# **Complexity lower bounds for algebraic computation trees**

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Several non-linear lower bounds are known in algebraic complexity unlike its boolean counterpart

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## **Computational models** Let *F* be a ground field.

- Input  $x = (x_1, \ldots, x_n) \in F^n$  is attributed to the root of tree T.
- To each node *v* of *T* (except leaves) a (testing) polynomial  $g_v \in F[X_1, \ldots, X_n]$  with  $deg(g_v) \leq d$  is assigned. The algebraic decision tree branches at node *v* according to whether  $g_v(x) = 0$ . In case  $F = \mathbb{R}$  one uses alternatively  $g_v(x) > 0$  as a condition of
- To each leaf *L* an output "*accept*" or "*reject*" is assigned.
- Denote by  $S_L \subset F^n$  the set of inputs which arrive at *L*. They are pairwise disjoint and the set *S* ⊂ *F <sup>n</sup>* **accepted by the algebraic decision tree** is the union of *S<sup>L</sup>* for all leaves *L* assigned with
- <span id="page-2-0"></span>The **complexity** *C<sup>d</sup>* is the depth of *T*.

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## **Algebraic** *d***-decision tree**

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- To each leaf *L* an output "*accept*" or "*reject*" is assigned.
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- <span id="page-4-0"></span>

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- To each leaf *L* an output "*accept*" or "*reject*" is assigned.
- <span id="page-7-0"></span>Denote by  $S_L \subset F^n$  the set of inputs which arrive at *L*. They are pairwise disjoint and the set *S* ⊂ *F <sup>n</sup>* **accepted by the algebraic decision tree** is the union of *S<sup>L</sup>* for all leaves *L* assigned with "*accept*".

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- <span id="page-8-0"></span>• The **complexity**  $C_d$  is the depth of T.

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- The **complexity**  $C_d$  is the depth of T.

The difference with algebraic decision trees is that testing polynomial *g*<sup>*y*</sup> is calculated as  $g_v = a \circ b$  where

*a*, *b* ∈ *F* ∪ {*X*<sub>1</sub>, . . . . , *X<sub>n</sub>*} ∪ {*g<sub>u</sub>*} where *u* runs the nodes on the path

 $\sqrt{m}$   $\rightarrow$   $\sqrt{m}$   $\rightarrow$   $\sqrt{m}$ 

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In particular,  $\deg(g_\mathsf{v}) \leq 2^{k_\mathsf{v}}$  where  $k_\mathsf{v}$  is the depth of  $\mathsf{v}.$ 

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In particular,  $\deg(g_{\rm\scriptscriptstyle V})\leq 2^{k_{\rm\scriptscriptstyle V}}$  where  $k_{\rm\scriptscriptstyle V}$  is the depth of  ${\rm\scriptscriptstyle V}.$ Denote by *C* the complexity of algebraic computation trees. For any constant *d* the model of computation trees is stronger than one of *d*-decision trees.

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Denote by *C* the complexity of algebraic computation trees. For any constant *d* the model of computation trees is stronger than one of *d*-decision trees.

We study the complexities  $C_d(S)$ ,  $C(S)$  of the membership problem to *S*.

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There are several lower bounds for algebraic computation trees over the field either  $F = \mathbb{C}$  or  $F = \mathbb{R}$  of the form  $C(S) \ge log_2(c_i) - n$ ,  $i = 1, 2, 3$  where

- *c*<sup>1</sup> is the maximum of the number of connected components in *S*
- **c**<sub>2</sub> is the Euler characteristic of *S* (Björner-Lovasz-Yao [1992]);
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- **•**  $c_2$  is the Euler characteristic of *S* (Björner-Lovasz-Yao [1992]);
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 $c_1$ ,  $c_2$   $<$   $c_3$ 

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**• EQUALITY is the set** 

 $\{(x_1, \ldots, x_n, y_1, \ldots, y_n) \subset F^{2n} : \{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}$ ;

- DISTINCTNESS is the set  $\{(x_1, \ldots, x_n) : x_i \neq x_j, i \neq j\};$
- KNAPSACK is the set  $\bigcup_{I \subset \{1,...,n\}} \{(x_1,\ldots,x_n) \,:\, \sum_{i \in I} x_i = 1\}.$

- 
- *C*(*DISTINCTNESS*) *n* · log*n (QuickSort (F* = R*), discriminant (F* = C*) Strassen [1973], Ben-Or [1983]);*
- $C(KNAPSACK) \geq \Omega(n^2)$  *(Ben-Or [1983]).*

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Let *F* be an algebraically closed field of characteristic  $p > 0$ . For a variety  $S\subset F^n$  introduce  $B(S)$  which replaces the sum of Betti numbers (over  $\mathbb C$  or  $\mathbb R$ ).

 $D$ enote  $N_k = N_k(S) = \sharp (S \cap \mathbb{F}_o^n)$ 

 $\textsf{Zeta function}~Z(S,t)=\exp(\sum_{1\leq k<\infty}N_kt^k/k)=P(t)/Q(t)$ is a rational function (due to Dwork, Deligne).

 $\bullet$  Define  $B(S) = \deg(P) + \deg(Q)$ .

<span id="page-27-0"></span>**Dima Grigoriev (CNRS)** 

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## **Zeta function**

- $\mathsf{Denote}\;\mathsf{N}_k=\mathsf{N}_k(\mathcal{S})=\sharp(\mathcal{S}\cap\mathbb{F}^n_p)$ *p k* ).
- $\textsf{Zeta function}~Z(S,t)=\exp(\sum_{1\leq k<\infty}N_kt^k/k)=P(t)/Q(t)$

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<span id="page-28-0"></span>**Dima Grigoriev (CNRS) [Complexity lower bounds: algebraic trees](#page-0-0) 14.6.11 7 / 23**

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<span id="page-29-0"></span>**Dima Grigoriev (CNRS) [Complexity lower bounds: algebraic trees](#page-0-0) 14.6.11 7 / 23**

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- $\bullet$  Define  $B(S) = \deg(P) + \deg(Q)$ .

<span id="page-30-0"></span>*Ben-Or [1994]).*

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• Define 
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B(S) = \deg(P) + \deg(Q)
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### **Theorem**

 $C(S) > a \cdot log(B(S)) - b \cdot n$  for some constants a, *b* (Ben-Or [1994])

<span id="page-31-0"></span>*Ben-Or [1994]).*

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## **Corollary**

<span id="page-32-0"></span> $C$ (*EQUALITY*)[,](#page-0-0)  $C$ (*DISTINCTNESS*)  $\times n \cdot \log n$  (*S[tr](#page-33-0)[a](#page-26-0)ss[e](#page-33-0)n* [\[1](#page-118-0)[97](#page-0-0)[3\]](#page-118-0), *Ben-Or [1994]).* **Dima Grigoriev (CNRS) [Complexity lower bounds: algebraic trees](#page-0-0) 14.6.11 7 / 23**

# **Complexity upper bound for membership to a linear complex**

An **arrangement** *S* ⊂ *F n* is a union of hyperplanes ∪*iH<sup>i</sup>* . A face of *S* is

*of S. Then*  $C(S) \leq O(n^3 \cdot \log N)$  *(Meyer auf der Heide [1985]).* 

<span id="page-33-0"></span>algebraic computation tree is non-uniform (de[pe](#page-32-0)[nd](#page-34-0)[s](#page-32-0)[o](#page-38-0)[n](#page-0-0)  $n$ [\)](#page-0-0)[.](#page-118-0)  $000$ 

# **Complexity upper bound for membership to a linear complex**

An **arrangement** *S* ⊂ *F n* is a union of hyperplanes ∪*iH<sup>i</sup>* . A face of *S* is an intersection of some of the hyperplanes. DISTINCTNESS and KNAPSACK are arrangements.

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One can generalize the theorem to arbitrary **linear complexes**, i. e.

<span id="page-34-0"></span>algebraic computation tree is non-uniform (de[pe](#page-33-0)[nd](#page-35-0)[s](#page-32-0)a[n](#page-0-0) *n*[\)](#page-0-0)[.](#page-118-0)...  $000$ 

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An **arrangement** *S* ⊂ *F n* is a union of hyperplanes ∪*iH<sup>i</sup>* . A face of *S* is an intersection of some of the hyperplanes. DISTINCTNESS and KNAPSACK are arrangements.

### **Theorem**

*Let S be either an arrangement or a (possibly unbounded) polyhedron (when F* = R*). Denote by N the number of faces (of all the dimensions) of S. Then*  $C(S) \leq O(n^3 \cdot \log N)$  *(Meyer auf der Heide [1985]).* 

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<span id="page-38-0"></span>

# **Membership to a polyhedron**

Let  $P \subset \mathbb{R}^n$  be a convex polyhedron (with *N* faces of all the dimensions). Then the sum of its Betti numbers equals 1.

*Complexity of linear decision trees*  $C_1(P)$  *> log<sub>2</sub>N (Rivest-Yao [1980])* 

 $C_d(P) \ge \Omega(\log N)$ , provided that  $N \ge (dn)^{\Omega(n)}$  (G.-Karpinski-Vorobjov

<span id="page-39-0"></span> $\Omega$ 

 $\sqrt{m}$   $\rightarrow$   $\sqrt{m}$   $\rightarrow$   $\sqrt{m}$ 

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### As a testing function *g<sup>v</sup>* at a node *v* of a Pfaffian *d*-decision tree appears a Pfaffian function.

Let  $v_0, \ldots, v_k$  be the nodes on the path from the root  $v_0$  to  $v_k$ .  $dg_{\mathsf{v}_k} = \sum_{1\leq j\leq n} h_{kj}(g_{\mathsf{v}_0},\ldots,g_{\mathsf{v}_k},X_1,\ldots,X_n)\cdot dX_j$  $\mathsf{for} \ \mathsf{some} \ \mathsf{polynomials} \ h_{kj} \in \mathbb{R}[Z_0, \ldots, Z_k, X_1, \ldots, X_n], \ \deg(h_{kj}) \leq d.$ 

*For a polyhedron P with N faces of all the dimensions*  $\frac{p^{\mathsf{faff}}}{d}(P) \geq \Omega(\sqrt{\log{N}})$ , provided that  $\mathsf{N} \geq (\mathsf{d} n)^{\Omega(n^4 \cdot \log{d})}$ *(G.-Vorobjov [1994]).*

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**Examples of Pfaffian functions:** polynomials, exp, log (on the positive half-line),  $\sin$  (on the interval  $(-\pi,\pi),\, \sqrt{X}$  (on the positive half-line),  $X^{-1}$  on  $\mathbb{R}\setminus\{0\}.$ 

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### **Theorem**

*Let S* ⊂ R *<sup>n</sup> be semi-pfaffian (defined by inequalities of the form*  $g > 0$  for Pfaffian functions g. Denote by  $c_3$  the sum of its Betti  $g \geq 0$  for *F* lamal randicities g. Before by  $\sigma_3$  and sam of the numbers. Then  $C_d^{\text{pfaff}} \geq \Omega(\sqrt{\log c_3})$  (G.-Vorobjov [1994]);

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 $f(Z) = \prod_{1 \leq i \leq n} (Z-x_i), \ h(Z) = \prod_{1 \leq i \leq n} (Z-y_i).$  Sets  $\{x_1, \ldots, x_n\} \equiv \{y_1, \ldots, y_n\} \Leftrightarrow f \equiv h;$ 

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One can also consider a continuous distribution of trees.

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### **Speed-up by probabilistic computation trees**

- $C^{prob}(EQUALITY) \asymp n$  (Bürgisser-Karpinski-Lickteig [1992]);
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Let  ${T_i}_{1 \leq i \leq s}$  be a collection of either algebraic *d*-decision or, respectively computation trees with attributed probabilities  $p_1, \ldots, p_s, \, \sum_{1 \leq i \leq s} p_i = 1.$  It is called **probabilistic algebraic** *d***-decision** or, respectively **probabilistic computation tree**. A probabilistic tree accepts set  $\mathcal{S} \subset \mathcal{F}^n$  if for any input  $x \in \mathcal{F}^n$  the output is correct with probability  $> 2/3$ .

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- <span id="page-54-0"></span>• Choose randomly  $z_0 \in F$  and test whether  $f(z_0) = h(z_0)$ . If yes, return "*accept*", else "*reject*".

The topological methods and differential-geometric methods fail for probabilistic trees (another evidence is the latter example).

Let *S* ⊂ *F <sup>n</sup>* be a polyhedron (when *F* = R) or an arrangement determined by *m* hyperplanes having  $N \geq m^{\Omega(n)}$  faces.

*2)*  $C^{prob}(S) \ge \Omega(\log N)$  *(G. [1997]).* 

*2) Cprob*(*DISTINCTNESS*) *n* · log*n, Cprob*(*KNAPSACK*) ≥ Ω(*n* 2 ) *(G.).*

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 $P(D|C) \subset P^{\text{rob}}(D \cup \text{ISTINCTIONS}) \asymp n \cdot \log n, \ C^{\text{prob}}(K \cap \text{IFSACK}) \ge \Omega(n^2)$  (G.).

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EQUALITY is a union of *n*-dimensional planes in *F* 2*n* (not an arrangement), thus the Theorem is not applic[abl](#page-58-0)e,  $\overline{a}$ ,  $\overline{a}$ ,  $\overline{a}$ ,  $\overline{a}$ ,  $\overline{a}$ ,  $\overline{a}$ 

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*1)*  $C_d^{prob}(S) \ge \Omega(\log N)$  *for constant d (G.-Karpinski-Meyer auf der Heide-Smolensky [1995]); 2)*  $C^{prob}(S)$  ≥ Ω(logN) *(G. [1997]).* 

### **Corollary**

*1*)  $C_d^{prob}(DISTINCTNESS) \times n \cdot log n, C_d^{prob}(KNAPSACK) \ge \Omega(n^2)$ *(G.-Karpinski-Meyer auf der Heide-Smolensky [1995]);*  $2)$   $C^{prob}($   $DISTIMESS) \asymp n \cdot \log n$ ,  $C^{prob}($   $KNAPSACK) \geq \Omega(n^2)$   $(G_{\cdot}).$ 

<span id="page-60-0"></span>EQUALITY is a union of *n*-dimensional planes in *F* 2*n* (not an arrangement), thus the Theorem is not applic[abl](#page-59-0)e,  $\overline{a}$ ,  $\overline{a}$ ,  $\overline{a}$ ,  $\overline{a}$ ,  $\overline{a}$ ,  $\overline{a}$ 

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**Dima Grigoriev (CNRS) [Complexity lower bounds: algebraic trees](#page-0-0) 14.6.11 12 / 23**

<span id="page-61-0"></span>

# **Probabilistic computation trees over algebraically closed fields of zero characteristic**

### Let  $S = H_1 \cup \cdots \cup H_m \subset F^n$  be an arrangement.

*containing*  $|cm|$  *hyperplanes, has*  $\geq$  *N faces (of all the dimensions). Then*  $C^{prob}(S) \ge \Omega(\log N - 2 \cdot n)$  *(G. [1997]).* 

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*Assume that for a certain c* > 0 *any subarrangement*  $H_{i_1} \cup \cdots \cup H_{i_{1},m}$ *containing*  $|cm|$  *hyperplanes, has*  $\geq$  *N faces (of all the dimensions). Then*  $C^{prob}(S) \ge \Omega(\log N - 2 \cdot n)$  *(G. [1997]).* 

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- $C^{prob}(DISTINCTNESS) \asymp n \cdot log n;$
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 $\left\{ \begin{array}{ccc} \square & \rightarrow & \left\{ \bigcap \emptyset \right\} & \rightarrow & \left\{ \begin{array}{ccc} \square & \rightarrow & \left\{ \bigcap \emptyset \right\} & \rightarrow & \square \end{array} \right. \end{array} \right.$ 

If a decision tree admits arbitrary polynomials as testing functions we call it **topological tree** and its complexity denote by  $C_{top} \leq C$ ,  $C_d$ (Shub-Smale).

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- ORTANT  $= \mathbb{R}^n_+ = \{(x_1, \ldots, x_n) : x_1 \geq 0, \ldots, x_n \geq 0\} \Leftrightarrow$
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### **MAX**

- ORTANT =  $\mathbb{R}^n_+$  = {( $x_1, ..., x_n$ ) :  $x_1 ≥ 0, ..., x_n ≥ 0$ } ⇔
- $\bullet$  MAX= = { $(x_1, ..., x_n) : x_1 = \max\{x_1, ..., x_n\}$ } is a weaker problem than
- MAX: to compute  $\max\{x_1, \ldots, x_n\}$  by means of a modification of a decision (or computation) tree in which the outputs of leaves are functions (polynomials) rather than labels "*accept*" or "*reject*".

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### $C_1(MAX) \leq n-1$

- *Can*(*MAX* =) = *n* − 1 *(Rabin [1972];*
- *C prob*  $d_d^{\text{proo}}(MAX =) \ge n/(2 \cdot d), \quad C^{\text{prob}}(MAX =) \ge n/4 \text{ (G.-K.-S.)}$ *[1995]);*
- $C_{top}^{prob}(MAX =) \le C_n^{prob}(MAX =) \le O(log^2 n)(G.-K.-S.$  [1995]);
- $C_{top}^{prob}(MAX) \leq C_{n}^{prob}(MAX) \leq O(\log^{2} n)$  *(Ben-Or [1996];*  $\leq O(\log^5 n)$  *G.-K.-S.* [1995]).

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 $\mathcal{A} \cap \mathcal{B} \rightarrow \mathcal{A} \subset \mathcal{B} \rightarrow \mathcal{A} \subset \mathcal{B} \rightarrow \mathcal{B}$ 

## **Topological complexity of the range searching**

For  $f_1, \ldots, f_m \in \mathbb{R}[X_1, \ldots, X_n]$  we say that a decision/computation tree *T* solves the RANGE SEARCHING problem if any two inputs  $x, y \in \mathbb{R}^n$ with different sign vectors  $(\text{sgn}(f_1), \ldots, \text{sgn}(f_n))(x) \neq (\text{sgn}(f_1), \ldots, \text{sgn}(f_n))(y)$ arrive to different leaves of *T*. Denote by *N* the number of sign vectors.

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#### **Theorem**

 $C_{top}$  (RANGE SEARCHING)  $\asymp$  logN (G. [1998]).

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## **Lower bound for probabilistic analytic trees**

We have seen that for MAX tossing a coin can (exponentially) speed-up computation. Here is an example of a set for which it is not the case.

Let an integer  $q \neq 2^m$ . Consider set  $\text{MOD}_q = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \prod_{1 \leq i \leq n} x_i \neq 0, q \mid \sharp \{ i : x_i < 0 \} \}.$ 

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#### **Theorem**

 $C_{an}^{prob}(MOD_q) \ge \Omega(\sqrt{n})$  *(G.-Karpinski-Smolensly [1995]).* 

<span id="page-78-0"></span> $\Omega$ 

So far, we studied the depth of decision/computation trees. Since *depth*  $\geq$   $log(size)$ , the bounds on the size are stronger.  $\mathsf{EXACT}_n = \{ (x_1, \ldots, x_{2 \cdot n}) \in \mathbb{R}^{2 \cdot n} : \prod_{1 \leq i \leq 2 \cdot n} x_i \neq 0, \, \sharp \{ i : x_i < 0 \} = n \}.$ 

#### **Open questi[o](#page-80-0)n**: size (MAX) for binary decision/go[m](#page-78-0)[p](#page-79-0)[ut](#page-86-0)[at](#page-0-0)[io](#page-118-0)[n t](#page-0-0)[re](#page-118-0)[es](#page-0-0)[?](#page-118-0)  $\Omega$

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#### **Open questi[o](#page-81-0)n**: size (MAX) for binary decisio[n/c](#page-79-0)o[m](#page-78-0)[p](#page-79-0)[ut](#page-86-0)[at](#page-0-0)[io](#page-118-0)[n t](#page-0-0)[re](#page-118-0)[es](#page-0-0)[?](#page-118-0)  $000$

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#### **Theorem**

*For analytic decision trees size<sub>an</sub>* $(EXACT_n) \geq 2^n/n$  *(G.-K.-S. [1995]).* 

Alternatively to usual (binary) trees one can consider **ternary** decision

#### **Open questi[o](#page-82-0)n**: size (MAX) for binary decisio[n/c](#page-80-0)o[m](#page-78-0)[p](#page-79-0)[ut](#page-86-0)[at](#page-0-0)[io](#page-118-0)[n t](#page-0-0)[re](#page-118-0)[es](#page-0-0)[?](#page-118-0)  $000$

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**Ternary**  $size_{d}(MAX =) \geq 2^{c_{d} \cdot n}$  for  $d$ -decision trees,  $c_{d} > 0$ 

#### **Open questi[o](#page-83-0)n**: size (MAX) for binary decisio[n/c](#page-81-0)o[m](#page-78-0)[p](#page-79-0)[ut](#page-86-0)[at](#page-0-0)[io](#page-118-0)[n t](#page-0-0)[re](#page-118-0)[es](#page-0-0)[?](#page-118-0)  $000$

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### **Theorem**

**Ternary** size<sub>d</sub>(MAX =)  $\geq$  2 $^{c_d \cdot n}$  for d-decision trees,  $c_d > 0$ *(G.-Karpinski-Yao [1994])*

**Open questi[o](#page-85-0)n**: size (MAX) for binary decisio[n/c](#page-83-0)o[m](#page-78-0)[p](#page-79-0)[ut](#page-86-0)[at](#page-0-0)[io](#page-118-0)[n t](#page-0-0)[re](#page-118-0)[es](#page-0-0)[?](#page-118-0)  $000$ 

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### **Communication tree**

Let  $F = \mathbb{R}$  (also  $F = \mathbb{C}$  can be studied). Let input variables be divided into 2 groups:  $X_1, \ldots, X_{n_1}$  and  $Y_1, \ldots, Y_{n_2}$ . An input  $(x, y) \in \mathbb{R}^{n_1+n_2}$ .

To each node *v* with an even (respectively, odd) depth a **calculating polynomial**  $a_v \in \mathbb{R}[X_1, \ldots, X_{n_1}]$  (respectively,  $b_v \in \mathbb{R}[Y_1, \ldots, Y_{n_2}])$  is

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Also a family of **testing polynomials** {*gvi*}1≤*i*≤*N<sup>v</sup>* is assigned. There are 2 players; the first one has access to *x* and the second to *y*. Let  $v_0, \ldots, v_k = v$  be the nodes on the path from the root  $v_0$  to  $v_k = v$ and *k* be odd (for definiteness).

<span id="page-89-0"></span>**Communication complexity** *CC* is the depth of the communication

### **Communication tree**

Let  $F = \mathbb{R}$  (also  $F = \mathbb{C}$  can be studied). Let input variables be divided into 2 groups:  $X_1, \ldots, X_{n_1}$  and  $Y_1, \ldots, Y_{n_2}$ . An input  $(x, y) \in \mathbb{R}^{n_1+n_2}$ . To each node *v* with an even (respectively, odd) depth a **calculating polynomial**  $a_v \in \mathbb{R}[X_1, \ldots, X_{n_1}]$  (respectively,  $b_v \in \mathbb{R}[Y_1, \ldots, Y_{n_2}]$ ) is attached.

Also a family of **testing polynomials** {*gvi*}1≤*i*≤*N<sup>v</sup>* is assigned. There are 2 players; the first one has access to *x* and the second to *y*. Let  $v_0, \ldots, v_k = v$  be the nodes on the path from the root  $v_0$  to  $v_k = v$ and *k* be odd (for definiteness).

Then at *v* the second player computes  $b_v(y)$ , transmits it to the first player and branches according to the vector

<span id="page-90-0"></span> $\{ \text{sgn}(g_{\nu i}(a_{\nu_0}(x), b_{\nu_1}(y), a_{\nu_2}(x), \dots, a_{\nu_{k-1}}(x), b_{\nu_k}(y) \}_{1 \leq i \leq N_{\nu_k}}\}$ 

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<span id="page-92-0"></span>

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\mathbb{R}_{>}^{n_1+n_2} = \{(x,y) \in \mathbb{R}^{n_1+n_2} : \forall i,j \, x_i > 0, \, y_j > 0\} \subset \mathbb{R}_{+}^{n_1+n_2} = \overline{\mathbb{R}_{>}^{n_1+n_2}}.
$$

- 
- $CC^{prob}(\mathbb{R}_{>}^{n_1+n_2}) \leq 4$ ,  $CC^{prob}(\mathbb{R}_{+}^{n_1+n_2}) \leq log^{O(1)}(n_1+n_2)$ .

Arrangement  $Q = \bigcup_{1 \leq i \leq n} \{X_i + Y_i = 0\} \subset \mathbb{R}^{2 \cdot n}$ .

# branching condition {sgn(*gvi*(*av*<sup>0</sup> (*x*), *av*<sup>2</sup> (*x*), . . . , *[a](#page-92-0)vk*[−](#page-94-0)[1](#page-92-0) [\(](#page-93-0)*[x](#page-99-0)*[\)](#page-100-0)[,](#page-0-0) *[y](#page-0-0)*[\)\)](#page-118-0)[}](#page-0-0)[1](#page-0-0)[≤](#page-118-0)*i*[≤](#page-118-0)*[N](#page-0-0)[v](#page-118-0)*

<span id="page-93-0"></span>

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\mathbb{R}_{>}^{n_1+n_2} = \{(x,y) \in \mathbb{R}^{n_1+n_2} : \forall i,j \, x_i > 0, \, y_j > 0\} \subset \mathbb{R}_{+}^{n_1+n_2} = \overline{\mathbb{R}_{>}^{n_1+n_2}}.
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#### **Proposition**

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CC(\mathbb{R}_{>}^{n_1+n_2}), CC(\mathbb{R}_{+}^{n_1+n_2}) = n_1 + n_2;
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Polyhedron  $S = \{(x, y) \in \mathbb{R}^{2 \cdot n} : \forall i \, x_i + y_i > 0\}.$ Arrangement  $Q = \bigcup_{1 \leq i \leq n} \{X_i + Y_i = 0\} \subset \mathbb{R}^{2 \cdot n}$ .

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<span id="page-94-0"></span>

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### **Lower bound on probabilistic communication complexity**

 $\text{Polyhedron } S = \{ (x, y) \in \mathbb{R}^{2 \cdot n} : \forall i \, x_i + y_i > 0 \}.$ Arrangement  $Q = \bigcup_{1 \leq i \leq n} \{X_i + Y_i = 0\} \subset \mathbb{R}^{2 \cdot n}$ .

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**Problem**; to obtain a lower bound for communication trees with branching condition {sgn(*gvi*(*av*<sup>0</sup> (*x*), *av*<sup>2</sup> (*x*), . . . , *[a](#page-96-0)vk*[−](#page-98-0)[1](#page-92-0) [\(](#page-93-0)*[x](#page-99-0)*[\)](#page-100-0)[,](#page-0-0) *[y](#page-0-0)*[\)\)](#page-118-0)[}](#page-0-0)[1](#page-0-0)[≤](#page-118-0)*i*[≤](#page-118-0)*[N](#page-0-0)[v](#page-118-0)*

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#### **Theorem**

 $CC^{prob}(S)$ ,  $CC^{prob}(Q)$ ,  $CC^{prob}(EQUALITY)$ ,  $CC^{prob}(KNAPSACK) > n$ *(G. [2006]).*

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**Dima Grigoriev (CNRS) [Complexity lower bounds: algebraic trees](#page-0-0) 14.6.11 20 / 23**

<span id="page-99-0"></span>

## To the root of a **network** input  $x \in \mathbb{R}^n$  is attributed.

To each node *v* of depth *k* lead edges from (at most) two nodes  $v_1$ ,  $v_2$ of depth  $k - 1$ .

To each node *v* a **computing polynomial**  $f_v \in \mathbb{R}[X_1, \ldots, X_n]$  and a **boolean indicator** *b<sup>v</sup>* are attached.

<span id="page-100-0"></span>

## Th[e](#page-99-0) **parallel complexi[t](#page-99-0)y** *PC* is the depth of the [ne](#page-101-0)t[w](#page-100-0)[o](#page-107-0)[r](#page-108-0)[k.](#page-0-0)

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Informally,  $b_v = 1$  ("*active*") means that one of the processors is located in *v*. We impose the condition that the number of "*active*" *v* of depth  $k$  is bounded by  $p \leq 2^k.$ 

<span id="page-101-0"></span>

## Th[e](#page-100-0) **parallel complexi[t](#page-99-0)y** *PC* is the depth of the [ne](#page-102-0)t[w](#page-100-0)[o](#page-107-0)[r](#page-108-0)[k.](#page-0-0)

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*f*<sub>*v*</sub> = *A* ◦ *B* where ◦ ∈ {+, ×} and *A*, *B* ∈ *F* ∪ {*X*<sub>1</sub>, . . . , *X*<sub>n</sub>} ∪ {*f*<sub>*v*<sub>1</sub></sub>, *f*<sub>*v*<sub>2</sub></sub>}

- 
- <span id="page-102-0"></span>

## Th[e](#page-101-0) **parallel complexi[t](#page-99-0)y** *PC* is the depth of the [ne](#page-103-0)t[w](#page-100-0)[o](#page-107-0)[r](#page-108-0)[k.](#page-0-0)

To the root of a **network** input  $x \in \mathbb{R}^n$  is attributed.

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To *v* two boolean functions  $B_{v1}$ ,  $B_{v2}$  are also attached.

**•** if both  $v_1$ ,  $v_2$  are "*passive*" then *v* is "*passive*" as well;

- if  $v_1$ ,  $v_2$  are "active" then  $b_v = B_{v2}(\text{sgn}(f_{v_1}(x)), \text{sgn}(f_{v_2}(x)))$ ;
- <span id="page-103-0"></span>if only  $v_i$ ,  $i = 1, 2$  is "*active*" then  $b_v = B_{v1}(\text{sgn}(f_{v_i}(x)))$ .

## Th[e](#page-102-0) **parallel complexi[t](#page-99-0)y** *PC* is the depth of the [ne](#page-104-0)t[w](#page-100-0)[o](#page-107-0)[r](#page-108-0)[k.](#page-0-0)

To the root of a **network** input  $x \in \mathbb{R}^n$  is attributed.

To each node *v* of depth *k* lead edges from (at most) two nodes  $v_1$ ,  $v_2$ of depth  $k - 1$ .

To each node *v* a **computing polynomial**  $f_v \in \mathbb{R}[X_1, \ldots, X_n]$  and a **boolean indicator** *b<sup>v</sup>* are attached.

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 $f_v = A \circ B$  where  $\circ \in \{+, \times\}$  and  $A, B \in F \cup \{X_1, \ldots, X_n\} \cup \{f_{v_1}, f_{v_2}\}$ 

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## Th[e](#page-103-0) **parallel complexi[t](#page-99-0)y** *PC* is the depth of the [ne](#page-105-0)t[w](#page-100-0)[o](#page-107-0)[r](#page-108-0)[k.](#page-0-0)

To the root of a **network** input  $x \in \mathbb{R}^n$  is attributed.

To each node *v* of depth *k* lead edges from (at most) two nodes  $v_1$ ,  $v_2$ of depth  $k - 1$ .

To each node *v* a **computing polynomial**  $f_v \in \mathbb{R}[X_1, \ldots, X_n]$  and a **boolean indicator** *b<sup>v</sup>* are attached.

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Th[e](#page-104-0) **parallel complexi[t](#page-99-0)y** *PC* is the depth of the [ne](#page-106-0)t[w](#page-100-0)[o](#page-107-0)[r](#page-108-0)[k.](#page-0-0)

To the root of a **network** input  $x \in \mathbb{R}^n$  is attributed.

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- <span id="page-106-0"></span>if only  $v_i$ ,  $i = 1, 2$  is "*active*" then  $b_v = B_{v1}(\text{sgn}(f_{v_i}(x)))$ .

Exactly one node *w* with the largest depth is "*active*", sgn( $f_w(x)$ ) is treated as an output of the parallel network.

## Th[e](#page-105-0) **parallel complexi[t](#page-99-0)y** *PC* is the depth of the [ne](#page-107-0)t[w](#page-100-0)[o](#page-107-0)[r](#page-108-0)[k.](#page-0-0)

To the root of a **network** input  $x \in \mathbb{R}^n$  is attributed.

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To each node *v* a **computing polynomial**  $f_v \in \mathbb{R}[X_1, \ldots, X_n]$  and a **boolean indicator** *b<sup>v</sup>* are attached.

Informally,  $b_v = 1$  ("*active*") means that one of the processors is located in *v*. We impose the condition that the number of "*active*" *v* of depth  $k$  is bounded by  $p \leq 2^k.$ 

 $f_v = A \circ B$  where  $\circ \in \{+, \times\}$  and  $A, B \in F \cup \{X_1, \ldots, X_n\} \cup \{f_{v_1}, f_{v_2}\}$ 

To *v* two boolean functions  $B_{v1}$ ,  $B_{v2}$  are also attached.

**•** if both  $v_1$ ,  $v_2$  are "*passive*" then *v* is "*passive*" as well;

- if  $v_1$ ,  $v_2$  are "active" then  $b_v = B_{v2}(\text{sgn}(f_{v_1}(x)), \text{sgn}(f_{v_2}(x)))$ ;
- if only  $v_i$ ,  $i = 1, 2$  is "*active*" then  $b_v = B_{v1}(\text{sgn}(f_{v_i}(x)))$ .

Exactly one node *w* with the largest depth is "*active*", sgn( $f_w(x)$ ) is treated as an output of the parallel network.

The **parallel complexity** *PC* is the depth of th[e](#page-106-0) [ne](#page-108-0)[t](#page-99-0)[w](#page-100-0)[o](#page-107-0)[r](#page-108-0)[k.](#page-0-0)

<span id="page-107-0"></span>
$\mathsf{For\ set\ } \mathcal{S} \subset \mathbb{R}^n$  parallel complexity  $\mathit{PC}(S) \geq \Omega(\sqrt{(\mathrm{log} N) / n})$  where  $N$  is

- either the sum of Betti numbers of *S* (Mulmuley [1994], Montaña-Morais-Pardo [1996].)
- or the number of faces of *S* of all the dimensions when *S* is a polyhedron, provided that  $N\geq n^{\Omega(n)}$  (G. [1996])

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### **Corollary**

*Let S* ⊂ R <sup>2</sup> *be an m-gon. Then*

 $\bullet$   $C(S) \times \log m$  (Steele-Yao [1982], Meyer auf der Heide [1985]);

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**Open problem**: obtain an upper bound on the parallel complexity for

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**Open problem**: obtain an upper bound on the parallel complexity for more general semi-algebraic sets (rather than for linear complexes).

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