Faster Polynomial Multiplication via Discrete Fourier Transforms

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Polynomial multiplication

Given

$$a(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad b(x) = b_0 + b_1 x + \cdots + b_n x^n,$$

Compute

$$c(x) = c_0 + c_1 x + \cdots + c_{2n} x^{2n} = a(x)b(x).$$

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For all $0 \leq i \leq 2n$, compute

$$c_i = \begin{cases} a_0 b_i + a_1 b_{i-1} + \cdots + a_i b_0, & 0 \leq i \leq n, \\ a_{i-n} b_n + a_{i-n+1} b_{n-1} + \cdots + a_n b_{i-n}, & n < i \leq 2n. \end{cases}$$

In what model?

- Arithmetic circuits with binary "$+$", "$-$", "$\cdot$"
- Each binary gate has unit cost
- No divisions
- Constants from the field available at no cost
- Inputs are the coefficients of the polynomials to be multiplied
- Outputs are the coefficients of the product polynomial
- Interested in a circuit for degree n polynomial multiplication of the minimal size

History and state of the art

School method: $O(n^2)$

Karatsuba 1960: $O(n^{\log_2 3}) = O(n^{1.585})$

Toom 1963: $n^{1+O(1/\sqrt{\log n})} = O(n^{1+\epsilon})$, for any fixed $\epsilon > 0$

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Schönhage 1977: $O(n \log n \log \log n)$, over field of char $= 2$

Kaminski 1988, Cantor-Kaltofen 1991: $O(n \log n \log \log n)$, over arbitrary **algebras**

Over $\mathbb{C}$ or $\mathbb{R}$: $O(n \log n)$

All general lower bounds: $\Omega(n)$.

Multiplication in $O(n \log n)$

Given

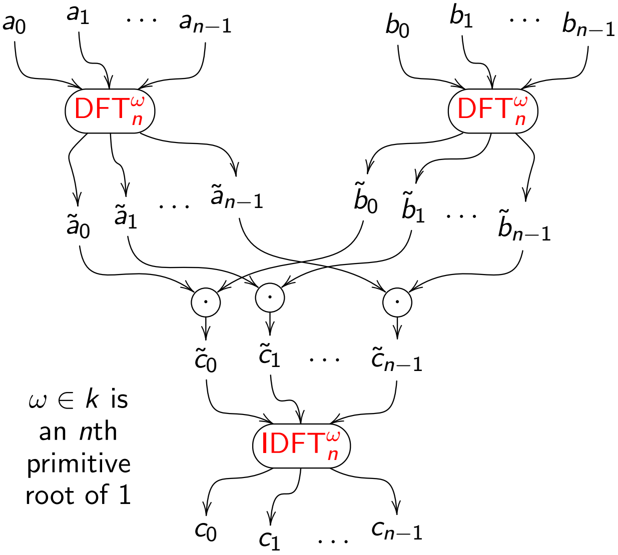
$$a(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1},$$
$$b(x) = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1},$$

Compute

$$c(x) = c_0 + c_1 x + \cdots + x_{n-1} x^{n-1} = a(x) b(x) \pmod{x^n - 1}.$$

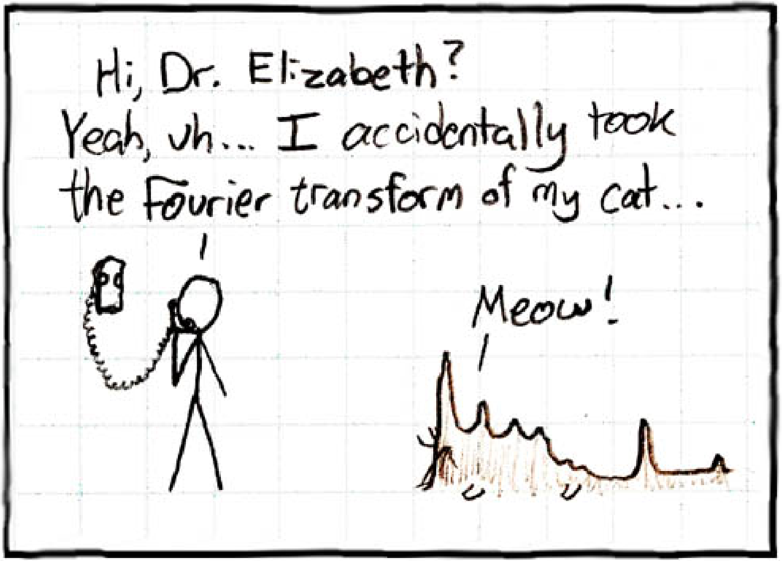
(Can always choose a larger n and pad polynomials with zeroes to reduce the ordinary polynomial multiplication to the product in $k[x]/(x^n - 1)$.)

Multiplication in $O(n \log n)$



$\omega \in k$ is an nth primitive root of 1

Discrete Fourier transform



Discrete Fourier transform

- Maps a degree $n-1$ polynomial to its values at n distinct nth roots of unity:

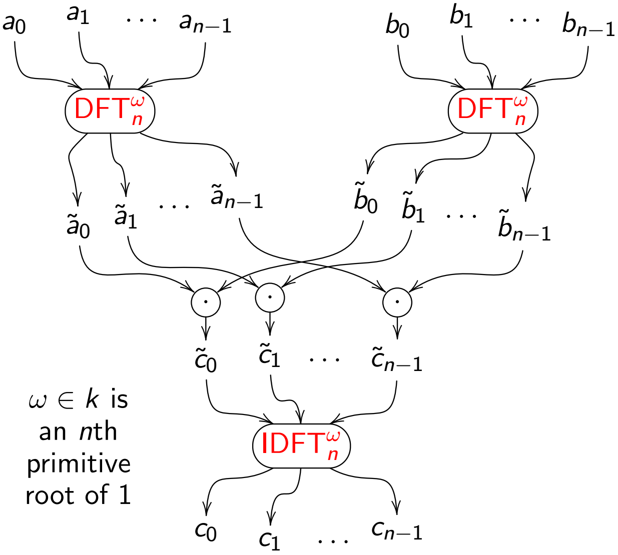
$$\tilde{a}_i := a(\omega^i) = \sum_{j=0}^{n-1} a_j \omega^{ij}, \qquad 0 \le i \le n-1$$

$$\mathrm{DFT}_n^{\omega} : (a_0, a_1, \ldots, a_{n-1}) \mapsto (\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_{n-1})$$

  (ω is a primitive nth root of unity)
- Linear transform: $\mathrm{DFT}_n^{\omega} : k[x] \to k^n$
- Isomorphism: $\mathrm{DFT}_n^{\omega} : k[x]/(x^n - 1) \to k^n$
- Can be *often* computed in $O(n \log n)$
- The inverse isomorphism is almost a DFT again:

$$\frac{1}{n} \mathrm{DFT}_n^{\omega^{n-1}} : (\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_{n-1}) \mapsto (a_0, a_1, \ldots, a_{n-1})$$

Discrete Fourier transform



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Discrete Fourier transform

- $L_k(n)$: the complexity of degree n polynomial multiplication over a field k
- $D_k(n)$: the complexity of computing length n DFT over k

$$L_k(n) \leq 3D_k(n) + 2n = O(n \log n).$$

Note: we need roots of unity.

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- More precisely: take a (ring) extension $\mathcal{A}_m$ of k of degree m over k with a 2ℓth root of unity $\omega \in \mathcal{A}_m$:
  - For example,
$$\mathcal{A}_m = k[x]/p_m(x),$$
  - $p_m(x) \in k[x]$ is a polynomial of degree m,
  - $p_m(x)$ vanishes on $\omega_{2\ell}$,
  - ($\omega_{2\ell}$ is a primitive 2ℓth root of unity in the algebraic closure of the field k.)

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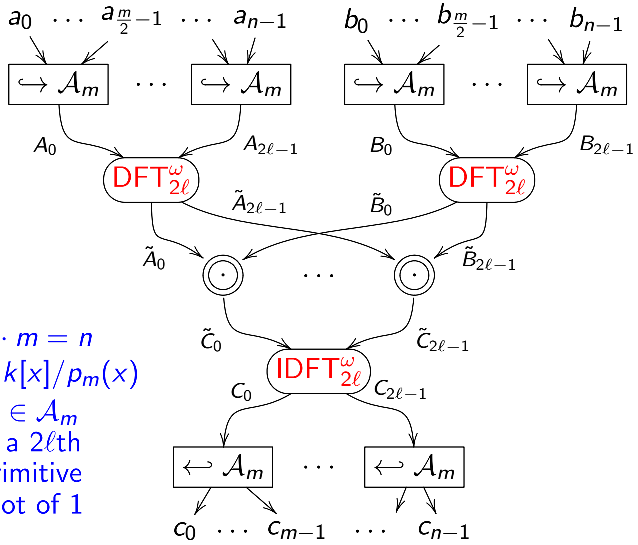
How can it help? See next slide.

Fast polynomial multiplication



$2\ell \cdot m = n$

$\mathcal{A}_m = k[x]/p_m(x)$

$\omega \in \mathcal{A}_m$
is a 2ℓth
primitive
root of 1

Fast polynomial multiplication

In this case

$$L_k(n) \leq 2\ell L_k(m) + 3D_{\mathcal{A}_m}(n) \cdot \text{complexity of aritrhmetics in } \mathcal{A}_m + \text{cost of embedding and unembedding in } \mathcal{A}_m$$

Our contribution #1:

- Formalize this kind of algorithms
- The relation between m and 2ℓ is a barrier for the algorithm's performance
- This relation depends heavily on the field properties
- The cost of the DFT can usually be made $O(n \log n)$
- Embedding and unembedding run usually in linear time, e.g., if $p_m(x)$ is sparse

Does it work?

Yes!!!

Schönhage-Strassen 1971: $\ell = m$

Schönhage 1977: $3\ell = 2m$ (+ a little trick)

Kaminski 1988: $\ell = \phi(m)$ (Euler's totient function)

Cantor-Kaltofen 1991: $\ell = m$ (and $\mathcal{A}_m$ is *a little* more complicated than $k[x]/p_m(x)$)

Slow fields

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Recall:

$$L_k(n) \leq 2\ell L_k(m) + 3D_{\mathcal{A}_m}(n) \cdot \text{complexity of arithmetics in } \mathcal{A}_m + \text{cost of embedding and unembedding in } \mathcal{A}_m$$

Ideally we want m to be small and ℓ to be large.

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Definition

For a field k, and n, s.t. $\text{char}\, k \nmid n$, let $f_k(n)$ be $[k(\omega_n) : k]$, the *degree function* of k.

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Definition

For a field k, and n, s.t. char $k \nmid n$, let $f_k(n)$ be $[k(\omega_n) : k]$, the *degree function* of k.

Our contribution #2:

- If $f_k(n) = o(\log \log n)$ for some *not too sparse set of n* then k is *fast* and $L_k(n) = o(n \log n \log \log n)$
- If $f_k(n) = \Omega(n^{1-\epsilon})$ for any fixed $\epsilon > 0$, then k is *slow* and any algorithm of that kind runs in $\Omega(n \log n \log \log n)$

More details

- To attach an ℓth root of unity we need an extension of degree at least $f_k(\ell)$
- The degree of the polynomial is then $\sim \ell \cdot f_k(\ell)$
- For the least solution i_0 of $i \cdot f_k(i) \geq n$, $f_k^{\surd}(n) := f_k(i_0)$
- The number of recursive steps is at least the number of $f_k^{\surd}(f_k^{\surd}(\cdots f_k^{\surd}(n)\cdots))$, until the value becomes $O(1)$
- This superposition depth will be denoted $(f_k^{\surd})^*(n)$
- The cost of all steps on a single recursion level is determined by the complexity of the DFTs, and is $\Theta(n \log n)$

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- This superposition depth will be denoted $(f_k^{\surd})^*(n)$
- The cost of all steps on a single recursion level is determined by the complexity of the DFTs, and is $\Theta(n \log n)$
- The total cost is estimated as

$$\Omega(n \log n) \cdot (f_k^{\surd})^*(n)$$

“Lower bound”

For the rational field $\mathbb{Q}$, for all n

$$f_{\mathbb{Q}}(n) = \phi(n) \geq c \cdot \frac{n}{\log \log n},$$

and

$$(f_{\mathbb{Q}}^{\sqrt{}})^{*}(n) = \Omega(\log \log n).$$

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Complexity of any DFT-based multiplication algorithm is then

$$\Omega(n \log n \log\log n).$$

It follows that over $\mathbb{Q}$ we need another kind of an algorithm.

Summary

- Uniform treatment of all known asymptotically fastest polynomial multiplication algorithms w.r.t. the total complexity
- A way to improve the total complexity upper bounds over certain fields
- Impossibility to improve Schönhage-Strassen over any fields (and rings or algebras) of characteristic 0
- In particular, no light at the end of the tunnel for polynomial multiplication over $\mathbb{Q}$

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- A way to improve the total complexity upper bounds over certain fields
- Impossibility to improve Schönhage-Strassen over any fields (and rings or algebras) of characteristic 0
- In particular, no light at the end of the tunnel for polynomial multiplication over $\mathbb{Q}$
- Over fields of positive characteristic,

Thank you for attention!