The complexity of inversion of explicit Goldreich's function by DPLL algorithms

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Goldreich's function

$f : \{0,1\}^n \to \{0,1\}^n$

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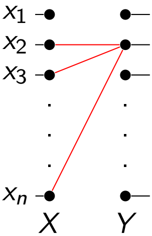

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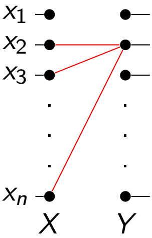

$P(x_{i1}, x_{i2}, \dots, x_{id})$

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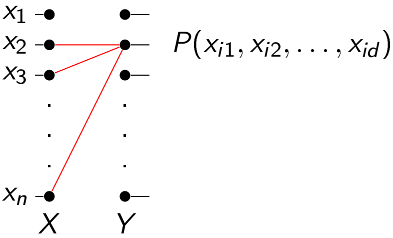

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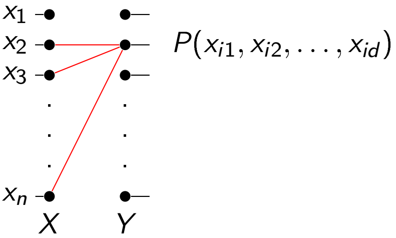

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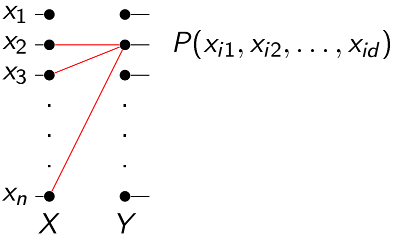

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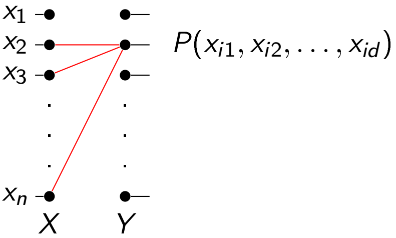

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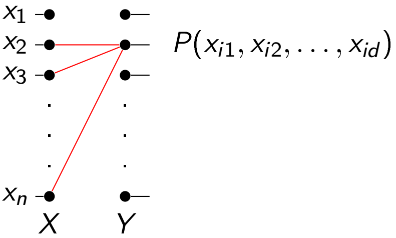

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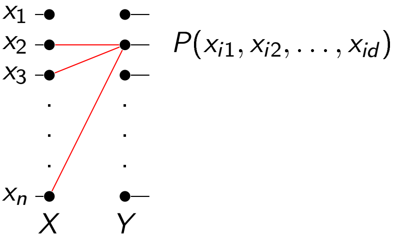

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- f is computed by constant depth circuit;
- [Applebaum, Ishai, Kushilevitz 2006] If one-way functions exist then there is a one-way function that can be computed by constant depth circuit.

DPLL algorithms

ϕ

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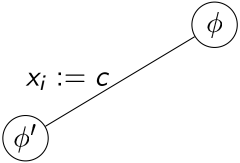
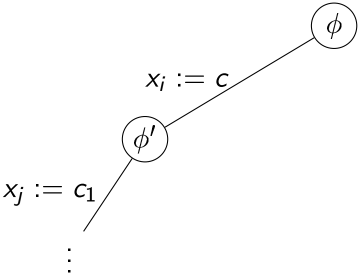
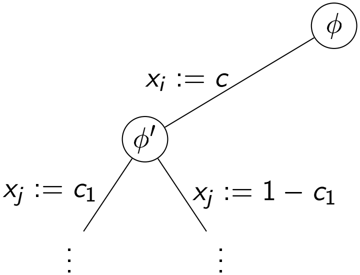
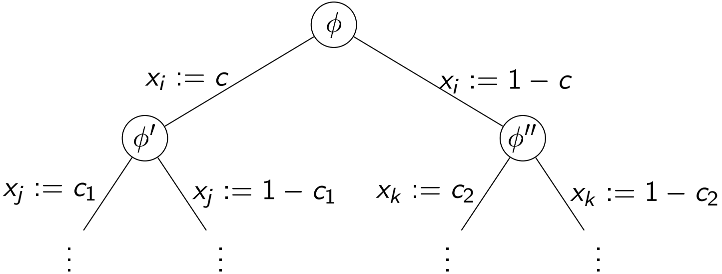



ϕ
$x_i := c$
ϕ'
$x_j := c_1$
$\vdots$

DPLL algorithms




ϕ
$x_i := c$
$x_i := 1 - c$
ϕ'
ϕ''
$x_j := c_1$
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$x_k := c_2$
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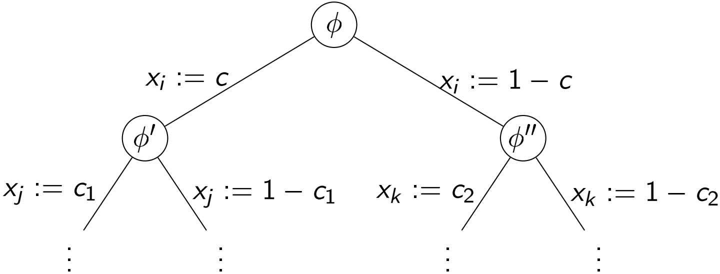

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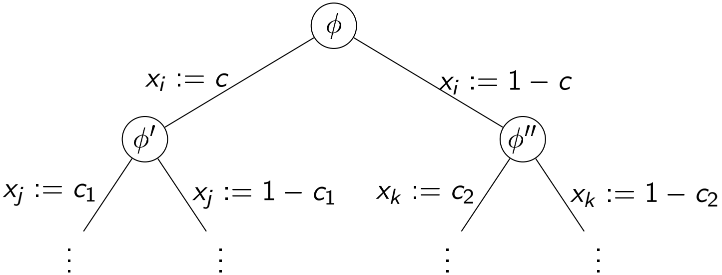

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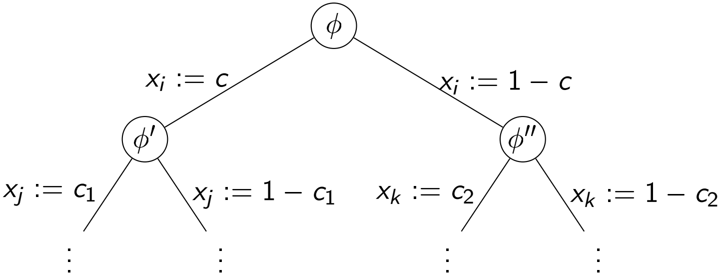

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- Simplification rules:
  - unit clause elimination;
  - pure literal rule.

Lower bounds for DPLL algorithms

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- Unsatisfiable formulas
  - Exponential lower bounds for resolution refutations of unsatisfiable formulas translate to backtracking algorithms.
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  - [Nikolenko 2002], [Achilioptas, Beame, Molloy 2003-2004] exponential lower bound for specific backtracking algorithms.
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  - Exponential lower bound for inversion of Goldreich's function by myopic [J. Cook et al. 2009] and drunken [Itsykson 2010] algorithms.

Drunken and myopic algorithms

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Drunken algorithms:

- **A**: any;
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There exists an explicit graph G such that every myopic or drunken DPLL algorithm makes at least $2^{n^{\Omega(1)}}$ steps on "$f(x) = f(a)$" for almost all $a \in \{0, 1\}^n$.

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- For myopic
  - we simplify previous proof and
  - $K = n^{1-\epsilon}$

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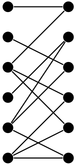

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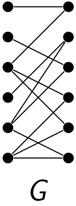

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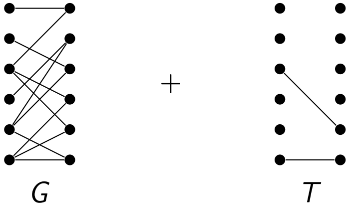

- G is an expander;
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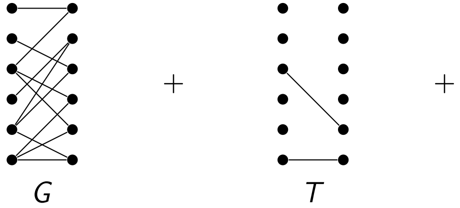
$$P(x_1, \dots, x_d) = {\color{red} x_1 \oplus x_2 \oplus \dots \oplus x_{d-k}} \oplus Q(x_{d-k+1}, \dots, x_d)$$



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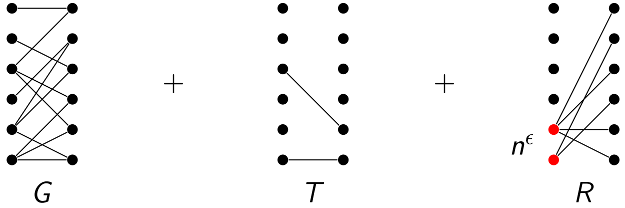

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- $G + T$ has full rank. $\forall y \in Y \subset T,\ deg(y) = 1$;
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- R contains nonlinear edges. $|\{x \mid x \in X,\ deg(x) \neq 0\}| \leq n^\epsilon$.
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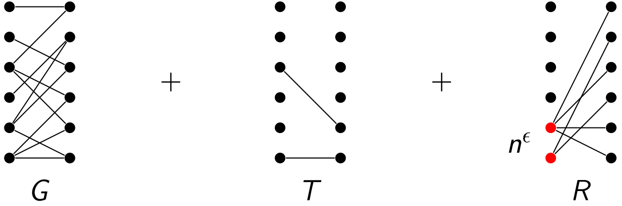

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  - $G + T + R$ is an expander.
  - Size of preimages no more than 2^{n^ϵ}.

We can invert $f_{G+T+R,P}$ in time $poly(n)2^{n^\epsilon}$, but this is still much!

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- With probability $1 - 2^{-\Omega(n)}$ after several steps current formula becomes unsatisfiable.

Plan of the proof

- Lower bounds for unsatisfiable formulas.
  - G is an expander.
  - P is almost linear.
  - Lower bounds for resolution proofs
- With probability $1 - 2^{-\Omega(n)}$ after several steps current formula becomes unsatisfiable.
  - G is an expander.
  - f is almost bijection.
  - Myopic algorithm can't recognize different absolute terms.