Pseudo-random graphs and bit probe schemes with one-sided error

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The problem under consideration:

bit probe scheme with one-sided error

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Our technique:

pseudo-random graphs

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Remark: $s = \Omega(n \log m)$

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   - good news: database of size $O(n \log m)$ bits
   - good news: randomization only to constructe the database
   - bad news: need to read $O(\log m)$ bits to answer a query

Buhrman–Miltersen–Radhakrishnan–Venkatesh [2001]

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Two features:

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- good news: read *one* bit to answer a query
- good news: memory $= O(n \log m)$
- bad news: exponential computations
- *some* news: two-sided errors
- bad news: need $\Omega(\frac{n^2 \log m}{\log n})$ for a one-sided error

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Do we cheat ?

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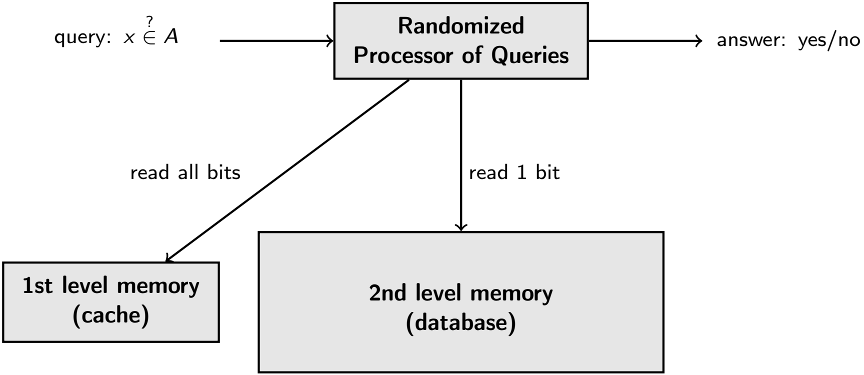
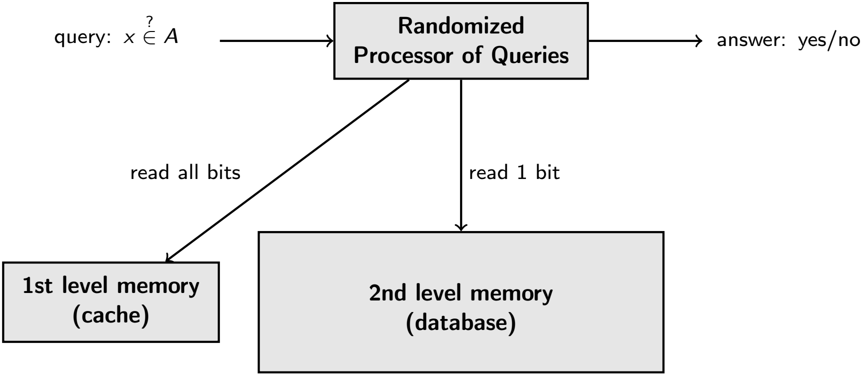
Do we cheat ? Yes, we have changed the model !
We allow *cached* memory of size $\mathrm{poly}(\log m)$.

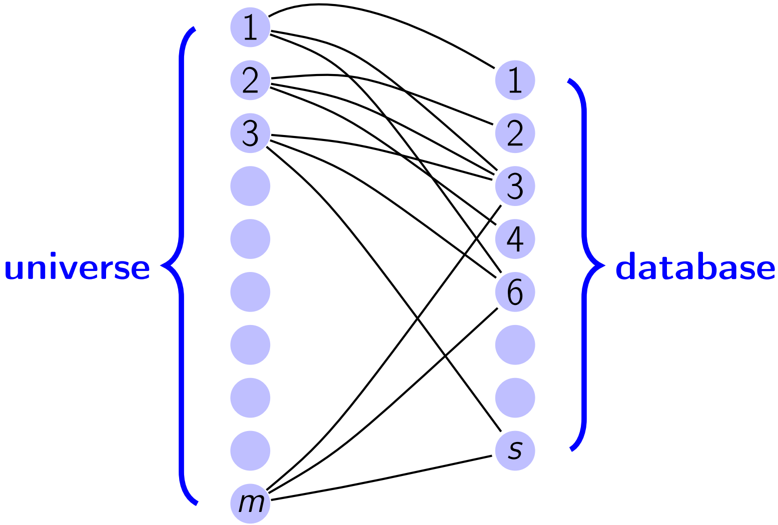

query: $x \stackrel{?}{\in} A$
Randomized
Processor of Queries
answer: yes/no
read all bits
read 1 bit
1st level memory
(cache)
2nd level memory
(database)



Theorem. For any n-element set A from an m-element universe there exists a randomized bit-probe scheme with one-sided error, with cache of size $O(\log^c m)$ and database of size $O(n \log^2 m)$.

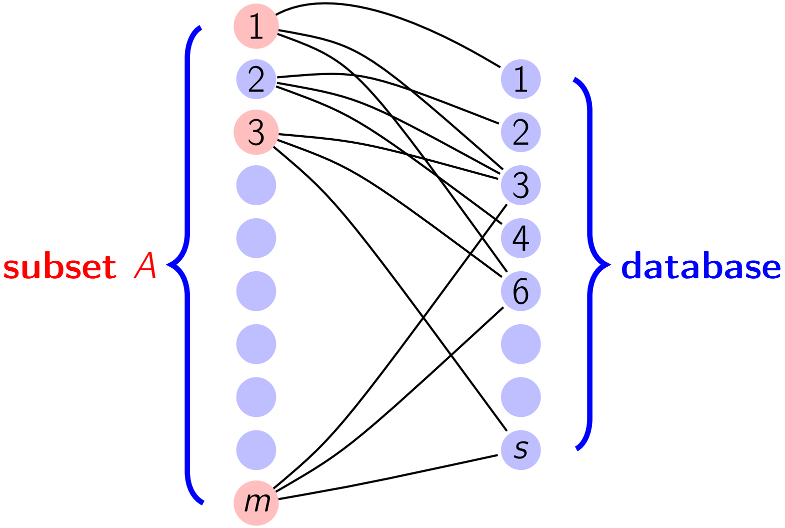
the left part: m vertices; degree $d = O(\log m)$

the right part: $s = O(n \log^2 m)$ vertices

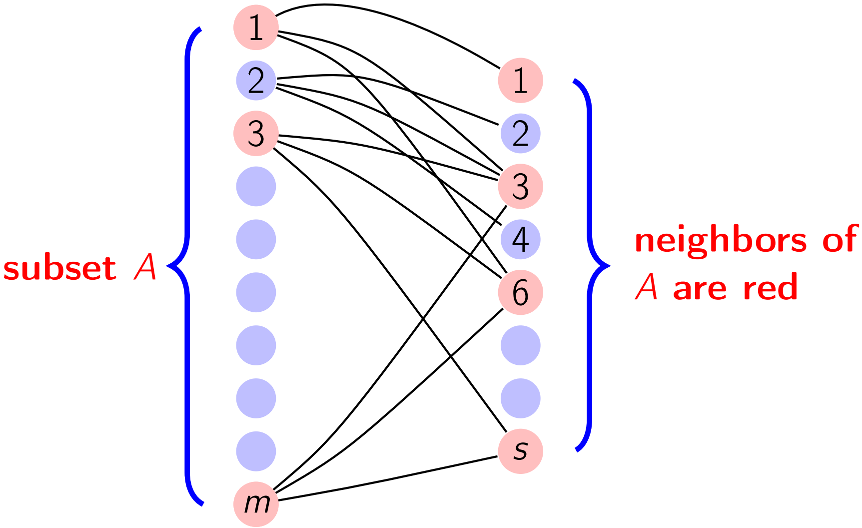


in the left part: set A of n vertices

the right part: $s = O(n \log^2 m)$ vertices



A graph is suitable for A if
for every $x \notin A$ **most** neighbors of x are blue



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We need a good PRG...

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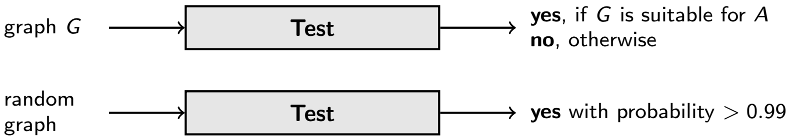
graph G
Test
yes, if G is suitable for A
no, otherwise
random graph
Test
yes with probability > 0.99

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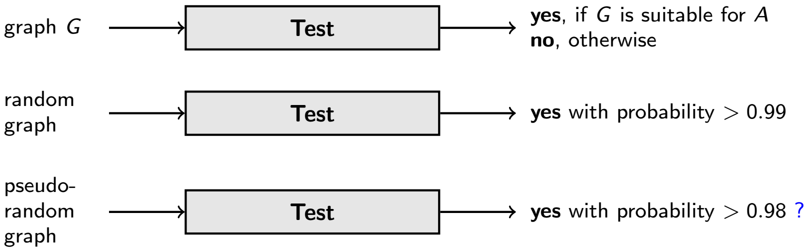
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pseudo-random graph
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yes with probability > 0.98 ?

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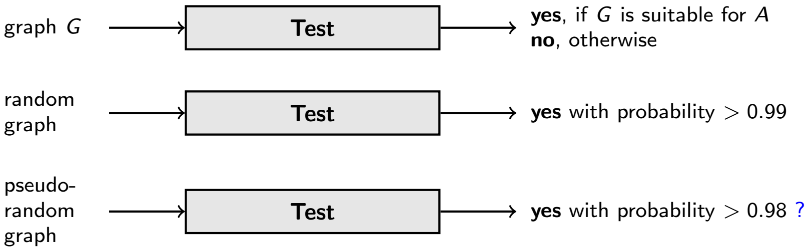

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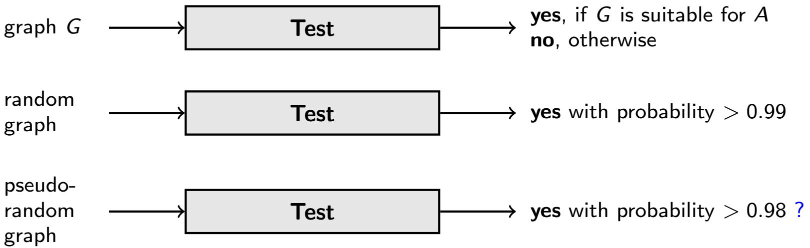

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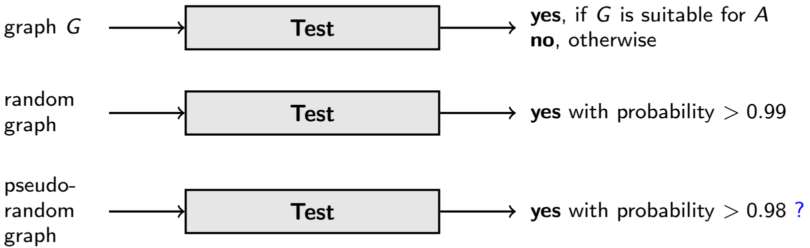

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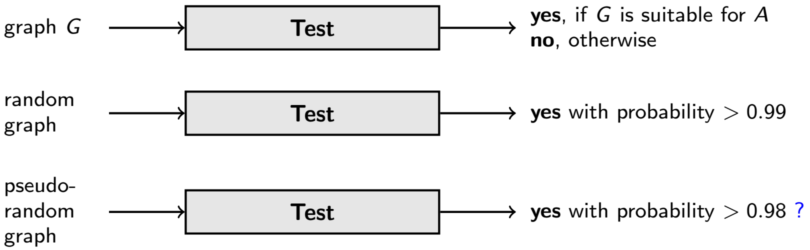

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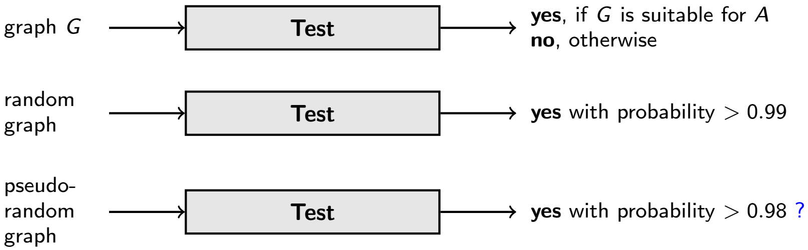

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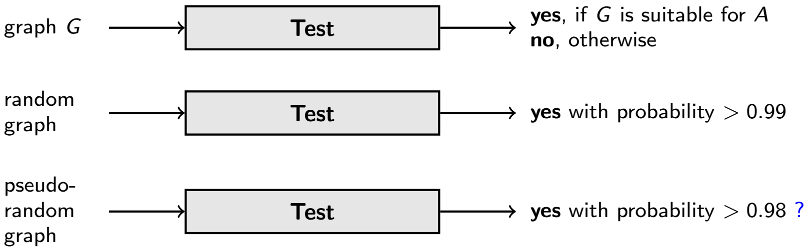

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Size of the seed $= \mathrm{poly}(\log m)$.

Conclusion: a **bit-probe scheme:**

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- one-sided error
- 1-st level "cached" memory $= \mathrm{poly}\log m$
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Beyond this talk: combine our construction with Guruswami–Umans–Vadhan

- read *two* bit to answer a query
- one-sided error
- 1-st level "cache" memory $= \mathrm{poly}\log m$
- 2-nd level memory $= n^{1+\delta}\mathrm{poly}\log m$
- computations in time $\mathrm{poly}(n, \log m)$

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Thank you! Questions?