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IMBEDDING THEOREMS FOR TURING MACHINES OF DIFFERENT DIMENSIONS AND KOLMOGOROV'S ALGORITHMS

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In this paper we consider a step-by-step simulation of the behavior of Kolmogorov's algorithm (K.a.) (see [1], [2]) on many-dimensional Turing machines (T.m.) (see [3]) and of the behavior of a many-dimensional T.m. on a T.m. of lower dimension. One can find the concept of a configuration for a T.m. in [3] and define that of a configuration for a K.a. analogously. We shall denote the class of all k-dimensional T.m. (K.a.) by TM^k (KA). We shall assume that Turing machines and Kolmogorov algorithms have inputs and outputs.

Let Φ and Ψ be certain classes of machines (TM^k or KA) and let t_1 and t_2 be general recursive functions. The relation $\Phi(t_1) \subset \Psi(t_2)$ will mean that the behavior of any machine $\phi \in \Phi$ whose time signalizer is majorized by t_1 can be simulated step-by-step by a machine $\psi \in \Psi$ whose time signalizer is majorized by t_2. A possible precise definition of the concept of a step-by-step simulation can be found in [5]. Such a definition is not needed in order to understand the results of this paper, but it is necessary for an understanding of the negative results formulated below (see (C), (D)).

The following two "imbedding theorems" constitute the basic contents of our paper. Let us fix a natural number $k \geq 2$ and a general recursive function t.

Theorem 1. $KA(t) \subset TM^{k+1}(t^{1+1/k})$.

Theorem 2. $TM^{k+1}(t) \subset TM^k(t^{1+1/k})$.

Let $\epsilon > 0$, let I be the identity mapping and let k and m be natural numbers. We cite the following known

Imbedding Theorems.

(A) $TM^m(t) \subset KA(t)$ *(see* [4]).

(B) $TM^m(t) \subset TM^1(t^{2-1/m+\epsilon})$ *(see* [5]).

Corollary. $TM^m(t) \subset TM^{k+1}(t^{1+1/k})$ *(follows from* (A), (B) *and Theorem* 1).

The following results on the impossibility of imbedding certain classes are known:

(C) $KA(I) \not\subset TM^m(I^{1+1/m-\epsilon})$ *(see* [3]);

(D) $TM^m(I) \not\subset TM^k(I^{1-1/m+1/k-\epsilon})$ *(see* [3]).(1)

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(1) In [3], p. 43, the first part of the sentence beginning on line 33 of the second column should read: "The arguments of Section III imply that the worst-case computation time for a single interrogation is at least proportional to $k^{m/i}$." If the corresponding further corrections are made, the indicated result is obtained.

Proof of Theorem 1. Let us fix a K.a. K and simulate its behavior step-by-step on a $(k+1)$-dimensional T.m. M in such a manner that the first t steps of K are simulated over a time $\asymp t^{1+1/k}$.

Let s be an upper bound for the number of vertices in an active neighborhood (the term "active part of a state" corresponds to this term in [2]). To each vertex α of the graph-memory of the K.a. K (in [2] the term "state" corresponds to this term), we assign the natural number (address) $A(\alpha)$ in such a way that the vertices which first appear at any given step (there are not more than s of them) are assigned the smallest addresses which have not yet been used. The mapping A is one-to-one.

We shall represent the behavior of K in a special way below. A family of oriented trees $\{\Gamma_0, \Gamma_1, \dots\}$ with marked arcs is constructed inductively. Γ_0 consists of a single vertex, which is its root. If Γ_n has already been constructed, Γ_{n+1} is constructed as follows: a new vertex R, which will be the root of Γ_{n+1}, is added to Γ_n; an arc is drawn from R to the root of Γ_n and marked with a zero; we then introduce another new vertex R_1, draw an arc from R to R_1, mark it with a one and construct a copy of Γ_n with root R_1; the result is Γ_{n+1}.

To each 1-vertex v of the tree Γ_n there corresponds a unique oriented path of Γ_n going from the root of Γ_n to v; denote this path by $B(v)$. The marks along $B(v)$ form a word of length n in the alphabet $\{0, 1\}$, which we shall regard as a number in binary notation. We shall call it the numerical image of v and denote it by $N(v)$. In the process of constructing $\{\Gamma_0, \Gamma_1, \dots\}$, the numerical image of every 1-vertex remains fixed. The mapping N is invertible.

To each vertex α of the graph-memory of the K.a. K we assign the 1-vertex $N^{-1}(A(\alpha))$ of some Γ_n and the list $F(\alpha)$ of addresses (in binary notation) of those vertices to which α is joined by edges. We consider the list $F(\alpha)$ to be "pasted" to the vertex $N^{-1}(A(\alpha))$. The memory of the K.a. K is completely described by the tree with the "pasted" lists. We shall realize this representation of K's behavior on M.

In modeling K, M constructs a successive packing of the trees $\Gamma_1, \Gamma_{k+1}, \dots, \Gamma_{kn+1}$ in the k-dimensional sublattice π of its memory and writes out the contents of $F(\alpha)$ (for every vertex α) perpendicular to π, placing the beginning of this list where the vertex $N^{-1}(A(\alpha))$ has been packed. The simulation of the pth step of the K.a. can be divided by convention into two stages. During the first stage, the packing of Γ_{kn+1}, which is already in the lattice π, is extended to a packing of $\Gamma_{k(n+1)+1}$ if only $(p-1)s \leq 2^{kn+1} < ps$. The necessary changes in the lists $F(\alpha)$ are carried out during the second stage.

First stage. We shall assume one of the two directions of each axis of the m-dimensional lattice to be positive, and shall number the axes with the integers $1, \dots, m$. The packing of Γ_1 is carried out in such a way that both arcs are packed in the positively directed unit vectors of axes 1 and 2. Suppose $\Gamma_{k(n-1)+1}$ has already been packed in the k-dimensional cube Q of the lattice π, whose sides are parallel to the axes and have length $L = (k+2)2^{n-1} - (k+1)$, where the root of the tree $\Gamma_{k(n-1)+1}$ is packed in the vertex $\mu(Q)$ of Q whose coordinates have the smallest values (in comparison with the corresponding coordinates of the other points of Q).

As a preliminary, let us describe the process of successive packing of Γ ... Γ

in 2, 3, ...-dimensional lattices, respectively, such that the following conditions hold for every natural number $m \geq 2$:

(a) if the vertices $\mu(Q(0)), \ldots, \mu(Q(2^m - 1))$ of the m-dimensional cubes $Q(0), \ldots, Q(2^m - 1)$ with sides of length L are pasted onto all the packings of the 1-vertices of Γ_m in such a way that $Q(N(v))$ is attached to the 1-vertex v, then $Q(0), \ldots, Q(2^m - 1)$ and the packing of Γ_m will be pairwise disjoint;

(b) the union of the packing of Γ_m and the cubes $Q(0), \ldots, Q(2^m - 1)$ is contained in a cube Q' whose sides have length $2L + m + 1$;

(c) the root of the tree Γ_m is packed in the vertex $\mu(Q')$ of Q'.

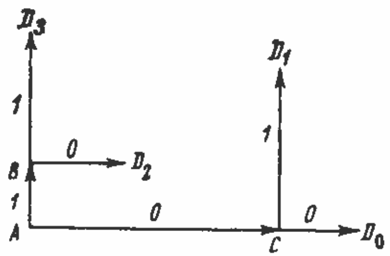

Figure 1

We pack the tree Γ_2 in the plane as in Figure 1, where

$$|AB| = |BD_2| = |CD_0| = 1; \quad |BD_3| = |CD_1| = |AC| - 1 = L + 1.$$

Suppose that Γ_{m-1} has already been packed in the lattice Λ of an $(m-1)$-dimensional cube with sides of length $2L + m$. Then the packing of Γ_m in an m-dimensional lattice is carried out as follows. Let the root of the tree Γ_{m-1} be packed in the vertice R'. We pack the root of Γ_m in a vertex R such that the vector RR' is equal to a unit vector of the first axis, and draw the arc RR'. From the point R we erect the segment RR'' of length $L + 1$ perpendicular to Λ (in the positive direction of the mth axis). We now carry out the packing of a copy of Γ_{m-1} in the $(m-1)$-dimensional sublattice parallel to Λ and passing through RR'' in such a way that the root of Γ_{m-1} is packed in R''. Conditions (a)–(c) hold for this packing of Γ_m. As a result of it, each arc of the tree Γ_m is packed in the positive direction of some axis.

Let us now return to the packing of Γ_{kn+1} in the k-dimensional lattice π. With the aid of the construction described above, we pack Γ_k in π in such a way that the packing of a 1-vertex with numerical image 0 of the tree Γ_k coincides with the already existing packing of the root of the tree $\Gamma_{k(n-1)+1}$. We mark the packings of all vertices of Γ_k in π with a special sign. After this, we carry out the packing of a copy of $\Gamma_{k(n-1)+1}$ in each of the cubes $Q(1), \ldots, Q(2^k - 1)$ (see (a)), marking the packings of 1-vertices with a separate sign. Thus we have constructed a packing of Γ_{kn+1} in a cube Q' with sides of length $2L + k + 1 = (k+2) \cdot 2^n - (k+1)$ such that $\mu(Q') = R$.

The packing of Γ_k in a k-dimensional lattice described above requires not more than $\times 2^k L$ steps of the T. m. M. If $g(n)$ denotes the time needed to construct a packing of Γ_{kn+1} with the aid of M, our construction yields the following recursion relation:

$$g(n+1) \leq 2^k g(n) + c \cdot 2^k((k+2) \cdot 2^{n-1} - (k+1))$$

(c is independent of n and k), from which it follows that $g(n) \leq c_1 \cdot 2^{nk}$, where c_1 does not depend on n. After simulating p steps of the K. a. K, the T. m. M performs at most $c_1 \cdot 2^{nk} \leq c_2 p$ steps during the first stage, where n is such that $2^{(n-1)k} \leq sp \leq 2^{nk}$ (here and in the sequel c with a subscript is independent of p).

Second stage. In making changes in the lists $F(\alpha)$ during the simulation of the pth step of the K.a. K, it is necessary to look over at most s 1-vertices corresponding to active neighborhoods, read in each of them a list of length at most $s \log_2 sp$ and then write down the contents of the new lists $F(\alpha)$ in the appropriate 1-vertices according to the transformation rules for K (see [2]). In order to find a 1-vertex α on the basis of its address, we make use of the signs we placed. The time needed for this is equal to the length of the packing of $B(\alpha)$, which does not exceed $c_3 \cdot 2^n \leq c_4 p^{1/k}$, since the packing of each arc is parallel to the positive direction of an axis.

Therefore, the total number of steps which M must perform in simulating p steps of the K.a. K does not exceed $c_5 p^{1+1/k}$, which proves Theorem 1 in view of linear speed-up for a T.m. (see [6]).

Proof of Theorem 2. One may assume with no loss of generality that the memory of the $(k+1)$-dimensional T. m. M' under consideration is contained in the set of boxes $U = \{(a_1, \ldots, a_{k+1})\}$, where $a_{k+1} > a_i \geq 0$ for $i = 1, \ldots, k$.

A one-to-one mapping h of the set U into a k-dimensional lattice will be constructed, and the contents of each box $(a_1, \ldots, a_{k+1})$ will be written down in the box $h(a_1, \ldots, a_{k+1})$ during the simulation of the behavior of the T.m. M'.

For all natural numbers l, m, x $(l \geq m \geq 1)$, we construct a mapping $h_{l,m,x}$ of the lattice of the m-dimensional cube $Q = \{(x_1, \ldots, x_m)\mid 0 \leq x_i < x\}$ into the lattice of the l-dimensional rectangular parallelepiped with $m-1$ sides of length $[x^{1+1/l}] + 2x$ and $l - m + 1$ sides of length $[x^{1/l}]$ such that if $h_{l,m,x}(x_1, \ldots, x_m) = (a_1, \ldots, a_l)$, then

1) $0 \leq a_m, \ldots, a_l < [x^{1/l}]$;

2) for every $i = 1, \ldots, m-1$,

$$[x_{i+1} x^{1/l}] + 2x_{i+1} \leq a_i < [(x_{i+1}+1) x^{1/l}] + 2(x_{i+1}+1). \tag{1}$$

We also construct a T.m. $M_{l,m}$ which computes $h_{l,m,x}(x_1, \ldots, x_m)$ on the basis of $x, x_1, \ldots, x_m$ with time signalizer equal to $c \log_2 x$, where c is independent of $x, x_1, \ldots, x_m$.

The construction is carried out by induction on m. For $m = 1$ it is not difficult to construct a mapping $h_{l,1,x}$ of a segment of length x into an l-dimensional cube with sides of length $[x^{1/l}]$ and a T. m. $M_{l,1}$ which computes $\{h_{l,1,x}\}_{x=1}^{\infty}$ with the required signalizer. If $m + 1 \leq l$, the transition from m to $m+1$ is accomplished as follows. Let $Q^m_{p,x} = \{(x_1, \ldots, x_m, p)\mid 0 \leq x_i < x, i = 1, \ldots, m\}$, $p = 0, \ldots, x-1$. For each $p = 0, \ldots, x-1$, we map the m-dimensional cube $Q^m_{p,x}$ by means of $h_{l,m,x}$ into the parallelepiped $\{(a_1, \ldots, a_m, \ldots, a_l)\mid 0 \leq a_{m+1}, \ldots, a_l < [x^{1/l}]$; $0 \leq a_1, \ldots, a_{m-1} < [x^{1+1/l}] + 2x$; $[px^{1/l}] + 2p \leq a_m < [(p+1)x^{1/l}] + 2(p+1)\}$.

Fusing all these mappings for $p = 0, \ldots, x-1$ we obtain the mapping $h_{l,m+1,x}$. For the construction of $M_{l,m+1}$, it is necessary to compute $[ix^{1/l}]$ for an appropriate i and make use of the $M_{l,m}$ which has already been constructed.

We map $Q^k_{x,x}$ by means of $h_{k,k,x}$ into the k-dimensional parallelepiped

$$P_x = \left\{ (a_1, \ldots, a_k) \;\middle|\; \sum_{0 \leq l < x} [l^{1/k}] \leq a_k < \sum_{0 \leq l \leq x} [l^{1/k}]; \right.$$
$$\left. 0 \leq a_i < [x^{1+1/k}] + 2x;\; i = 1, \ldots, k-1 \right\}.$$

Fusing all such mappings for $x = 1, 2, \ldots,$ we obtain the required h.

Suppose that one of the heads of the T.m. M' reads the box X at the tth step and passes over to the neighboring box X'. Using the T.m. $M_{k,k}$, the T.m. M computes $h(X)$ and $h(X')$ and moves the simulating head from $h(X)$ to $h(X')$ (the inequality (1) and the definition of P_x yield the estimate $\|h(X) - h(X')\| < O(t^{1/k})$). Thus the tth step of the T.m. M' is simulated by $O(t^{1/k})$ steps of the T.m. M, which proves Theorem 2.

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ON MAXIMAL PATHS AND CYCLES IN A GRAPH

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In this paper we determine, for ordinary graphs, the minimum numbers of edges which guarantee:

A) the existence in an n-vertex graph of a simple path of length k;

B) the existence in an n-vertex graph of a simple cycle of length $\geq k$;

C) the existence in a connected n-vertex graph of a simple path of length k;

D) the existence in a 2-connected n-vertex graph of a simple cycle of length $\geq k$.

As will be seen below, the critical problem is D). The solutions to problems A)–C) will be obtained as corollaries of the solution to D).

Problems A)–C) were posed by Erdős and Gallai [1]. They obtained solutions for a number of special cases (A, n a multiple of k; B, $n-1$ a multiple of $k-2$; C, $k^2 - k + 6 < n$) and stated a conjecture on the solution of problems A)–C) in the general case. We show in this paper that this conjecture is correct for problems A and B, but not quite correct for problem C. The solution of problem B for $n < 2k$ is contained in the results of [2]. In the monograph [3], p. 224, are announced solutions to B when $n-2$ is a multiple of $k-2$, and to D under assumptions of the form $k << n$.

Our terminology basically follows [4]. By a graph we understand a finite unoriented graph without loops and multiple edges. We shall only consider simple cycles and paths (i.e. without self-intersections) without further comment. As usual, the length of a path or cycle is defined as the number of edges it contains. A graph is called 2-connected, or a block, if it remains connected upon the removal of any vertex. We use the following notation: $|X|$ is the number of elements in a finite set X; $V(G)$ and $E(G)$ are the sets of vertices and edges of a graph G; $d_G(x)$ is the order of a vertex x in the graph G; $c(G)$ is the length of the largest simple cycle in the graph G.

If P is a subgraph of G, then $\langle P \rangle$ will denote the subgraph of G generated by the vertices of the graph P.

We also introduce the following functions:

$$A(n, k) = {}^1/_2 n(k-1) - {}^1/_2 r(k-r) \quad \text{for} \quad n = qk + r,\ 0 \leq r < k,$$
$$B(n, k) = {}^1/_2 n(k-1) - {}^1/_2 r(k-r) \quad \text{for} \quad n = q(k-2) + r,\ 0 < r \leq k-2,$$
$$f(n, k, l) = {}^1/_2 l(l-1) + (n-l)(k-l),$$
$$C(n, k) = \max\{f(n, k, [k/2]+1),\ f(n, k, k-1)\},$$
$$D(n, k) = \max\{f(n, k, [k/2]+1),\ f(n, k, k-2)\}.$$

Then the following statements hold.

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