Work of A.A. Markov on stepwise semantics of constructive mathematics

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November 14, 2002

1 Introduction

This paper presents an account on a work which essentially concluded construction of the logical base for the work of Russian constructivist school. [1] This school founded by A.A.Markov (1903-1979) insisted on development of mathematics using only effective, mechanizable means. Some information on its history and main features can be found in the papers and books by A.A. Markov cited in the References and in [40, 39, 20]. Here we describe Markov's approach to semantics of negative arithmetical formulas. Important requirement of Markov-style constructivism was that the objects should be coded by natural numbers. This directed main attention to theories formalizable in the language of first order arithmetic.

Logic of Markov's school can be described using three basic principles: recursive realizability, Markov principle and classical logic for sentences containing no constructive problems, i.e. $\exists, \vee$-free sentences [38, 40]. This is a complete description from the classical point of view. Markov presents his approach in quite different way. He considers formal language ($L_{2\omega+1}$ in [36] or $Я_{\omega+1}$ in [34]) having the same expressive power as the language of first-order arithmetic with all logical connectives $\&, \vee, \supset, \forall, \exists, \forall \leq, \exists \leq$ (the two latter quantifiers are bounded). Arbitrary formulas of this language are transformed into formulas of the form

$$\exists x A \tag{1}$$

where A is an *almost negative* formula: only decidable formulas can occur after $\exists, \vee$. Transformation into the form (1) is done by the algorithm SH [36,

[1] The Russian version of this work was written as a comment to several papers (cf. [29]-[18]) most of them to be published by Nauka Publishers, Moscow in Collected Works by A.A. Markov.

§12] based on the ideas of N. Shanin [39] and equivalent (under a suitable coding in the intuitionistic arithmetic HA) to recursive realizability introduced by S.C. Kleene [17]. This step and related logical principles have been intensively discussed at the early stages of the Russian constructivist school, and by the mid-sixties they were sufficiently clear. Markov's discussion of the algorithm SH is relatively brief.

It is easy to transform the formula A in (1) into a *negative* formula A' which contains neither $\vee$ nor bounded $\exists$-quantifier (so that HA derives $\neg\neg A' \leftrightarrow A'$) using *Markov principle*: it is enough to replace $\exists x M$ by $\neg\forall x \neg M$ and $A \vee B$ by $\neg(\neg A \& \neg B)$. But Markov's goal was to construct a semantics for almost negative formulas in a bottom-up fashion and simultaneously justify *the principle of constructive selection* (his name for *Markov principle*). In the following we consider only semantics of almost negative formulas sliced into stages (steps) according to implication nesting: recall that negation is defined via implication and $\perp$. It is known that in the presence of the universal quantifier $\forall$ implication complexity corresponds to quantifier complexity (number of quantifier alternations) used in the investigations of classical arithmetic. Using notation $\perp = (0 \neq 0)$, $\neg A = A \supset \perp$, $\dot{\exists} x A = \neg\forall x \neg A$ a formula $\forall x_1 \dot{\exists} x_2 \ldots \forall x_k \dot{\exists} x_{k+1} M$ containing k alternations of quantifiers is translated into a formula $\forall x_1 (\forall x_2 (\ldots (\forall x_{k+1} \neg M \supset \perp) \ldots \supset \perp) \supset \perp)$ containing k "essential" implications. Importance of implication in constructive mathematics was observed early enough [14, 19] and by the beginning of 1960-ies there were hopes to find its interpretation in almost negative formulas based on a deductive understanding sketched in papers by P. Lorenzen (cf. [21]) and based on the notions of derivable and admissible rule. For a given formal system $\mathbf{C}$ a rule

$$\frac{A}{B} \tag{2}$$

is *admissible* if $\mathbf{C} \vdash A$ implies $\mathbf{C} \vdash B$ where $\vdash$ is the derivability symbol. The rule (2) is *derivable* (in the system $\mathbf{C}$) if there is a deduction of B from A by the rules of $\mathbf{C}$. Every derivable rule is obviously admissible, but the converse is not true generally. A good example is the disjunction property of intuitionistic formal systems: the rule

$$\frac{\neg A \to B \vee C}{(\neg A \to B) \vee (\neg A \to C)}$$

for closed formulas A, B, C is admissible in the intuitionistic propositional calculus, but is not derivable there. Otherwise the formula

$$(\neg A \to B \vee C) \to ((\neg A \to B) \vee (\neg A \to C))$$

would be intuitionistically derivable. If $\mathbf{C}$ is a standard logical system, i.e. the set of derivable formulas is recursively enumerable, the derivability of the rule (2) can be expressed by an arithmetical formula

$$\exists x Proof(x, \mathcal{G}(A)) \to \exists y Proof(y, \mathcal{G}(B)) \tag{3}$$

where $\mathcal{G}(A), \mathcal{G}(B)$ are Godelnumbers (numerical codes) of the formulas A, B and $Proof$ is the familiar proof predicate. If the calculus $\mathbf{C}$ is complete for a class of formulas containing A and B then (3) can be rewritten as

$$True(A) \rightarrow True(B)$$

where $True(F)$ means that F is true. This underlies A. A. Markov's definition of implication as admissibility of a rule for the formulas of a basic language containing essentially Σ_1^0-formulas (language L_1 in [36] or language Я_1 in [29]). By the principles codified by recursive realizability, formula (3) is interpreted as existence of an effective function transforming x into y:

$(\exists f \in Rec)(\forall x(Proof(x, \mathcal{G}(A)) \rightarrow Proof(f(x), \mathcal{G}(B))))$ where $f \in Rec$ means that a natural number f is a code of a unary total recursive function.

For implication $F \supset G$ which is not immediately reducible to implications of Σ_1^0-formulas A.A. Markov suggested interpretation by derivability of the corresponding rule $F \vdash G$. Such a definition he used mainly in a situation when there exists a logical system $\mathbf{C}$ (which is not recursively enumerable in general) sound for some previously defined notion of truth $True(H)$ for the formulas H of the same complexity as F and G. In such a case it was always understood that all true (in the sense of the predicate $True$) formulas H are added to the axioms of $\mathbf{C}$). In other words, a formula $F \supset G$ is true if $\mathcal{F}, F \vdash G$ by the rules of the system $\mathbf{C}$ where $\mathcal{F} = \{F : True(F)\}$. Formulas $\forall x A[x]$ beginning with an unbounded universal quantifier are reduced to simpler formulas using ω-rule (called also Carnap rule):

$$\frac{\ldots A[n] \ldots all\ n}{\forall x A[x]}$$

A.A. Markov required that every application of this rule should be constructive (there should be a uniform general method) and introduces by a generalized inductive definition a notion of truth based on this rule. Such a treatment is equivalent to the notion of a recursive derivation with ω-rule if one admits a principle of generalized induction corresponding to this inductive definition.

The only remaining special rule of the stepwise semantics is Markov principle. It can be stated as an implication

$$\neg\neg\exists x M \supset \exists x M \tag{4}$$

where M is a formula containing no unbounded quantifiers. For the premise $\neg\neg\exists x M$ of the implication (4) the truth turns out to be equivalent to derivability in a system (to be denoted by $\mathbf{C}_2$) with effective ω-rule and *intuitionistic* logic. This allowed to use the following important observation [23, 36]:

Theorem 1 *Markov rule*

$$\frac{\neg\neg\exists x M}{\exists x M} \tag{5}$$

where M is an arithmetical formula without unbounded quantifiers, is admissible in the system $\mathbf{C}_2$.

Similar results for other systems were known in the literature [18]. It was a possibility to justify the rule at a low stage of the introduced hierarchy that was important to A. A. Markov. Another important feature was a method of the proof which anticipated later proofs by A. Dragalin [3] and H. Friedman [5]. The latter two proofs use much weaker means (Kalmar elementary functions are sufficient) to transform the proofs than earlier proofs which used cut elimination. A.A. Markov sketched his proof only for systems with the ω-rule described below, but its specialization for the system $\mathbf{C_2}$ [23] would be elementary in the same sense as the proofs by Friedman and Dragalin.

After admissibility of the rule (5) is established, A.A. Markov postulates it at the next level as a basic rule, so that the implication $\neg\neg\exists xM \supset \exists xM$ turns out to be true. Final result is a stepwise definition of the constructive truth for almost negative arithmetical formulas which is equivalent to the standard classical definition if one uses classical logic (cf. [38]).

2 One-quantifier systems

2.1 The first approximation

Details of the definitions changed in the process of investigation. The first publication [22, 23] contained only two lowest levels. The language L_1 in [23] deals with strings in a finite alphabet and has the same expressive power as an arithmetical language $\mathcal{L}_1$ including all Kalmar-elementary functions, equations, inequations (i.e. negated equations), $\&, \vee$, bounded quantifiers $(\forall \leq t), (\exists \leq t)$ and unbounded existential quantifier $\exists$. Recall that Kalmar-elementary functions are constructed from the constants, projections, $+$, $\times$ and $[a/b]$ by bounded sums, bounded products and substitution.

Formulas of $\mathcal{L}_1$ are essentially Σ^0_1-formulas. The truth definitions for closed L_1-formulas in [23] is standard. A calculus C_1 described in [23] is complete for L_1. We describe a similar calculus which is complete for the arithmetical language $\mathcal{L}_1$. Let $|t|$ stand for the numerical value of the closed term t, the letter n stand for arbitrary numeral, i.e. expression of the form $0 + 1 + \ldots + 1$, and $F[v/t]$ is the result of substituting term t for all free occurrences of a variable v in F with standard precautions. $F[t]$ stands for $F[x/t]$.

Calculus C_1:

Axioms:

True closed equations $t = r$ and true closed inequations $t \neq r$

Inference rules:

Introduction rules for $\&, \vee, \exists$:

$$\frac{F \quad G}{F\&G} \qquad \frac{F}{F \vee G} \qquad \frac{G}{F \vee G} \qquad \frac{F[t]}{\exists xA}$$

Introduction rules for bounded quantifiers:

$$\frac{F[n]}{(\exists x \leq n)F} \qquad \frac{(\exists x \leq n)F}{(\exists x \leq n+1)F} \qquad \frac{(\exists x \leq |t|)F}{(\exists x \leq t)F}$$

$$\frac{F[0]}{(\forall x \leq 0)F} \qquad \frac{(\forall x \leq n)F \;\; F[n+1]}{(\forall x \leq n+1)F} \qquad \frac{(\forall x \leq |t|)F}{(\forall x \leq t)F}$$

It is easy to see that these rules are sound for the (obvious understanding of the) language $\mathcal{L}_1$. By induction on the formula it is easy to show that these rules are complete: every true formula is derivable. The language L_2 in [23] has the same expressive power as the arithmetical language $\mathcal{L}_2$ which has as its formulas all formulas of $\mathcal{L}_1$, all implications $F \supset G$ where $F, G \in \mathcal{L}_1$ and arbitrary conjunctions of formulas in $\mathcal{L}_2$. Hence nested implication are not allowed. The truth of a closed implication $F \supset G$ is defined as admissibility of the rule

$$\frac{F}{G}$$

in the calculus $\mathbf{C}_1$, conjunction is understood in a natural way. By the completeness of the calculus $\mathbf{C}_1$ this definition is equivalent to a standard classical definition. For the formulas of the language L_2 A.A. Markov defines in [23] a sound calculus C_2. Main technical feature of that calculus is a reformulation of standard axioms and rules of the intuitionistic logic and arithmetic adapted to the restricted language considered here. Except axioms corresponding to the standard defining equations for computable functions and usual properties of bounded and unbounded quantifiers the calculus C_2 contains two induction rules for proving properties of strings in a finite alphabet. Corresponding rule for the arithmetical language would have a form

$$(Ind)\ \frac{F[0] \supset G[0] \quad F[x+1] \supset (F[x] \vee G[x+1]) \quad F[x+1] \& G[x] \supset G[x+1]}{F[x] \supset G[x]}$$

where $F, G \in \mathcal{L}_1$. Let's note that both premises and the conclusion of the rule Ind are formulas of the language $\mathcal{L}_2$. On the other hand, in traditional formalism the rule Ind is equivalent to the standard induction rule for the formula $F \supset G$, which contains implication more complicated that the formulas of the language $\mathcal{L}_2$:

$$(Ind+)\ \frac{F[0] \supset G[0] \quad (F[x] \supset G[x]) \supset (F[x+1] \supset G[x+1])}{F[x] \supset G[x]}$$

Indeed, the premises of the rules (Ind) and $(Ind+)$ are equivalent in the intuitionistic predicate calculus with Markov principle:

$$[(F[x] \supset G[x]) \supset (F[x+1] \supset G[x+1])] \leftrightarrow$$

$$[(F[x+1] \supset (F[x] \vee G[x+1]))\&(F[x+1] \& G[x] \supset G[x+1])]$$

This trick of reducing logical complexity by implicit use of relations to be justified later is characteristic for Markov's constructive mathematics and its predecessors (intuitionism, finitism). A.A. Markov does not explain his choice of the induction rule, but notes in [22] impossibility of a complete calculus for the formulas of the language $\mathcal{L}_2$. He also states there that Markov principle holds for refutability in C_2, which probably means admissibility of Markov rule. Although [22, 23] contain no indication to the proof of the latter statement, it is plausible that Markov had in mind the same method that was later used in [31] to prove admissibility of Markov rule in the systems with ω-rule.

2.2 Systems with ω-rule

2.2.1 The language $\mathcal{L}_2$

As pointed out already, the set of the true formulas of the language L_2 is not recursively enumerable: true formulas of the form $\neg\exists xM (\leftrightarrow \forall x\neg M)$ with M containing at most bounded quantifiers form complete Π^0_1-set. The truth definition for the implications $\exists xM \supset \exists xK$ appeals to the notion of a uniform general method or algorithm. This notion was used in [24, 25] to construct a system of rules for the language L_2 containing ω-rule. Informal justification of such approach using (what Markov called) *intuition of generality* is given in [27, §7], [36, §2].

The rules for the language $\mathcal{L}_2$ similar to the rules from [24, 25, 36] are intuitionistic rules for $\supset, \&, \vee$ adapted to avoid nested implications and ω-rule:

$$(R1)\ \frac{F \quad F\supset G}{G} \qquad (R2)\ \frac{F\supset G \quad G\supset H}{F\supset H} \qquad (R3)\ \frac{G}{F\supset G}$$

$$(R4)\ \frac{F\supset G \quad F\supset H}{F\supset G\&H} \qquad (R5)\ \frac{F\supset H \quad G\supset H}{F\vee G\supset H} \qquad (R6)\ \frac{K \quad L}{K\&L}$$

$$(R7)\ \frac{K\&L}{K} \qquad (R8)\ \frac{K\&L}{L} \qquad (R9)\ \frac{\ldots I[n]\supset G \ldots n=0,1,\ldots}{\exists xI\supset G}$$

Here F, G, H are $\mathcal{L}_1$-formulas, i.e. formulas of the language $\mathcal{L}_1$, K, L are $\mathcal{L}_2$-formulas, $\exists xI$ is a closed $\mathcal{L}_1$-formula. Let's recall that the derivations according to these rules can use (as axioms) arbitrary true $\mathcal{L}_1$-formulas. It is obvious that the rules $R1 - R9$ are sound (or semantically acceptable in A.A. Markov's terminology): from true formulas one can derive only true formulas.

2.2.2 Languages $\mathcal{L}_3 - \mathcal{L}_\omega$

Formulas of the arithmetical language $\mathcal{L}_3$ which will have the same expressive power as the language L_3 in [24, 25, 36] are implications $J \supset K$ of the $\mathcal{L}_2$-formulas and arbitrary conjunctions of $\mathcal{L}_3$-formulas. In other words single nesting of implication is allowed. The truth of an implication $J \vdash K$ is defined as derivability $J \vdash K$ by the rules $R1 - R9$ using arbitrary true $\mathcal{L}_2$-formulas.

It turns out that the language $\mathcal{L}_3$ (as well as the wider languages $\mathcal{L}_n, \mathcal{L}_\omega$ introduced below) is equivalent to $\mathcal{L}_2$, and it is sufficient to have at most one ω-rule in every branch of the derivation. Justification of that statement is combined in [24, 25] with the admissibility proof for Markov rule. The proof is based on the following observation which is implicit in A.A. Markov's constructions.

Every propositional combination of $\mathcal{L}_1$-formulas can be transformed into a classically equivalent $\mathcal{L}_2$-formula, i.e. into a conjunction of implications of $\mathcal{L}_1$-formulas. A.A. Markov defines such a transformation $\mathcal{R}$. Main non-intuitionistic steps of $\mathcal{R}$ are

$$\mathcal{R}((I \supset F) \supset G) = (I \vee G)\&(F \supset G)$$

$$\mathcal{R}((I \supset F) \supset (H \supset G)) = (H \supset I \vee G)\&(F\&H \supset G)$$

Remaining steps like

$$\mathcal{R}(F \supset (G \supset H)) = F\&G \supset H$$

are defined in a natural way.

Theorem 2 *A formula $K \in \mathcal{L}_3$ is true iff and only if the formula $\mathcal{R}(K)$ is true.*

This proposition is stated in [24, 25] only for the case when a justification of the truth of $K \supset L$ uses only bounded number of ω-rules in every branch, but it is obviously true (together with its proof) without this restriction (cf. [31], theorem 11). The proof is done by induction on the derivation with ω-rule. The detailed proof by cases was never published, but none of these cases presents any difficulty. Theorem 2 amounts to completeness of the rule $\mathcal{R}(K) \vdash K$ for the language $\mathcal{L}_3$. Important instance

$$\mathcal{R}((\exists x M \supset \bot) \supset \bot) = ((\exists x M \vee \bot)\&(\bot \supset \bot)) \leftrightarrow \exists x M$$

justifies Markov principle.

The language $\mathcal{L}_{n+1},\ n+1 > 3$ where n times nested implications are allowed is defined in a natural way. The truth of $\mathcal{L}_{n+1}$-implication is defined as derivability by the rules R1-R9 plus the rules:

$$R10\ \frac{D}{\mathcal{R}(D)} \qquad R11\ \frac{\mathcal{R}(D)}{D}$$

Then a union is formed:

$$\mathcal{L}_\omega = \bigcup_{n>0} \mathcal{L}_n\ ,$$

and $\mathcal{L}_\omega$ is reduced to $\mathcal{L}_2$ by the transformation $\mathcal{R}$. Hence the truth of an $\mathcal{L}_\omega$ -formula in the stepwise semantics is the same as the truth of its translation into $\mathcal{L}_2$, and so it coincides with usual arithmetic truth. In particular familiar classical propositional logic is valid in $\mathcal{L}_2$.

3 The language $\mathcal{L}_{\omega+1}$

The last step made in [24, 25] is adding initial universal quantifiers to $\mathcal{L}_\omega$-formulas. These quantifiers are understood in a natural way: $\forall x K$ means that there exists a uniform general method allowing to establish the truth of every formula $K[n]$. The language $\mathcal{L}_{\omega+1}$ having the same expressive power as the language $L_{\omega+1}$ from [36] contains $\mathcal{L}_\omega$ and allows unrestricted use of $\forall$ and &. The rules R1 (modes ponens), R6-R8 (introduction and elimination of &) in the section 2.2.1, which are denoted below by $\Pi(\omega+1)1-4$ as well as the rules

$$\Pi(\omega+1)5\ \frac{\forall x(F\supset B)}{\exists xF\supset B}\qquad \Pi(\omega+1)6\ \frac{\forall xG}{G[n]}\qquad \Pi(\omega+1)7\ \frac{\ldots G[n]\ldots}{\forall xG}$$

are sound for $\mathcal{L}_{\omega+1}$. It is easy to see that this system of rules denoted below by $\Pi(\omega+1)$ is also complete for deriving true $\mathcal{L}_{\omega+1}$-formulas from the true $\mathcal{L}_\omega$-formulas, and it is possible to have a finite bound for the number of ω-rules in any branch: the bound is the maximum nesting of universal quantifiers. Indee, a true conjunction is obtained by the rule R8, a true $\forall$-formula is obtained by ω-rule, and in this way every true $\mathcal{L}_{\omega+1}$-formula is reduced to true $\mathcal{L}_\omega$-formulas which are initial (axioms). Since this reduction preserves equivalence in a traditional sense, the stepwise semantics coincides with traditional truth for $\mathcal{L}_{\omega+1}$-formulas.

3.1 Languages $\mathcal{L}_{\omega+n}, n\geq 2$

The language $L_{\omega+1}$ exactly corresponds to the level Π^0_2 of the arithmetical hierarchy. The next step increases the complexity by 1. Its iteration leads to formulas with arbitrary nesting of implications and universal quantifiers, so that all almost negative formulas are obtained.

The formulas of the language $\mathcal{L}_{\omega+n+1},\ n>0$ are constructed from $\mathcal{L}_{\omega+n}$-formulas by one application of implication and by unrestricted application of $\&,\forall$ to such implications. In fact Markov used $\supset N$ to indicate the level-N implication, but we drop N. Semantics is defined as before: $\forall xF$ means availability of a uniform general method allowing to establish the truth of every formula $F[n]$. An implication $A\supset B$ where $A,B\in\mathcal{L}_{\omega+n}$ means derivability $A\vdash B$ by the rules $\Pi_{\omega+n}$ described below (using arbitrary true $\mathcal{L}_{\omega+n}$-formulas). The rules $\Pi_{\omega+n}$ for $n>1$ include (modulo technical details) the rules R1-R4,R6-R8 from the section 2.2.1 (modus ponens, the transitivity of implication, adding a redundant premise, introduction of conjunction into the conclusion of an implication, introduction and elimination of conjunction) as well as the following rules:

$$\frac{\ldots A\supset G[n]\ldots}{A\supset\forall xG}\ \forall^+\qquad \frac{\ldots H[n]\ldots}{\forall xH}\ \forall^+\qquad \frac{\forall xH}{H[n]}\ \forall^-\qquad \frac{A\&I\supset B}{A\supset\mathcal{R}(I,B)}\ (exp)$$

where $\mathcal{R}(I,B)$ for $I \in \mathcal{L}_{\omega+n-2}, B \in \mathcal{L}_{\omega+n-1}$ is the result of of transforming implication $A \supset (B \supset C)$ into a $\mathcal{L}_{\omega+n}$-formula by moving universal quantifiers and conjunctions forward and replacing $A \supset (B \supset D)$ by $A\&B \supset D$. In other words, the rule (exp) replaces exportation $A\&I \supset B \vdash A \supset (I \supset B)$ which has a conclusion outside the language $\mathcal{L}_{\omega+n}$. In fact the rules $\Pi_{\omega+n}$ were stated for the implication $\supset (N-1)$ and are admissible for implications $\supset M$ $(M < N-1)$. We do not stress these distinctions. It is easy to see that the rules $\Pi_{\omega+n}$ are sound both for $n = 1$ and for $n > 1$. It is possible to show that they are complete: every formula of the language $\mathcal{L}_{\omega+n+1}, n \geq 1$ which is true in the traditional sense is also true in the stepwise semantics. We only sketch the argument to clarify the role of every rule. Let us note that arbitrary $\mathcal{L}_{\omega+n+1}$-formula is reduced to implications of $\mathcal{L}_{\omega+n}$-formulas by analysing it using the rules of &- and $\forall$-introduction. For example a formula $K\&L$ is true iff both K and L are true. It remains to prove the following assertion.

Lemma 1 *If $n \geq 1$ and the formula $A \supset B$ where $A, B \in \mathcal{L}_{\omega+n}$ is true in the traditional sense, then $A \vdash B$ by the rules $\Pi_{\omega+n}$.*

Proof . Analysis using the rules of &- and $\forall$-elimination allows to reduce our task (non-constructively) to deriving the relation below which is true in the traditional sense:

$$E \supset F \vdash G \supset H \tag{6}$$

where $E, F, G, H \in \mathcal{L}_{\omega+n-1}$. One can assume that the formula $G \supset H$ is false, since otherwise (6) is an axiom. Then $E \supset F$ is false too, and hence E is true and F is false. For example if a formula $A = \forall x D$ is false then for some n the formula $D[n]$ is false and $A \vdash D[n]$ by $\forall$-elimination. The formula $F \supset H \in \mathcal{L}_{\omega+n}$ is true and for $n \geq 2$ the required derivation is as follows:

$$\cfrac{\cfrac{\cfrac{E \supset F \quad E}{F} \qquad F \supset H}{H}}{G \supset H}$$

For $n = 1$ the true formula $(E \supset F) \supset (G \supset H)$ is in $\mathcal{L}_\omega$, and hence (6) is obtained by the rule $\Pi(\omega+1)1$(*modus ponens*) which concludes the argument.

More constructive proof for equivalence of stepwise and traditional semantics would use natural deduction calculus with ω-rule as an equivalent of the traditional semantics, and induction on the normal (cut-free) proofs in this calculus (cf. [38, 3]). Note that the rule $\Pi(\omega+1)9$ is not used in our proof, and as noted by Gimon [6], A.A.Markov showed that this rule can be omitted.

4 Languages $\mathcal{L}_{2\omega}, \mathcal{L}_{2\omega+1}$

The language $L_{2\omega}$ from [36] is the union of all $L_{\omega+n}, n \geq 0$, and under traditional interpretation it coincides with the language of the classical arithmetic.

In particular, for every formula $A \in L_{2\omega}$ the formula $\neg\neg A \supset A$ is true in the stepwise semantics. Hence the classical propositional calculus is sound both for $L_{2\omega}$ and for the language $\mathcal{L}_{2\omega} = \bigcup_n \mathcal{L}_{\omega+n}$. The arithmetical language $\mathcal{L}_{2\omega+1}$ corresponding to the language $L_{2\omega+1}$ from [36] is just the language of the first order arithmetic. All logical connectives $\&, \vee, \supset, \forall, \exists$ can be applied here without restrictions. This language is reduced to the language $\mathcal{L}_{2\omega}$ (and hence to one of $\mathcal{L}_{\omega+n}, n \geq 0$) by "elucidation of the constructive problem". Arbitrary sentence A is transformed by an algorithm SH (Section 1) inro a formula $\exists x A'(x)$ where $A'(x) \in \mathcal{L}_{2\omega}$, and then the truth of A is defined as the existence of a natural number n such that $A'(n)$ is true.

This concludes our description of Markov's stepwise semantics.

5 The languages Я_α

In his papers [28]-[34] A.A. Markov presented a new version of the stepwise semantics equivalent to one in [36]. Let's describe the relation between these two versions and the deductive means used in these papers.

The language Я_0 in [28] is the language of bounded quantifiers. Corresponding arithmetical language consists exactly of formulas of the language $\mathcal{L}_1$ (Section 2.1) which do not contain unbounded quantifiers. The truth of Я_0-sentences is decidable and the law of excluded middle is true for Я_0-sentences. The language Я_1 from [29] is equivalent to L_1: it is the result of adding $\exists$ to the language Я_0 (unrestricted application of $\vee, \&$ is allowed too).

The language Я_2 from [30] is equivalent to $L_{\omega+1}$ as turns out later. By the definition Я_2 includes Я_1, implications of Я_1-formulas and is closed under $\&, \forall$. Hence a complete calculus is impossible for Я_2 and its semantics is defined via derivability with ω-rule. Up to this point all arguments can be formalized in the first order intuitionistic arithmetic. The paper [30] contains a new rule for informal arguments which may be called induction on ω-derivations: *if $Y(F)$ is a property of formulas F of the language Я_2, the property Y is preserved under all applications of the rules and it holds for all closed true Я_2-formulas and for a given formula K, then every formula derivable from K has the property Y.*

This rule is used for the soundness proof: every Я_2-formula derivable from true Я_2-formulas is true.

The language Я_3 from [31] is equivalent to the language $L_{\omega+2}$. A. A. Markov notes that one can prove soundness of Markov rule by induction on ω-derivations. In fact, more general statement is formulated: the rule

$$\frac{A \& I \supset B}{A \supset \mathcal{R}_2(I, B)}$$

where the algorithm $\mathcal{R}_2$ is similar to the algorithm $\mathcal{R}$ in section 2.2.2, is admissible.

The languages $Я_n (n \geq 4)$ from [32] are equivalent to the languages $L_{\omega+n-1}$ from [36]. Needed properties of true formulas are proved by induction on derivations.

The language $Я_\omega$ from [33] is a union of languages $Я_n$, and hence it is equivalent to the language $L_{2\omega}$ from [36]. Finally, the language $Я_{\omega+1}$ from [34] is equivalent to the language $L_{2\omega+1}$ from [36], i.e. to the language of first order arithmetic with realizability interpretation. In connection with the soundness theorem for the ω-rule it is mentioned that its proof is based on Church thesis. Summing up we see that Markov uses first order intuitionistic arithmetic, induction on ω-derivations and Church thesis (in the presence of general inductive definitions).

The last paper in the series [28]-[35] states completeness theorem for the classical predicate logic in the form given by Bernays:

For every consistent formula F of the first order predicate calculus there is a substitution σ of arithmetical formulas for predicate variables such that σF is true. The substitution σ can be chosen in the language $Я_2$.

It is noted that the proof is constructive and uses ideas of Henkin and Hasenjäger.

6 Further work

The work of A.A. Markov on stepwise semantics was continued by his students. They studied extensions of the language of arithmetic contained in the language of ramified analysis, where the language of a rank r allows quantifiers over sets definable by formulas of lower rank. Semantics which was obtained in this way turned out to be equivalent to the traditional semantics.

A. Dragalin [4] proved that the levels of stepwise semantics form a strict hierarchy.

M. Kanovich [15, 16] extended stepwise semantics to languages with quantifiers over arithmetical sets.

E.S. Burgina [1] sketched an extension of the stepwise semantics to ramified analysis of all finite ranks. The truth of the formula $\forall x A$ with a variable x of rank r is defined as the truth of all substitution instances $A[x/\mathbf{q}xB]$ where $\mathbf{q}xB$ is a closed abstract of a rank $\leq r$.

M. Dombrowski-Kabanchenko [2] extends stepwise semantics to all constructive levels: he constructs languages L_x for all $x \in O$, so that every hyperarithmetical set is representable in one of L_x.

V. Gimon [7, 8] investigates one of the lowest levels, viz. propositional combinations of $Я_1$-formulas.

F. Gorgy, A. Sahyoun and V.Gimon investigate in [6, 9, 10, 11, 12, 13] independence of rules introduced by A.A. Markov for the stepwise system.

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