An extended tree-width notion for directed graphs related to the computation of permanents

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Outline

1. Introduction
2. Basics; triangular tree-width
3. Permanents of bounded ttw matrices
4. Conclusion

1. Introduction

Restriction of many hard graph-theoretic problems to subclasses of bounded tree-width graphs becomes efficiently solvable

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Well known examples important here, see Courcelle, Makowsky, Rotics:

- permanent of square matrix $M = (m_{ij})$ of bounded tree-width; here, the tree-width of M is the tree-width of the underlying graph G_M which has an edge (i, j) iff $m_{ij} \neq 0$
  Valiant: computing permanent of general 0-1 matrices is #P-hard; if matrices have entries from field $\mathbb{K}$ of characteristic $\neq 2$ the permanent polynomials build a VNP-complete family
- Hamiltonian cycle decision problem

Important: though above G_M is directed its tree-width is taken as that of the undirected graph, i.e., an edge (i,j) is present if at least one entry m_{ij} or m_{ji} is non-zero;

thus, tree-width of G_M does not reflect a case of lacking symmetry where $m_{ij} \neq 0$ but $m_{ji} = 0$.
However, that might have impact on computation of the permanent

Example: for an upper triangular (n,n)-matrix M the tree-width of G_M is $n-1$; its permanent nevertheless is easy to compute.

Goal: introduce new tree-width notion called triangular tree-width for directed graphs such that

- for square matrices M bounding the triangular tree-width of G_M allows to compute $perm(M)$ efficiently
- the (undirected) tree-width of G_M is greater than or equal to its triangular tree-width; examples where the former is unbounded whereas the latter is not do exist.

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Definition (Tree-width)

$G = \langle V, E\rangle$ graph, k-tree-decomposition of G is a tree $T = \langle V_T, E_T\rangle$ such that:

(i) For each $t \in V_T$ a subset $X_t \subseteq V$ of size at most $k+1$.

(ii) For each edge $(u, v) \in E$ there is a $t \in V_T$ s.t. $\{u, v\} \subseteq X_t$.

(iii) For each vertex $v \in V$ the set $\{t \in V_T | v \in X_T\}$ forms a (connected) subtree of T.

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Trees have tree-width 1, cycles have twd 2

Tree-width of a matrix $M = (m_{ij})$ over $\mathbb{K}$: (undirected) tree-width of (directed) incidence graph G_M

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weight of edge $(i,j) = m_{ij}$;

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Permanent $\quad perm(M) := \sum_{e \text{ cycle cover}} \prod m_{i,j}^{e_{i,j}}$

Valiant's conjecture: Computation of permanent hard

Basic idea for defining triangular tree-width

Let $G_M = (V, E)$ with $V = \{1, \ldots, n\}$ and consider an order on the vertices, f.e., $1 < 2 < \ldots < n$;

if a cycle contains an increasing edge w.r.t. the order, it must contain as well a decreasing edge.

for computation of permanent more important than $twd(G_M)$ is the tree-width of both the graph of increasing and that of decreasing edges

find optimal order with respect to bounding both tree-width parameters

Definition (Triangular tree-width ttw)

M square matrix, $G_M = (V, E)$ with $V = \{1, \dots, n\}$, $\sigma : V \to V$ a permutation;

a) $G_\sigma^{inc} = (V, E_\sigma^{inc})$ graph of increasing edges w.r.t. order defined by σ, G_σ^{dec} accordingly; loops are located in E_σ^{dec}

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c) $ttw(G_M) := \min\limits_{\sigma \in S_n} \max\{twd(G_\sigma^{inc}), twd(G_\sigma^{dec})\}$.

Basic observations:

- if M has a symmetric structure of non-zero entries, i.e., $m_{ij} \neq 0 \;\Leftrightarrow\; m_{ji} \neq 0$, then $ttw(G_M) = twd(G_M)$

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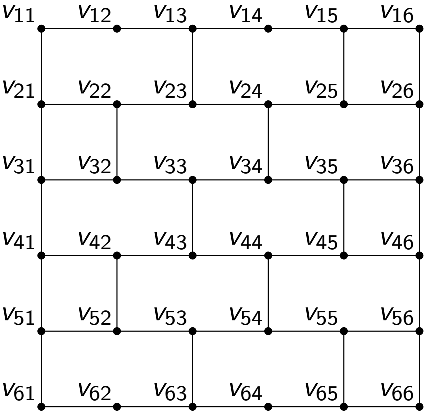
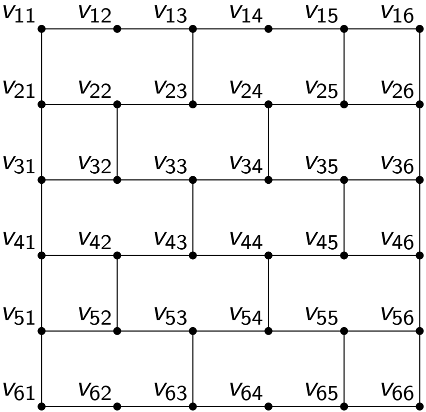
- if M has a symmetric structure of non-zero entries, i.e., $m_{ij} \neq 0 \Leftrightarrow m_{ji} \neq 0$, then $ttw(G_M) = twd(G_M)$
- computation of the triangular tree-width of a directed graph is NP-hard
- triangular tree-width extends the tree-width notion; in particular, there are families of graphs for which the former is bounded whereas the latter is not



The grid graph G_n **for** $n = 6$; direction of edges from left to right and from top to bottom



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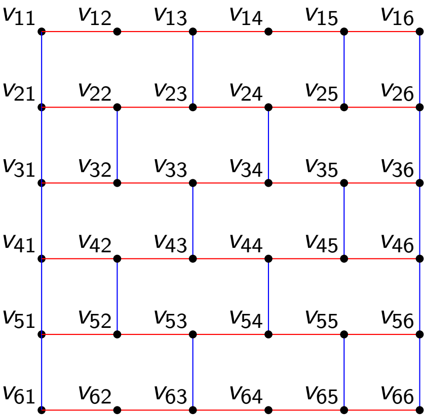
$twd(G_n)$ tends to ∞ for $n \to \infty$ Thomassen



order vertices from left to right and from bottom to top; the red edges are increasing, blue ones are decreasing and $ttw(G_{n,\sigma}) = 1$.

3. Permanents of bounded ttw matrices

Theorem (Main theorem)

Let $\{M_i\}_{i \in I}$ be a family of matrices of bounded triangular tree-width at most $k \in \mathbb{N}$. For every member M of the family, given corresponding tree-decompositions of the graphs of increasing and of decreasing edges, perm(M) can be computed in polynomial time in the size of M. The computation is fixed parameter tractable w.r.t. k.

Typically, proving such theorems employs climbing a tree-decomposition bottom up, see, f.e., Flarup & Koiran & Lyaudet;

once a vertex has been removed during this process from a bag of the tree-decomposition it has not to be considered any longer

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Main problem: we have to deal with two tree-decompositions. In general, a vertex that has been removed in one can occur further up in the other. Thus, it cannot be removed in the usual way and backtracking cannot be avoided.

Solution of this problem: guarantee that one of the two decompositions bounds the number of occurences of each vertex in a bag to say $10 \cdot k$ many.

Definition (Perfect decompositions)

$G = (V, E)$ directed has a perfect triangular tree-decomposition of width k if there is a permutation σ and two corresponding tree-decompositions $T_\sigma^{inc}, T_\sigma^{dec}$ of width k for $G_\sigma^{inc}, G_\sigma^{dec}$, respectively, and none of the vertices of G occurs in more than $10k$ many bags of T_σ^{dec}.

STEP 1: Show main theorem for perfect decompositions

Climb decomposition T_σ^{inc} of G_σ^{inc} bottom up and construct partial cycle covers together with their weights;

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each time a vertex i disappears when climbing up in T_σ^{inc} all information about i given in T_σ^{dec} is incorporated; since i occurs in at most $10 \cdot k$ bags of T_σ^{dec} again this results in at most $\tilde{f}(k)$ many types;

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thus, i can be removed in both decompositions

STEP 2: Removing the perfectness assumption

Theorem

For M, G_M and permutation σ with triangular tree-decomposition $T_\sigma^{inc}, T_\sigma^{dec}$ one can construct a new matrix $\tilde{M}$, a corresponding graph $G_{\tilde{M}}$ and a permutation $\tilde{\sigma}$ such that $perm(\tilde{M}) = perm(M)$, $G_{\tilde{M}}$ is of bounded triangular tree-width witnessed by $\tilde{\sigma}$ and $(\tilde{T}_\sigma^{inc}, \tilde{T}_\sigma^{dec})$ is a perfect triangular decomposition.

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Proof: New graph is obtained by adding at most linearly many vertices and edges in such a way that original vertices occurring too often in bags of T_σ^{dec} are partially replaced by new ones; replacement can be done in such a way that the cycle covers of the old and those of the new graph are in a 1-1 correspondence maintaining the weights

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The Hamiltonian cycle decision problem is efficiently solvable for families of matrices that are of bounded triangular tree-width k. Here, a corresponding tree-decomposition has to be given.

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Proof: Main difference is in removing the perfectness assumption; here, the transformation leads to a slight modification of the HC problem.

4. Conclusions

Triangular tree-width notion tailored to permanent problem;

are there other graph related problems for which the result holds?

More general: does it hold for all monadic-second order definable problems?

We conjecture not

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Is triangular tree-width related to other parameters for directed graphs?

Such parameters are for example directed tree-width (Johnson & Robertson & Seymour & Thomas) and entanglement (Berwanger & Grädel)