Maltsev digraphs

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Digraphs

A directed graph or digraph is a relational structure $G = (V, E)$ with domain V, the vertices, and relation $E \subseteq V \times V$, the directed edges or arcs.

A polymorphism f of G is an n-ary operation in V that is a homomorphism $f : G \times \cdots \times G \longrightarrow G$.

A maltsev polymorphism of G is a 3-ary polymorphism, m, that satisfies $m(x, y, y) = m(y, y, x) = x$ for all $x, y \in V$.

Constraint Satisfaction Problem

or Homomorphism Problem

Given two digraphs $G = (V(G), E(G)), H = (V(H), E(H))$ is there a homomorphism $h : G \longrightarrow H$?

h satisfies $(a, b) \in E(G) \Rightarrow (h(a), h(b)) \in E(H)$.

If a homomorphism exists we write $G \longrightarrow H$.

The CSP with a fixed H is denoted $CSP(H)$, and can be viewed as $\{G : G \longrightarrow H\}$.

CSP

Example

CSP(K_2) is the graph 2-colourability problem.

More generally CSP(H), with H a digraph is the H-colouring problem. This is a much studied problem in graph theory.

Problem: Classify $CSP(H)$ with respect to computational (and descriptive) complexity.

Polymorphisms and complexity

Good polymorphisms imply tractability.

Lack of polymorphisms NP-complete problems.

Certain polymorphism imply membership in a specific complexity class.

Example (Dalmau, Krokhin)

If H is preserved by a majority operation then $CSP(H)$ is in NL.

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Example (Dalmau, Larose)

If H is preserved by a maltsev operation + extra condition then $CSP(H)$ is in L.

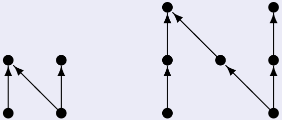
Polymorphisms are not easy to understand from a combinatorial point of view.

Aim: combinatorial description of digraphs with certain algebraic properties.

Characterization

Theorem

A digraph is preserved by a maltsev polymorphism iff it does not contain an N of any length



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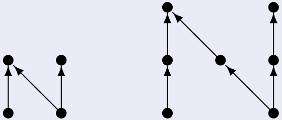

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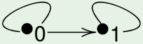
The graph 

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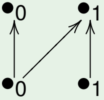


Recognition in polynomial time

The non existence of Ns of length 1 is equivalent in the adjacency matrix to the following property: when 2 rows, or columns, share a 1 then they are identical.

It folllows from that maltsev digraphs can be recognized in polynomial time on the number of vertices, $O(n^4)$.

CSP

Theorem

Let H be a maltsev digraph. If H is acyclic then it retracts onto a directed path. Otherwise it retracts onto the disjoint union of directed cycles.

It then follows that, for a maltsev digraph H, $CSP(H)$ is in **L**.

List Homomorphism problem

The list H-colouring problem, given an input graph G and for each vertex $v \in V(G)$ a list $L_v \subseteq V(H)$, asks if there is $f : G \longrightarrow H$ s.t. $f(v) \in L_v$.
It is equivalent to $CSP(H_u)$, where H_u is H together with all unary relations $\{S : S \subseteq V(H)\}$.

In this case we need conservative polymorphisms, i.e. $m(x, y, z) \in \{x, y, z\}$.

Characterization

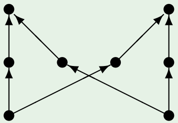
Theorem

A digraph admits a conservative maltsev polymorphism iff

$$\left.\begin{array}{llll} x \longrightarrow & \cdots \longrightarrow & x_{k-1} \longrightarrow & u \\ y \longrightarrow & \cdots \longrightarrow & y_{k-1} \longrightarrow & u \\ y \longrightarrow & \cdots \longrightarrow & z_{k-1} \longrightarrow & v \end{array}\right\} \Rightarrow \begin{cases} \exists\ x \ \rightarrow\ \cdots\ \rightarrow\ w_{k-1}\ \rightarrow\ v \text{ with} \\ w_i \in \{x_i, y_i, z_i\} \text{ for each } i. \end{cases}$$

Example

The digraph



has a maltsev polymorphism, but not a conservative maltsev polymorphism.

We can then deduce that, when H is a maltsev digraph, $LHOM(H)$ is in **L**.

Cycles and paths

Theorem

Let H be a (conservative) maltsev digraph. Then H is preserved by (conservative) polymorphisms $t_1, t_2, \dots, t_k$ (not necessarily distinct) satisfying a single equational sequence

$$t_1(x_{1,1}, x_{1,2}, \dots, x_{1,n_1}) \approx \cdots \approx t_k(x_{k,1}, x_{n,2}, \dots, x_{k,n_k}) \approx x,$$

where $\{x_{1,1}, \dots, x_{1,n_1}\} = \cdots = \{x_{k,1}, \dots, x_{k,n_k}\}$ and $x \in \{x_{1,1}, \dots, x_{1,n_1}\}$, iff the directed path/cycle is preserved by these polymorphisms.

Maltsev iff Pixley iff Minority implies Majority

$$m(x, y, y) = m(y, y, x) = m(x, y, x) = x$$
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2-semilattice polymorphisms

A binary polymorphism c is commutative if $c(x, y) = c(y, x)$.

A disjoint union of directed cycles admits a binary conservative commutative polymorphism iff it contains no even cycles.

Theorem

A maltsev digraph with a conservative commutative polymorphism is a conservative maltsev digraph.

These do not retract to cycles of even length.