d(s+jR) > d(s) + HjR \tag{18}$$

for some $j > V((1-H)V+n+1)$, and take the minimal such $j$. There exists a path $T$ of the length $s+jR-n$ in $G$ for which $n(T)+d(T)=d(s+jR)$. For $0 \leq i \leq V((1-H)V+n+1)$ denote by $T_i$ the beginning of the $T$ of the length $s+iR-n$. One can represent the path $T=T_i\overline{T_i}$ as a concatenation of two paths.

There exists a subsequence $0 \leq i_0 < i_1 < \cdots < i_{(1-H)V+n+1} \leq V((1-H)V+n+1)$ such that each path $T_{i_l}$, $0 \leq l \leq (1-H)V+n+1$ ends with the same vertex $v$ of $G$. Assume that there exists $0 \leq l \leq (1-H)V+n$ for which

$$d(T_{i_{l+1}}) \leq d(T_{i_l}) + H(i_{l+1}-i_l)R. \tag{19}$$

Then we consider a concatenation $\overline{T} := T_{i_l}\overline{T_{i_{l+1}}}$ being a path of the length $s+jR-n-(i_{l+1}-i_l)R$ in $G$. We obtain

$$d(\overline{T}) = d(T) + d(T_{i_l}) - d(T_{i_{l+1}}) > d(s) + HjR - H(i_{l+1}-i_l)R$$

due to (18), (19), and we get a contradiction with the choice of the minimal $j$ (see (18)).

Thus, for every $0 \leq l \leq (1-H)V+n$ we have

$$d(T_{i_{l+1}}) \geq d(T_{i_l}) + H(i_{l+1}-i_l)R + 1.$$

Summing up these inequalities for $0 \leq l \leq (1-H)V+n$ we conclude that

$$d(T_{i_{(1-H)V+n+1}}) - d(T_{i_0}) \geq H(i_{(1-H)V+n+1} - i_0)R + (1-H)V + n + 1$$

which contradicts to Lemmata 4.1, 4.2. □

Note that $V < (Mn)^n$ (see Definition 2.1) and $R < \exp(V)$. Lemmata 4.1, 4.2, 5.2, 5.3 entail the following theorem.

**Theorem 5.4** *For $s > (V^2+1)(V+n)+V((1-H)V+n+1)^2$ the tropical Hilbert function $d_a(s)$ of the integer vector $a=(a_0,\ldots,a_n)$ with an amplitude at most $M$ (4) fulfils the following equality:*

$$d_a(s+R) = d_a(s) + HR.$$

*for some integer $R < \exp((O(Mn))^n)$ where $H := H_a$ is the tropical entropy of the vector $a$.*

We call a function (from the natural numbers to themselves) *quasi-linear* if it is a sum of a linear function and a periodic function with an integer period.

**Corollary 5.5** *The tropical Hilbert function*

$$d(s) = Hs + r(s)$$

*is quasi-linear for $s > (Mn)^{O(n)}$ where $r(s)$ is a periodic function with an integer period less than $\exp((O(Mn))^n)$.*

**Example 5.6** *Following [3], [4] one can show that*

- $d_{(0,0,0)}(s) = \lceil s/3 \rceil$;
- $d_{(0,1,0)}(s) = \lceil s/4 \rceil$.

**Remark 5.7** *In case when the tropical entropy $H = H(a) = 0$ Lemma 4.2 implies that $d(s) \equiv const$ for sufficiently large $s$, taking into account that $d(s)$ is a non-decreasing function. Recall (see [3]) that Newton polygon $\mathcal{N}(a) \subset \mathbb{R}^2$ for a vector $a = (a_0, \dots, a_n)$ is defined as the convex hull of the rays $\{(i, y) : y \geq a_i\}$ for $0 \leq i \leq n$. We say that the vector $a$ is regular [3] if each point $(i,\, a_i)$ with $a_i < \infty$ is a vertex of $\mathcal{N}(a)$, and the indices $i$ for which $a_i < \infty$ constitute an arithmetic progression. It was proved in [3, Corollary 5.7] that $H(a) = 0$ iff $a$ is regular. For regular $a$ in case when each $(i,\, a_i)$, $0 \leq i \leq n$ is a vertex of $\mathcal{N}(a)$ one can deduce from [2, Corollary 4.9] that $d(s) = s$ for $s \leq n$ and $d(s) = n$ for $s \geq n$.*

# 6 Tropical boolean vectors

As we already mentioned it would be interesting to extend the results of the paper to arbitrary vectors $a$ involving infinite coordinates. The first step to implementing this idea can be considered as the construction of an appropriate graph $G_a$ (cf. section 2) for the case when $a$ is a tropical boolean vector (see the Introduction). In this case, the construction looks simpler and contains less technical details comparing to the case considered in the previous sections 2, 3.

## 6.1 Construction of a graph for tropical boolean vectors

We call a vector $a = (a_0, \dots, a_n)$ *tropical boolean vector* if for all $0 \leqslant i \leqslant n$ it holds either $a_i = 0$ or $a_i = \infty$, and $a_0 = a_n = 0$.

Below we construct a directed graph $G := G_a$. First we define the vertices of $G$.

**Definition 6.1** *Every its vertex $v$ corresponds to an (open in its linear hull) polyhedron $P := P_v \subset \mathbb{R}^n$ with the condition that for each pair of coordinates $y_r, y_t, 1 \leq r, t \leq n$ a system of equations and strict inequalities defining $P$ contains either $y_r = y_t$ or $y_r < y_t$.*

These linear restrictions set the order on the coordinates $y_1, \dots, y_n$. The polyhedra $\{P_v\}$ constitute a partition of $\mathbb{R}^n$. When $P_v$ is empty we ignore $v$. Now we define the edges of $G$.

**Definition 6.2** *There is an edge $(v, w)$ in $G$ iff there exist vectors $(y_0, \dots, y_{n-1}) \in P_v, (y_1, \dots, y_n) \in P_w$ such that the sequence $(y_0, \dots, y_n) \in \mathbb{R}^{n+1}$ satisfies the vector $a$.*

Similar to subsection 2.2 for a vertex $v$ of $G$ define $S := S_v$ as a set of $0 \leq t \leq n-1$ such that $y_t = a_t + y_t = \min_{0 \leq j \leq n-1}\{a_j + y_j\}$. In other words, $t \in S$ iff $a_t = 0$ and $y_t \leq y_j$ for each $0 \leq j \leq n-1$ such that $a_j = 0$. The definition of $S$ does not depend on a choice of a point $(y_0, \dots, y_{n-1}) \in P_v$ (cf. Lemma 2.4). The following theorem is similar to Theorem 3.1.

**Theorem 6.3** *Let $v$ be a vertex of the graph $G := G_a$ and $(y_0, \dots, y_{n-1}) \in P_v$.*

*If a point $(z_0, \dots, z_{n-1}) \in P_v$ and a sequence $(z_0, \dots, z_n) \in \mathbb{R}^{n+1}$ satisfies the vector $a$ then $(z_1, \dots, z_n) \in P_w$ for some edge $(v, w)$ of $G$.*

*Conversely, let $(y_1, \dots, y_n) \in P_w$ for an edge $(v, w)$ of $G$, and the sequence $(y_0, \dots, y_n) \in \mathbb{R}^{n+1}$ satisfy the vector $a$. If $t \in S$ for some $0 \leq t \leq n-1$ then $y_n \geq y_t$.*

*i) Let $t \in S$ for some $1 \leq t \leq n-1$ and $y_r = y_n$ for some $1 \leq r \leq n-1$. Assume that a point $(z_0, \dots, z_{n-1}) \in P_v$. If a point $(z_1, \dots, z_{n-1}, z) \in P_w$ then $z = z_r$. The point $(z_1, \dots, z_{n-1}, z_r) \in P_w$, and the sequence $(z_0, \dots, z_{n-1}, z_r) \in \mathbb{R}^{n+1}$ satisfies the vector $a$.*

*ii) Let $t \in S$ for some $1 \leq t \leq n-1$. Assume that $y_{r_1} < y_n$ for some $1 \leq r_1 \leq n-1$ and for every $1 \leq r \leq n-1$ neither $y_{r_1} < y_r \leq y_n$ nor $y_n \leq y_r$ holds. Then for any point $(z_0, \dots, z_{n-1}) \in P_v$ if a point $(z_1, \dots, z_{n-1}, z) \in P_w$ then $z_{r_1} < z$ and for every $1 \leq r \leq n-1$ neither $z_{r_1} < z_r \leq z_n$ nor $z_n \leq z_r$ holds. For any $z_{r_1} < z_n \in \mathbb{R}$ the point $(z_1, \dots, z_n) \in P_w$ and the sequence $(z_0, \dots, z_n) \in \mathbb{R}^{n+1}$ satisfies the vector $a$.*

*iii) Let $t \in S$ for some $1 \leq t \leq n-1$. Assume that $y_{r_1} < y_n < y_{r_2}$ for some $1 \leq r_1, r_2 \leq n-1$, and for every $1 \leq r \leq n-1$ neither $y_{r_1} < y_r \leq y_n$ nor $y_n \leq y_r < y_{r_2}$ holds. Then for any point $(z_0, \dots, z_{n-1}) \in P_v$ if a point $(z_1, \dots, z_{n-1}, z) \in P_w$ then $z_{r_1} < z < z_{r_2}$ and for every $1 \leq r \leq n-1$ neither $z_{r_1} < z_r \leq z_n$ nor $z_n \leq z_r < z_{r_2}$ holds. For any $z_n \in \mathbb{R}, z_{r_1} < z_n < z_{r_2}$ the point $(z_1, \dots, z_n) \in P_w$ and the sequence $(z_0, \dots, z_n) \in \mathbb{R}^{n+1}$ satisfies the vector $a$.*

*iv) Let $S = \{1\}$. Then $y_n = y_0$. For any point $(z_0, \dots, z_{n-1}) \in P_v$ the point $(z_1, \dots, z_{n-1}, z_0) \in P_w$ and the sequence $(z_0, \dots, z_{n-1}, z_0) \in \mathbb{R}^{n+1}$ satisfies the vector $a$.*

**Proof**. An informal idea of the proof is to transfer inequalities on the differences between the coordinates $y$ to the corresponding inequalities on the coordinates $z$, and back.

Let $(z_0, \dots, z_{n-1}) \in P_v$ and a sequence $(z_0, \dots, z_{n-1}, z_n) \in \mathbb{R}^{n+1}$ satisfy the vector $a$. Assume that $t \in S$ for some $1 \leq t \leq n-1$, then

$$z_t = a_t + z_t = \min_{0 \leq j \leq n}\{a_j + z_j\}. \tag{20}$$

First, consider the case when $t \in S$ for some $1 \le t \le n-1$ and $z_n = z_r$ for some $1 \le r \le n$ (cf. i)). Then the sequence $(y_0, \dots, y_{n-1}, y_n = y_r) \in \mathbb{R}^{n+1}$ also satisfies the vector $a$. Indeed, (20) implies that $y_t = a_t + y_t = \min_{0 \le j \le n}\{a_j + y_j\}$. Therefore, due to Definition 6.2 there exists an edge $(v, w)$ of $G$ such that $(y_1, \dots, y_{n-1}, y_n = y_r) \in P_w$. Hence $(z_1, \dots, z_{n-1}, z_n = z_r) \in P_w$ as well. This proves the first statement of the theorem in the case under consideration.

Now consider the case when $t \in S$ for some $1 \le t \le n-1$ and $z_n > z_r$ for each $1 \le r \le n-1$ (cf. ii)). Then for any $y > \max_{1 \le j \le n-1}\{y_j\}$ the sequence $(y_0, \dots, y_{n-1}, y) \in \mathbb{R}^{n+1}$ satisfies the vector $a$. Indeed, (20) implies that $y_t = a_t + y_t = \min\{\min_{0 \le j \le n-1}\{a_j + y_j\}, y\}$. Due to Definition 6.2 there exists an edge $(v, w)$ (independent of $y$) of $G$ such that $(y_1, \dots, y_{n-1}, y) \in P_w$. Hence $(z_1, \dots, z_{n-1}, z_n) \in P_w$ as well. This proves the first statement of the theorem in the case under consideration.

The case when $t \in S$ for some $1 \le t \le n-1$ and $z_{r_1} < z_n < z_{r_2}$ for some $1 \le r_1, r_2 \le n-1$ such that for each $1 \le r \le n-1$ neither $z_{r_1} < z_r \le z_n$ nor $z_n \le z_r < z_{r_2}$ holds (cf. iii)) can be studied in a similar manner as the previous case.

Finally, consider the case $S = \{1\}$ (cf. iv)). Then $z_0 < z_l$ for each $1 \le l \le n-1$ for which $a_l = 0$. Therefore, $z_n = z_0$. Hence the sequence $(y_0, \dots, y_{n-1}, y_0) \in \mathbb{R}^{n+1}$ satisfies the vector $a$. Due to Definition 6.2 there exists an edge $(v, w)$ of $G$ such that $(y_1, \dots, y_{n-1}, y_0) \in P_w$. Therefore $(z_1, \dots, z_{n-1}, z_0) \in P_w$ as well. This proves the first statement of the theorem.

One can directly verify the second statement of the theorem. □

**Corollary 6.4** *The edges of the graph $G$ do not depend on choices of points $(y_0, \dots, y_{n-1}) \in P_v$.*

**Remark 6.5** *Let an edge $(v, w)$ fulfill the assumptions of one of the items Theorem 6.3 i), ii), iii) and a point $(z_0, \dots, z_{n-1}) \in P_v$. Then for any $z \in \mathbb{R}$ such that $(z_1, \dots, z_{n-1}, z) \in P_w$ the sequence $(z_0, \dots, z_{n-1}, z) \in \mathbb{R}^{n+1}$ satisfies the vector $a$. In contrast, in case of Theorem 6.3 iv) only for the value $z = z_0$ it holds that the sequence $(z_0, \dots, z_{n-1}, z)$ satisfies the vector $a$ (cf. Theorem 3.1).*

## 6.2 The polyhedron of tropical recurrent sequences yielded along a path of the graph

Consider an arbitrary path $T$ of a length $k$ with vertices $v_0, \dots, v_k$ in the graph $G_a$. Similar to subsection 2.2 we describe a recursive process yielding along $T$ tropical recurrent sequences satisfying the vector $a$. For the first vertex $v_0$ take any vector $(y_1, \dots, y_n) \in P_{v_0}$. Assume by recursion that a tropical recurrent sequence $(y_1, \dots, y_{k+n})$ is already yielded along $T$. Then $(y_{k+1}, \dots, y_{k+n}) \in P_{v_k}$. Take an edge $(v_k, w)$ of $G$ and denote by $T_w$ the extension of $T$ by

$(v_k, w)$. We choose $y_{k+n+1} \in \mathbb{R}$ such that $(y_{k+2}, \dots, y_{k+n+1}) \in P_w$ and the sequence $(y_{k+1}, \dots, y_{k+n+1}) \in \mathbb{R}^{n+1}$ satisfies the vector $a$. Thus, the tropical recurrent sequence $(y_1, \dots, y_{k+n+1})$ is yielded along $T_w$. Theorem 6.3 justifies that a required $y_{k+n+1}$ exists and moreover, Theorem 6.3 describes all possible $y_{k+n+1}$. This completes the description of the recursive process.

Denote by $Q_T \subset \mathbb{R}^{k+n}$ the set of all the tropical recurrent sequences yielded along $T$ by the described recursive process. One can define $Q_T$ by imposing linear inequalities for each edge of $T$. Say, for an edge $(v_i, v_{i+1}), 0 \le i \le k-1$ we impose that the point $(y_{i+1}, \dots, y_{i+n+1})$ belongs to $P_{v_i}$, the point $(y_{i+2}, \dots, y_{i+n+2})$ belongs to $P_{v_{i+1}}$. This suffices for edges $(v_i, v_{i+1})$ fulfilling the items Theorem 6.3 i), ii), iii). In case of Theorem 6.3 iv) one has to impose an extra condition that $y_{i+1} = y_{i+n+2}$, i.e. the sequence $(y_{i+1}, \dots, y_{i+n+2}) \in \mathbb{R}^{n+1}$ satisfies the vector $a$. Thus, $Q_T$ is (an open in its linear hull) polyhedron.

If an edge $(v_i, v_{i+1})$ fulfills one of the items Theorem 6.3 i), iv) we call the edge *rigid*, otherwise, if the edge fulfills one of the items Theorem 6.3 ii), iii) we call the edge *augmenting*. Similar to subsection 2.2 when the edge $(v_k, w)$ is rigid the value of $y_{k+n+1}$ is unique, while when the edge is augmenting the values of $y_{k+n+1}$ vary in an open interval. Therefore, when the edge $(v_k, w)$ is rigid the polyhedron $Q_{T_w}$ is homeomophic to $Q_T$, while when the edge is augmenting the polyhedron $Q_{T_w}$ is homeomophic to $Q_T \times \mathbb{R}$.

Conversely, Theorem 6.3 implies that any tropical recurrent sequence satisfying the vector $a$ emerges along a suitable path of $G$ in the described above recursive process. Thus, the tropical prevariety of all tropical recurrent sequences of a length $k+n$ satisfying the vector $a$ coincides with the union of polyhedra $Q_T$ over all the paths of the length $k$ in $G$.

For a path $T$ in the graph $G$ denote by $d(T)$ the number of augmenting edges in $T$. By $n(T) \le n$ denote the number of the pairwise distinct coordinates in $(y_1, \dots, y_n) \in P_{v_0}$ for the first vertex $v_0$ of $T$. We summarize the proved above in the following theorem which is analogous to the Theorem 3.5 for the case when $a$ is a tropical boolean vector.

**Theorem 6.6** *For any tropical boolean vector $a := (a_0, \dots, a_n)$ (i. e. $a_0 = a_n = 0$ and each $a_i$, $0 \le i \le n$ equals either $0$ or $\infty$) a finite directed graph $G := G_a$ is construced with the following properties. For an arbitrary path $T$ of a length $k$ in $G$ denote by $Q_T \subset \mathbb{R}^{k+n}$ the polyhedron of all the tropical recurrent sequences satisfying the vector $a$ and corresponding (as described above in this subsection) to the path $T$ in $G$. Then $\dim(Q_T) = d(T) + n(T)$. Moreover, the union of polyhedra $Q_T$ over all the paths $T$ of the length $k$ coincides with the tropical prevariety of all the tropical recurrent sequences of the length $k+n$ satisfying the vector $a$.*

Now let us notice that all the arguments presented in sections 4 and 5 for the graph constructed in section 2 are also true in the case of tropical boolean vectors. Indeed, both definitions of $n(T)$ and $d(T)$ and thereby, $d_a(s)$ coincide with the definitions for the case when $a$ has a finite amplitude. Moreover, an analogue of Theorem 3.5 holds in the tropical boolean case (Theorem 6.6). As all the statements from sections 4 and 5 (except of Theorem 5.4 and Corollary 5.5) depend only on $d_a(s)$ and on Theorem 3.5, we can formulate the following corollaries.

**Corollary 6.7** *Lemmata 4.1, 4.2, 5.1, 5.2, 5.3 and Corollary 4.3 hold when $a$ is a tropical boolean vector.*

**Proof**. Follows from the proofs of the mentioned statements. $\square$

**Corollary 6.8** *Theorem 5.4 and Corollary 5.5 hold when $a$ is a tropical boolean vector putting in the bounds $M = 1$.*

**Proof**. From subsection 6.1 it follows that $V$ is less than the amount of orders on an $n$-element set, hence it is less than $n^n$. Thus, we can put $M = 1$. in the bounds. The remaining part of the proof is literally as in the proofs of the mentioned statements. $\square$

# 7 Sharp bounds on the tropical entropy

## 7.1 Sharp lower bound on the positive entropy

In this section our main goal is to prove that if for a vector $a = (a_0, \dots, a_n) \in \mathbb{Z}^{n+1}$ its tropical entropy $H(a) > 0$ then $H(a) \geqslant \frac{1}{4}$. Together with the example [3, Example 5.5] demonstrating that $H(0, 1, 0) = 1/4$ (cf. also Example 5.6) we will conclude that this bound is sharp. This result is the answer to the hypothesis that was formulated in [3, Remark 5.6] (for the criterion of positivity of the tropical entropy see [3, Corollary 5.7], cf. also Remark 5.7).

**Theorem 7.1** *If a vector $a$ is not regular then $H(a) \geqslant \frac{1}{4}$.*

**Proof**. Consider Newton polygon $\mathcal{N}(a)$ of the vector $a$ (see Remark 5.7). It has several bounded edges and two unbounded edges. First, assume that there is a bounded edge of $\mathcal{N}(a)$ such that it contains at least three points of $a$, i. e. of the form $(i, a_i)$ (in this case we follow the proof of [3, Theorem 5.5]). Making a suitable affine transformation one can suppose w. l. o. g. that this edge lies on the abscissas axis and $(0, 0)$ is its left end-point. Consider the points of $a$ located on this edge: $I := \{(i, 0) : a_i = 0\}$, then $|I| \geqslant 3$ by our assumption. One can assume w.l.o.g. that the greatest common divisor $GCD(I)$ of the differences $i_1 - i_2$ of all the pairs of the elements $i_1,\ i_2 \in I$ of $I$ equals 1. Otherwise, one can consider separately all $GCD(I)$ arithmetic progressions with the difference $GCD(I)$.

Pick any three elements of $I$ not all with the same parity, say $0, 2i, j$ w.l.o.g. where $i \geqslant 1$ and $j$ being odd. Consider the following tropical recurrent sequence $z$ satisfying $a$ :

- $z_{2l+1} = 0$, for $0 \le l \in \mathbb{Z}$;
- $z_{2(2qi+r)} = 0$, for $0 \le q \in \mathbb{Z}$ and $0 \leqslant r < i$;
- $z_{2((2q+1)i+r)} \geqslant 0$, for $0 \le q \in \mathbb{Z}$ and $0 \leqslant r < i$.

Here and below while defining sequences $z_s$, we consider only non-negative subscripts $s$. Taking finite fragments $(z_1, \dots, z_N)$ with growing $N$ we conclude that $H(a) \geqslant \frac{1}{2+1+1} = \frac{1}{4}$.

Now we assume that no edge of $\mathcal{N}(a)$ contains a point of $a$ other than two vertices of this edge. We take an edge of $\mathcal{N}(a)$ with the biggest difference of indices of its vertices. Due to a suitable affine transformation we suppose w.l.o.g. that these vertices are $(0, 0)$ and $(n_0, 0)$. There exists $i \in J$ such that $n_0$ does not divide $i$, since $a$ is not regular. Among such $i$ we pick $i_0$ for which $c := a_{i_0}$ is minimal. Then $c > 0$. Denote $k = GCD(n_0, i_0)$. When $\frac{n_0}{k}$ is even we consider the sequence $\{z_i\}_{0 \le i \in \mathbb{Z}}$:

- $z_{qn_0 - 2ji_0 + i} = 0$, when $0 \leqslant 2j \leqslant \frac{n_0}{k}$;
- $z_{2qn_0 - (2j+1)i_0 + i} = c$, when $0 < 2j + 1 < \frac{n_0}{k}$;
- $z_{(2q+1)n_0 - (2j+1)i_0 + i} \geqslant c$, when $0 < 2j + 1 < \frac{n_0}{k}$,

for $0 \le q \in \mathbb{Z}, 0 \leqslant i < k$. This sequence satisfies $a$ and taking finite fragments $(z_1, \dots, z_N)$ with growing $N$ we conclude that $H(a) \geqslant \frac{1}{2+1+1} = \frac{1}{4}$. Thus further we suppose that $\frac{n_0}{k}$ is odd.

We denote the first (respectively, the last) index of $a$ by $b$ (respectively, by $e$). Thus, the projection of $\mathcal{N}(a)$ is the interval from $b$ to $e$ on the abscissas axis. Before we prove the statement of the theorem in general case let us prove the following lemma.

**Lemma 7.2** *If there exists $i_1 \neq i_0$ such that $n_0 \nmid i_1$ and $a_{i_1} = a_{i_0}$ then $H(a) \geqslant \frac{1}{4}$.*

**Proof**.

1. Let $n_0 | (i_1 - i_0)$. Then we consider a sequence $\{z_i\}_{i \in \mathbb{Z}}$ such that:
   - $z_{qn_0 - 2ji_0 + i} = 0$ when $0 \leqslant 2j < \frac{n_0}{k}$;
   - $z_{qn_0 - (2j+1)i_0 + i} \geqslant c$ when $0 < 2j + 1 < \frac{n_0}{k}$

for $0 \leq q \in \mathbb{Z}$, $0 \leqslant i < k$. This sequence satisfies $a$.

Indeed,

- For $m = qn_0 - 2ji_0 + i$ we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = 0$, the minimum is attained at indices $m$ and $m + n_0$.
- For $m = qn_0 - (2j+1)i_0 + i$ we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + i_0$ and $m + i_1$.

Taking finite fragments $(z_1, \ldots, z_N)$ with growing $N$ we conclude that $H(a) \geqslant \frac{1}{2}$ for even $\frac{n_0}{k}$ and $H(a) \geqslant \frac{\frac{\frac{n_0}{k}-1}{2} \cdot k}{n_0} \geqslant \frac{1}{3}$.

2. Let $n_0 \nmid (i_1 - i_0)$.

   (a) Assume that $\mathbf{k} = \mathbf{1}$ (since we consider the case where $\frac{n_0}{k}$ is odd, thus $n_0$ is odd).

   First, consider sequence $\{z_i\}_{0 \leq i \in \mathbb{Z}}$ such that:

   - $z_{qn_0 - 2ji_0} = 0$ when $0 \leqslant 2j \leqslant n_0$;
   - $z_{2qn_0 - (2j+1)i_0} = c$ when $0 < 2j + 1 < n_0$;
   - $z_{(2q+1)n_0 - (2j+1)i_0} \geqslant c$ when $0 < 2j + 1 < n_0$

   for $0 \leq q \in \mathbb{Z}$. This sequence satisfies $a$.

   Indeed,

   - For $m = qn_0 - 2ji_0$ we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = 0$, the minimum is attained at indices $m$ and $m + n_0$.
   - For $m = 2qn_0 - (2j+1)i_0$ we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + i_0$ and $m$.
   - For $m = (2q+1)n_0 - (2j+1)i_0$ we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + i_0$ and $m + n_0$.

   Now, we claim that there exists $0 < 2l + 1 < n_0$ such that $z_{qn_0 - (2l+1)i_0 + i_1} = 0$ for all $0 \leq q \in \mathbb{Z}$. Note, that if we found $2l' + 1$ such that $z_{qn_0 - (2l'+1)i_0 + i'_1} = 0$ for all $0 \leq q \in \mathbb{Z}$ for some $i'_1 \equiv i_1$, then $z_{qn_0 - (2l'+1)i_0 + i_1} = 0$ for all $0 \leq q \in \mathbb{Z}$.

   Recall that $GCD(i_0, n_0) = 1$ and $n_0 \nmid i_1$, therefore there exists $m_{i_1}$ such that $m_{i_1} i_0 \equiv i_1 \ (mod\ n_0)$ and $0 < m_{i_1} < n_0$.

   - If $m_{i_1}$ is odd then the required $2l + 1$ equals $n_0 - 2$. Indeed, $qn_0 - (n_0 - 2)i_0 + m_{i_1} i_0 = qn_0 - (n_0 - 2 - m_{i_1})i_0$. $0 \leqslant n_0 - 2 - m_{i_1} < n_0 - 2$ and $(n_0 - 1 - m_{i_1})$ is even, thus $z_{qn_0 - (n_0 - 2 - m_{i_1})i_0} = 0$.
   - If $m_{i_1}$ is even then the required $2l + 1$ equals $m_{i_1} - 1$. Indeed, $qn_0 - (m_{i_1} - 1)i_0 + m_{i_1} i_0 = qn_0 + i_0 = (q + i_0)n_0 - (n_0 - 1)i_0$. $0 < n_0 - 1 < n_0$ and $n_0 - 1$ is even, thus $z_{(q+i_0)n_0 - (n_0 - 1)i_0} = 0$.

Now consider a sequence $\{z_i\}_{0\leq i\in\mathbb{Z}}$ such that:

- $z_{qn_0-2ji_0} = 0$ when $0 \leqslant 2j \leqslant n_0$;
- $z_{2qn_0-(2j+1)i_0} = c$ when $0 < 2j+1 < n_0$ and $l \neq j$;
- $z_{2qn_0-(2l+1)i_0} \geqslant 0$;
- $z_{(2q+1)n_0-(2j+1)i_0} \geqslant c$ when $0 < 2j+1 < n_0$

for $0 \leq q \in \mathbb{Z}$. This sequence satisfies $a$. Indeed,

- For $m = qn_0 - 2ji_0$ we have $\min_{b\leqslant v\leqslant e}\{a_v + z_{v+m}\} = 0$, the minimum is attained at indices $m$ and $m + n_0$.
- For $m = 2qn_0 - (2j+1)i_0$ and $j \neq l$ we have $\min_{b\leqslant v\leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + i_0$ and $m$.
- For $m = 2qn_0 - (2l+1)i_0$ we have $\min_{b\leqslant v\leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + i_0$ and $m + i_1$.
- For $m = (2q+1)n_0-(2j+1)i_0$ we have $\min_{b\leqslant v\leqslant e}\{a_v+z_{v+m}\} = c$, the minimum is attained at indices $m + i_0$ and $m + n_0$.

Taking finite fragments $(z_1, \ldots, z_N)$ with growing $N$ we conclude that $H(a) \geqslant \frac{\frac{n-1}{2}+1}{2n} \geqslant \frac{1}{4}$.

(b) Now assume that $\mathbf{k} > \mathbf{1}$. Define $k_1 := GCD(i_1, n_0)$. W.l.o.g. we can consider that $k_1 \geqslant k$ (otherwise we can swap $i_0$ and $i_1$).

We will find $k$ different indices $l_1, \ldots l_k$ such that $z_{qn_0+l_j+i_1} = 0$ for all $0 \leq q \in \mathbb{Z}$ and for all $1 \leqslant j \leqslant k$ and $l_{j_1} \not\equiv l_{j_2}$ for all $j_1 \neq j_2$. Note, that if $i_1' \equiv i_1\ (mod\ n_0)$ and $z_{qn_0+l_j+i_1'} = 0$ for all $0 \leq q \in \mathbb{Z}$ and for all $1 \leqslant j \leqslant k$ then it is true for $i_1$. Thus, we can assume that $0 \leqslant i_1 < n_0$. We can represent $i_1$ as $s \cdot k + r$, where $0 \leqslant r < k$. We study two different cases:

i. $\mathbf{r = 0.}$
Denote $n' := \frac{n_0}{k}$, $i_0' = \frac{i_0}{k}$ and $i_1' := \frac{i_1}{k}$. Also denote $a' = (a_j)_{j\equiv 0\ (mod\ k)}$. Similar to the previous case ($k = 1$) we can consider sequence $\{z_i'\}_{0\leq i\in\mathbb{Z}}$ that provides the bound $H(a') \geqslant \frac{1}{4}$. Now take sequence $\{z_i\}_{0\leq i\in\mathbb{Z}}$ as follows:
- $z_{i\cdot k+r} = z_i'$, for $0 \leq i \in \mathbb{Z}$ and $0 \leqslant r < k$.

This provides us the bound $H(a) \geqslant \frac{1}{4}$.

ii. $\mathbf{r \neq 0}$
Note, that in this case $k_1 > k$ and thus $s \neq 0$ and $s + 1 \neq \frac{n_0}{k}$. Consider $a' = (a_j)_{j\equiv 0\ (mod\ k)}$. Sequence $\{z_i'\}_{0\leq i\in\mathbb{Z}}$ is defined as follows:
- $z'_{q'\frac{n_0}{k}-2j\frac{i_0}{k}} = 0$, where $0 \leqslant 2j < \frac{n_0}{k}$;
- $z'_{2q'\frac{n_0}{k}-(2j+1)\frac{i_0}{k}} = c$, where $0 < 2j+1 < \frac{n_0}{k}$;

- $z'_{(2q'+1)\frac{n_0}{k}-(2j+1)\frac{i_0}{k}} \geqslant c$, where $0 < 2j+1 < \frac{n_0}{k}$

for $0 \leq q' \in \mathbb{Z}$. Here we have three different cases:

A. $s \equiv \frac{i_0}{k}\ (mod\ \frac{n_0}{k})$.
From the proof for $k=1$ we know that there exists $0 < 2l+1 < \frac{n_0}{k}$ such that $z'_{q'\frac{n_0}{k}-(2l+1)\frac{i_0}{k}+(s+1)} = 0$ for all $0 \leq q' \in \mathbb{Z}$.
Consider $\{z_i\}_{0\leq i\in Z}$ as follows:
   - $z_{qn_0-2ji_0+i}$, where $0 \leqslant 2j < n_0$ and $0 \leqslant i < k$;
   - $z_{2qn_0-(2j+1)i_0+i} = c$, where $0 < 2j+1 < n_0$, $j \neq l$ and $0 \leqslant i < k$;
   - $z_{2qn_0-(2l+1)i_0+i} \geqslant c$, where $0 \leqslant i < k$;
   - $z_{(2q+1)n_0-(2l+1)i_0+i}$, where $0 < 2j+1 < n_0$ and $0 \leqslant i < k$

   for $0 \leq q \in \mathbb{Z}$. We claim that $\{z_i\}_{i\in\mathcal{Z}}$ satisfies $a$.
   Indeed,
   - For $m = qn_0 - 2ji_0 + i$ we have $\min_{b\leqslant v\leqslant e}\{a_v + z_{v+m}\} = 0$, the minimum is attained at indices $m$ and $m+n_0$.
   - For $m = 2qn_0-(2j+1)i_0+i$, $j \neq l$ we have $\min_{b\leqslant v\leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m$ and $m+i_0$.
   - For $m = (2q+1)n_0 - (2j+1)i_0 + i$, $j \neq l$ we have $\min_{b\leqslant v\leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m+n_0$ and $m+i_0$.
   - For $m = qn_0-(2l+1)i_0+i$ we have $\min_{b\leqslant v\leqslant e}\{a_v+z_{v+m}\} = c$, the minimum is attained at indices $m+i_1$ and $m+i_0$.

   Taking finite fragments $(z_1, \ldots, z_N)$ with growing $N$ we conclude that $H(a) \geqslant \frac{\frac{\frac{n}{k}-1}{2}\cdot k+k}{2n} \geqslant \frac{1}{4}$

B. $s+1 \equiv \frac{i_0}{k}\ (mod\ \frac{n_0}{k})$.
This case is the same as the previous one except that we need to find $0 < 2l+1 < \frac{n_0}{k}$ such that $z'_{q'\frac{n_0}{k}-(2l+1)\frac{i_0}{k}+s} = 0$ for all $0 \leq q' \in \mathbb{Z}$.
Taking finite fragments $(z_1, \ldots, z_N)$ with growing $N$ we conclude that $H(a) \geqslant \frac{\frac{\frac{n_0}{k}-1}{2}\cdot k+k}{2n_0} \geqslant \frac{1}{4}$

C. $s,\ s+1 \not\equiv \frac{i_0}{k}\ (mod\ \frac{n_0}{k})$.
From the proof for $k=1$ we know that there exist $0 < 2l+1,\ 2l'+1 < \frac{n_0}{k}$ such that $z'_{q'\frac{n_0}{k}-(2l+1)\frac{i_0}{k}+s} = 0$ and $z'_{q'\frac{n_0}{k}-(2l'+1)\frac{i_0}{k}+(s+1)} = 0$ for all $0 \leq q' \in \mathbb{Z}$.
Consider $\{z_i\}_{0\leq i\in\mathbb{Z}}$ as follows:

- $z_{qn_0-2ji_0+i}$, where $0 \leqslant 2j < n_0$ and $0 \leqslant i < k$;
- $z_{2qn_0-(2j+1)i_0+i} = c$, where $0 < 2j+1 < n_0$, $j \neq l,\ l'$ and $0 \leqslant i < k$;
- $z_{2qn_0-(2l+1)i_0+i} \geqslant c$, where $0 \leqslant i < k-r$;
- $z_{2qn_0-(2l+1)i_0+i} = c$, where $k-r \leqslant i < k$;
- $z_{2qn_0-(2l'+1)i_0+i} = c$, where $0 \leqslant i < k-r$;
- $z_{2qn_0-(2l'+1)i_0+i} \geqslant c$, where $k-r \leqslant i < k$;
- $z_{2qn_0-(2l+1)i_0+i} \geqslant c$, where $0 \leqslant i < k$;
- $z_{(2q+1)n_0-(2l+1)i_0+i}$, where $0 < 2j+1 < n_0$ and $0 \leqslant i < k$

for $0 \leq q \in \mathbb{Z}$. We claim that $\{z_i\}_{i \in \mathcal{Z}}$ satisfies $a$.
Indeed,

- For $m = qn_0 - 2ji_0 + i$ we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = 0$, the minimum is attained at indices $m$ and $m + n_0$.
- For $m = 2qn_0 - (2j+1)i_0 + i$, $j \neq l,\ l'$ we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m$ and $m + i_0$.
- For $m = 2qn_0 - (2l+1)i_0 + i$, $k - r \leqslant i < k$ we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m$ and $m + i_0$.
- For $m = 2qn_0 - (2l'+1)i_0 + i$, we have $0 \leqslant i < k - r$ $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m$ and $m + i_0$.
- For $m = (2q+1)n_0 - (2j+1)i_0 + i$, $j \neq l,\ l'$ we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + n_0$ and $m + i_0$.
- For $m = (2q+1)n_0 - (2l+1)i_0 + i$, $k - r \leqslant i < k$ we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + n_0$ and $m + i_0$.
- For $m = (2q+1)n_0 - (2l'+1)i_0 + i$, $0 \leqslant i < k - r$ we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + n_0$ and $m + i_0$.
- For $m = qn_0 - (2l+1)i_0 + i$, $0 \leqslant i < k - r$ we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + i_1$ and $m + i_0$.
- For $m = qn_0 - (2l'+1)i_0 + i$, $k - r \leqslant i < k$ we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + i_1$ and $m + i_0$.

Taking finite fragments $(z_1, \dots, z_N)$ with growing $N$ we conclude that $H(a) \geqslant \frac{\frac{\frac{n_0}{k}-1}{2} \cdot k + k}{2n_0} \geqslant \frac{1}{4}$.

□

Now we are returning to the proof of the theorem. Assume that there is no such $n \nmid i_i$, $i_0 \neq i_1$ that $a_{i_1} = a_{i_0}$. As in the proof of Lemma 7.2 we will consider two different cases.

1. $\mathbf{k = 1.}$

   Define the following sequence $\{z'_i\}_{0 \leq i \in \mathbb{Z}}$:

   - $z'_{qn_0 - 2ji_0} = 0$ when $0 \leqslant 2j < n_0$;
   - $z'_{2qn_0-(2j+1)i_0} = c$ when $0 < 2j+1 < n_0$;
   - $z'_{(2q+1)n_0-(2j+1)i_0} \geqslant c$ when $0 < 2j+1 < n_0$

   for $0 \leq q \in \mathbb{Z}$.

   We define $L_0 := \{b \leqslant v \leqslant e, n_0 \nmid v, v \neq 0, n_0,$ such that $z'_{qn_0-(n_0-1)i_0+v} = 0$ for all $q \in \mathbb{Z}\}$. Set $x := \min\,\{a_v \mid v \in L_0\}$. Also define $i_x$ by the equation $a_{i_x} = x$. If such $i_x$ is not unique then we choose any $i_x$ with such property.

   (a) First assume that $x \leqslant 2c$.

   In this case we define a sequence $\{z_i\}_{0 \leq i \in \mathbb{Z}}$ as follows:

   - $z_{qn_0-2ji_0} = 0$ when $0 \leqslant 2j < n_0$, $2j \neq n_0 - 1$;
   - $z_{2qn_0-(2j+1)i_0} = c$ when $0 < 2j+1 < n_0$;
   - $z_{(2q+1)n_0-(2j+1)i_0} \geqslant x$ when $0 < 2j+1 < n_0$
   - $z_{2qn_0-(n_0-1)i_0} = x$;
   - $z_{(2q+1)n_0-(n_0-1)i_0} \geqslant x$

   for $0 \leq q \in \mathbb{Z}$. We claim that $\{z_i\}_{i \in \mathcal{Z}}$ satisfies $a$.

   Indeed,

   - For $m = qn_0 - 2ji_0$, $2j \neq n_0 - 1$ we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = 0$, the minimum is attained at indices $m$ and $m + n_0$.
   - For $m = 2qn_0 - (2j+1)i_0$, $2j \neq n_0 - 1$ we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m$ and $m + i_0$.
   - For $m = (2q+1)n_0 - (2j+1)i_0$ we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m + n_0$ and $m + i_0$.
   - For $m = 2qn_0 - (n-1)i_0$, we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = x$, the minimum is attained at indices $m$ and $m + i_x$ (because $a_v + z_{v+m}$ is at least $x$ if $v \in L_0$ and $a_v + z_{v+m} \geqslant min\{c+c, c+x\} \geqslant x$ if $v \notin L_0$).
   - For $m = (2q+1)n_0 - (n_0-1)i_0$ we have $\min_{b \leqslant v \leqslant e}\{a_v + z_{v+m}\} = x$, the minimum is attained at indices $m + n_0$ and $m + i_x$ (because $a_v + z_{v+m}$ is at least $x$ if $v \in L_0$ and $a_v + z_{v+m} \geqslant min\{c+c, c+x\} \geqslant x$ if $v \notin L_0$).

Taking finite fragments $(z_1, \dots, z_N)$ with growing $N$ we conclude that $H(a) \geqslant \frac{\frac{n_0-1}{2}+1}{2n_0} \geqslant \frac{1}{4}$.

(b) Now we assume that $x > 2c$.

Denote $\min_{v\neq i_0,\ v\nmid n_0}\{a_v\}$ by $s$. Note, that $s > c$. Indeed, otherwise we can use lemma 7.2 and get the required bound. Denote $\min_{v\neq 0,n_0,\ v|n_0}\{a_v\}$ by $d$. Note, that $d > 0$. Finally, set $y := \min\{s + c, x, 2c + d\}$.

Define a sequence $\{z_i\}_{0\leq i\in\mathbb{Z}}$ as follows:

- $z_{qn_0-2ji_0} = 0$ when $0 \leqslant 2j < n_0$, $2j \neq n_0 - 1$;
- $z_{2qn_0-(2j+1)i_0} = c$ when $0 < 2j + 1 < n_0$;
- $z_{(2q+1)n_0-(2j+1)i_0} \geqslant y$ when $0 < 2j + 1 < n_0$
- $z_{4qn_0-(n_0-1)i_0} = 2c$;
- $z_{(4q+1)n_0-(n_0-1)i_0} = t_q$, where $t_q$ takes an arbitrary value from the interval $[2c, y]$;
- $z_{(4q+2)n_0-(n_0-1)i_0} = t_q$, where $t_q$ takes an arbitrary value from the interval $[2c, y]$;
- $z_{(4q+3)n_0-(n_0-1)i_0} = 2c$

for $0 \leq q \in \mathbb{Z}$. We claim that $\{z_i\}_{i\in\mathcal{Z}}$ satisfies $a$.

Indeed,

- For $m = qn_0-2ji_0$, we have $2j \neq n_0-1$ $\min_{b\leqslant v\leqslant e}\{a_v+z_{v+m}\} = 0$, the minimum is attained at indices $m$ and $m + n_0$.
- For $m = 2qn_0 - (2j + 1)i_0$, $2j \neq n_0 - 1$ we have $\min_{b\leqslant v\leqslant e}\{a_v + z_{v+m}\} = c$, the minimum is attained at indices $m$ and $m + i_0$.
- For $m = (2q+1)n_0-(2j+1)i_0$ we have $\min_{b\leqslant v\leqslant e}\{a_v+z_{v+m}\} = c$, the minimum is attained at indices $m + n_0$ and $m + i_0$.
- For $m = 4qn_0 - (n_0 - 1)i_0$, we have $\min_{b\leqslant v\leqslant e}\{a_v + z_{v+m}\} = 2c$, the minimum is attained at indices $m$ and $m + i_0$ (because $a_v + z_{v+m}$ is at least $x > 2c$ if $v \in L_0$, and $a_v + z_{v+m} \geqslant min\{s + c, 0 + t_q, d + 2c\} \geqslant 2c$ if $v \notin L_0$).
- For $m = (4q+1)n_0-(n_0-1)i_0$ we have $\min_{b\leqslant v\leqslant e}\{a_v+z_{v+m}\} = t_q$, the minimum is attained at indices $m$ and $m + n_0$ (because $a_v + z_{v+m}$ is at least $x > y \geqslant t_q$ if $v \in L_0$, and $a_v + z_{v+m} \geqslant min\{c + y, s + c, d + 2c\} \geqslant y \geqslant t_q$ if $v \notin L_0$).
- For $m = (4q+2)n_0-(n_0-1)i_0$, we have $\min_{b\leqslant v\leqslant e}\{a_v+z_{v+m}\} = 2c$, the minimum is attained at indices $m + n_0$ and $m + i_0$ (because $a_v + z_{v+m}$ is at least $x > 2c$ if $v \in L_0$, and $a_v + z_{v+m} \geqslant min\{s + c, 0 + t_q, d + 2c\} \geqslant 2c$ if $v \notin L_0$).
- For $m = (4q+3)n_0-(n_0-1)i_0$, we have $\min_{b\leqslant v\leqslant e}\{a_v+z_{v+m}\} = 2c$, the minimum is attained at indices $m$ and $m+n_0$ (because

$a_v + z_{v+m}$ is at least $x > 2c$ if $v \in L_0$, and $a_v + z_{v+m} \geqslant min\{c + y, s + c, 0 + t_q, d + 2c\} \geqslant 2c$ if $v \notin L_0$).

Taking finite fragments $(z_1, \dots, z_N)$ with growing $N$ we conclude that $H(a) \geqslant \frac{n_0-1+1}{4n_0} = \frac{1}{4}$.

2. $\mathbf{k > 1}$.

Consider the following sequence $\{z'_i\}_{0 \leq i \in \mathbb{Z}}$:

- $z'_{qn_0-2ji_0+i} = 0$ when $0 \leqslant 2j \leqslant n_0$;
- $z'_{2qn_0-(2j+1)i_0+i} = c$ when $0 < 2j+1 < n_0$;
- $z'_{(2q+1)n_0-(2j+1)i_0+i} \geqslant c$ when $0 < 2j+1 < n_0$

for $0 \leq q \in \mathbb{Z}$ and $0 \leqslant i < k$.

For $0 \leqslant i < k$ define $L_{0,i} := \{b \leqslant v \leqslant e,\ \ n_0 \nmid v,\ \ v \neq 0, n_0$ such that $z'_{qn_0-(n_0-1)i_0+i+v} = 0$ for all $0 \leq q \in \mathbb{Z}$ and such that $qn_0 - (n_0-1)i_0 + i + v \neq q'n_0 - (n_0-1)i_0 + i'$ for any $q'$ and for any $0 \leqslant i' < k\}$. Set $x_i := \min\{a_v \mid v \in L_{0,i}\}$. Define $i_{x,i}$ by the equation $a_{i_{x,i}} = x_i$.

Denote $\min_{v \neq i_0,\ v \nmid n_0}\{a_v\}$ by $s$. Note, that $s > c$. Indeed, otherwise we can use lemma 7.2 and get the required bound. Denote $\min_{v \neq 0, n_0,\ v | n_0}\{a_v\}$ by $d$. Note, that $d > 0$. For $0 \leqslant i < k$ set $y_i := min\{s + c, x_i, 2c + d\}$. Finally, define $M := \max_{0 \leqslant i < k}\{x_i, y_i\}$.

Define a sequence $\{z_i\}_{0 \leq i \in \mathcal{Z}}$ as follows:

- $z_{qn_0-2ji_0+i} = 0$ when $0 \leqslant 2j \leqslant n_0$, $2j \neq (n_0-1)$, where $0 \leqslant i < k$;
- $z_{2qn_0-(2j+1)i_0+i} = c$ when $0 < 2j+1 < n_0$, where $0 \leqslant i < k$;
- $z_{(2q+1)n_0-(2j+1)i_0+i} \geqslant M$ when $0 < 2j+1 < n_0$, where $0 \leqslant i < k$;

  For $0 \leqslant i < k$ set:

  (a) if $x_i \leqslant 2c$ then:
  - $z_{2qn_0-(n_0-1)i_0+i} = x_i$;
  - $z_{(2q+1)n_0-(n_0-1)i_0+i} \geqslant M$;

  (b) if $x_i > 2c$ then:
  - $z_{4qn_0-(n_0-1)i_0+i} = 2c$;
  - $z_{(4q+1)n_0-(n_0-1)i_0+i} = t_{q,i}$, where $t_{q,i}$ takes an arbitrary value from the interval $[2c, y_i]$;
  - $z_{(4q+2)n_0-(n_0-1)i_0+i} = t_{q,i}$, where $t_{q,i}$ takes an arbitrary value from the interval $[2c, y_i]$;

- $z_{(4q+3)n_0-(n_0-1)i_0+i} = 2c$

for $0 \le q \in \mathbb{Z}$. We claim that this sequence satisfies $a$. It is sufficient to check that a subsequence $\{z_{qn_0-(n-1)i_0+i'}\}_{0\le q\in\mathbb{Z}}$ does not change the minima in the subsequence $\{z_{qn_0-(n-1)i_0+i}\}_{0\le q\in\mathbb{Z}}$ with $i \ne i'$ in the definition of satisfiability of the vector $a$ (see (1)). The latter is true because $z_{qn_0-(n_0-1)i_0+i'} \geqslant c$ and thus $z_{qn_0-(n_0-1)i_0+i'} + a_v \geqslant c + s \geqslant x_i$ (if $x_i \leqslant 2c$) and $z_{qn_0-(n_0-1)i_0+i'} + a_v \geqslant c + s \geqslant y_i$ (if $x_i > 2c$).

Taking finite fragments $(z_1, \dots, z_N)$ with growing $N$ we conclude that in the worst case $H(a) \geqslant \frac{(\frac{n_0}{k}-1)k+k}{4n_0} = \frac{1}{4}$.

□

## 7.2 Sharp upper bound on the tropical entropy in case of a single bounded edge of Newton polygon

The last theorem is an upper bound on $H(a)$ in case of a single bounded edge of Newton polygon $\mathcal{N}(a)$. We conjecture that this bound holds for an arbitrary vector $a$. We mention that in [3] a weaker upper bound $1-1/n$ was established for an arbitrary vector $a$. Together with the result $H(a) = 1 - 2/(n+1)$ for a vector $a = (a_0, \dots, a_n)$ with $a_0 = \dots = a_n = 0$ [3, Example 5.2] it demonstrates the sharpness of the obtained upper bound.

**Theorem 7.3** *If Newton polygon for a has only one bounded edge then* $H(a) \leqslant 1 - \frac{2}{n+1}$.

**Proof**. For convenience we make a suitable affine transformation such that $a_0 = a_n = 0$.

Consider the polyhedral complex $D(s)$. It is a union of a finite number of polyhedra such that each of these polyhedra $Q$ satisfies the following conditions. For every $0 \leqslant j \leqslant s-n$ there exists a pair $0 \leqslant i_1 < i_2 \leqslant n$ such that

$$z_{j+i_1} + a_{i_1} = z_j + a_{i_2} = \min_{0\leqslant p\leqslant n}\{z_{p+j} + a_p\} \tag{21}$$

for any $(z_1, \dots, z_s) \in Q$.

For every $Q$ we consider the following *restriction graph* $RG(Q)$ :

- vertices are the indices of coordinates from 1 to $s$;
- there is an edge between vertices $i$ and $j$ if there is a linear condition of the form $y_i + \gamma = y_j$ which is true for all $(y_1, \dots, y_s) \in Q$.

Let us notice that $RG(Q)$ is the union of connected components where each component is the complete subgraph. Moreover, the dimension of $Q$ equals the number of components of $RG(Q)$ (cf. [4]).

Let us fix some $Q$ from the finite union above. For arbitrary $(t_1, \dots, t_s) \in Q$ we construct the following sequence by recursion:

- The first element of the sequence equals the least index $i_0$ such that $t_{i_0} = \min_{1 \leqslant f \leqslant s} t_f$;
- Let $i_v$ be the last current constructed element of the sequence. If $i_v + n > s$ then we terminate the process and declare $i_v$ to be the last constructed element of the sequence.
- If $i_v + n \leqslant s$ then we consider $\min_{0 \leqslant p \leqslant n}\{t_{i_v+p} + a_p\}$. According to the definition of a tropical sequence and the definition of $Q$ there exist $0 \leqslant p_1 < p_2$ such that $\min_{0 \leqslant p \leqslant n}\{z_{i_v+p} + a_p\} = z_{i_v+p_1} + a_{p_1} = z_{i_v+p_2} + a_{p_2}$ for all $(z_1, \dots, z_s) \in Q$. If $p_1 > 0$ then we set $i_{v+1} = i_v + p_1$ and $i_{v+2} = i_v + p_2$. Otherwise, we just set $i_{v+1} = i_v + p_2$.

  Note that there can be more than two indices where $\min_{0 \leqslant p \leqslant n}\{z_{i_v+p} + a_p\}$ is attained for all $(z_1, \dots, z_s) \in Q$. We pick some pair $p_1 < p_2$.

We will call this sequence an *equality row* for $(t_1, \dots, t_s)$. Now we claim two important statements:

- $$i_0 < n + 1 \tag{22}$$

  Indeed, suppose the contrary. Then consider $\min_{0 \leqslant p \leqslant n}\{t_{i_0-n+p} + a_p\}$. As $t_{i_0} = \min_{1 \leqslant f \leqslant s}\{t_f\}$ and $a_n = 0$ then this minimum equals $\min_{1 \leqslant f \leqslant s}\{t_f\}$ and there exist $p_1 < p_2 \leqslant n$ such that $t_{i_0-n+p_1} + a_{p_1} = t_{i_0-n+p_2} + a_{p_2} = \min_{1 \leqslant f \leqslant s}\{t_f\}$. As $a_p \geqslant 0$ then we obtain that $a_{p_2} = a_{p_1} = 0$ and $t_{i_0-n+p_1} = t_{i_0-n+p_2} = t_{i_0}$. However, $i_0 - n + p_1 < i_0$ and we get a contradiction with that $i_0$ is the least index such that $t_{i_0} = \min_{1 \leqslant f \leqslant s}\{t_f\}$.

- $$t_{i_v} = t_{i_0}, \tag{23}$$

  for all $i_v$ in the equality row.

  We prove this by recursion. For $i_0$ the statement is already true. Suppose we have proved this statement for $i_v$ and we consider $\min_{0 \leqslant p \leqslant n}\{t_{i_v+p} + a_p\}$ then either $t_{i_v} + a_0 = t_{i_{v+1}} + a_{p_2}$ equals this minimum or $t_{i_{v+1}} + a_{p_1} = t_{i_{v+2}} + a_{p_2}$. However, this minimum is less or equal to $t_{i_v} + a_0 = t_{i_v} = \min_{1 \leqslant f \leqslant s}\{t_f\}$. Recalling the fact that $a_p \geqslant 0$ for $0 \leqslant p \leqslant n$ we obtain that $a_{p_1} = a_{p_2} = 0$ and either $t_{i_{v+1}} = t_{i_v} = t_{i_0}$ or $t_{i_{v+2}} = t_{i_{v+1}} = t_{i_v} = t_{i_0}$.

Let us fix $(t_1, \dots, t_s) \in Q$ and its equality row $\{i_0, \dots, i_E\}$. Consider another arbitrary point $(t'_1, \dots, t'_s) \in Q$. We prove the following lemma:

**Lemma 7.4** *If for some $v$ it is true that $t'_{i_v} = \min_{0 \leqslant f \leqslant s}\{t'_f\}$ then for all $w \geqslant v$ it is true that $t'_{i_w} = t'_{i_v}$.*

**Proof of lemma**. Indeed, during the recursive construction of the equality row $i_v$ for $(t_1, \dots, t_s)$ there could appear one of the following three possibilities:

- $v = 0$. Then the processes of construction of equality row for $(t_1, \dots, t_s)$ and for $(t'_1, \dots, t'_s)$ completely coincide.

- We considered $\min_{0 \leqslant p \leqslant n}\{t_{i_{v-1}+p} + a_p\}$ which is equal to $t_{i_{v-1}+p_1} + a_{p_1} = t_{i_{v-1}+p_2} + a_{p_2}$ for some $p_1 < p_2$ and $i_v = i_{v-1} + p_2$. Then the processes of construction of equality row for $(t_1, \dots, t_s)$ and for $(t'_1, \dots, t'_s)$ completely coincide starting from the next step.

- We considered $\min_{0 \leqslant p \leqslant n}\{t_{i_{v-1}+p} + a_p\}$ which is equal to $t_{i_{v-1}+p_1} + a_{p_1} = t_{i_{v-1}+p_2} + a_{p_2}$ for some $p_1 < p_2$ and $i_v = i_{v-1} + p_1$. We recall that these equalities are true for arbitrary $(z_1, \dots, z_s) \in Q$ and so they are true for $(t'_1, \dots, t'_s)$. Thus $t'_{i_{v-1}+p_2}$ also equals $\min_{1 \leqslant f \leqslant s}\{t'_f\}$ and we come to the previous case.

Now we define $Q_b$ as

$\{(y_1, \dots, y_s) \in Q$ and $b$ is the least index such that $y_b = \min_{1 \leqslant f \leqslant s}\{t_f\}\}$

According to the statement 22 $Q = \bigcup_{b=1}^{n} Q_b$. Next we prove the crucial lemma.

**Lemma 7.5** *The number of connected components in the $RG(Q_b)$ is not greater than $s + 4 - \frac{2s}{n+1}$.*

**Proof of lemma**. According to the definition of $Q_b$ and according to lemma 7.4 for every $i_v > b$ from the equality row, $(b, i_v)$ is an edge in $RG(Q_b)$. We partition $[b, s]$ into disjoint intervals each of length $n + 1$ starting from $b$. Now we produce the following sequence $\{G'_r\}_{r=0}^{[\frac{s-q}{n+1}]}$ of subgraphs by recursion on an interval number:

- $G'_0$ is just $RG(Q_b)$ without edges;

- Suppose we have produced $G'_r$ and now we are considering $(r+1)$-th interval of length $(n+1)$. The interval contains at least one element $i_v$ from the equality row. If there are at least two elements from the equality row then for each $i_v$ from this interval we add an edge $(b, i_v)$ to the graph $G'_r$ and obtain $G'_{r+1}$.

  Otherwise, we consider $(r+1)$-th interval:

  $$[b + (n+1) \cdot r;\ \ b + (n+1) \cdot r + n].$$

  Consider $\min_{0 \leqslant p \leqslant n}\{y_{b+(n+1)\cdot r+p} + a_p\}$. According to the definition of the tropical sequence there exist $p_1 < p_2$ such that this minimum equals $y_{b+(n+1)\cdot r+p_1} + a_{p_1} = y_{b+(n+1)\cdot r+p_2} + a_{p_2}$ for all $(y_1, \ldots, y_s) \in Q_b$. Thus there is an edge from $RG(Q_b)$ whose vertices have indices from the $(r+1)$-th interval and at least one of them does not lie in the equality row. We call this edge a non-equality edge. Then we set $G'_{r+1}$ as $G'_r$ with one added edge $(b, i_v)$ and one added non-equality edge.

We claim that for every $r$ the number of components in $G'_r$ is at least by two less than $G'_{r-1}$. It follows from the fact that at each step all edges have at least one end-point which does not belong to the transitive closure of previous subgraph.

Thus we obtain that the number of components is less than $s - 2 \cdot [\frac{s-b}{n+1}] \leqslant s + 2 - 2\frac{s-b}{n+1} \leqslant s + 4 - 2\frac{s}{n+1}$.

Now we note that $\dim Q = \max_{1 \leqslant b \leqslant n}\{\dim Q_b\}$ and therefore, according to lemma 7.5 we obtain that $\dim Q \leqslant s + 4 - \frac{2s}{n+1}$. Tending to the limit on $s$ we obtain the required statement of the theorem. $\square$

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