Computing vertex-surjective homomorphisms to partially reflexive trees

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Outline

1. **Introduction**
   - Basic definitions
   - Our results
2. **Classification of computational complexity**
   - Polynomial cases
   - Hardness
3. **Open problems**

Homomorphisms

Definition (Homomorphisms)

A *homomorphism* from a graph G to a graph H is a mapping $f : V_G \to V_H$ that maps adjacent vertices of G to adjacent vertices of H, i.e., $f(u)f(v) \in E_H$ whenever $uv \in E_G$.

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Theorem (Hell-Nešetřil dichotomy theorem, 1990)

The H-Coloring problem is solvable in polynomial time if H is bipartite, and NP-complete otherwise.

Surjective homomorphisms

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Definition (Partially reflexive graphs)

We say that a vertex incident to a self-loop is *reflexive*, whereas vertices with no self-loop are said to be *irreflexive*. A graph that contains zero or more reflexive vertices is called *partially reflexive*. In particular, a graph is *reflexive* if all its vertices are reflexive, and a graph is *irreflexive* if all its vertices are irreflexive.

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Problem (Surjective H-Coloring)

The problem SURJECTIVE H-COLORING *tests whether a given graph G allows a surjective homomorphism to a graph H.*

Related problems

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- Let H be an induced subgraph of a graph G. A homomorphism f from a graph G to H is a *retraction* from G to H if $f(h) = h$ for all $h \in V_H$.

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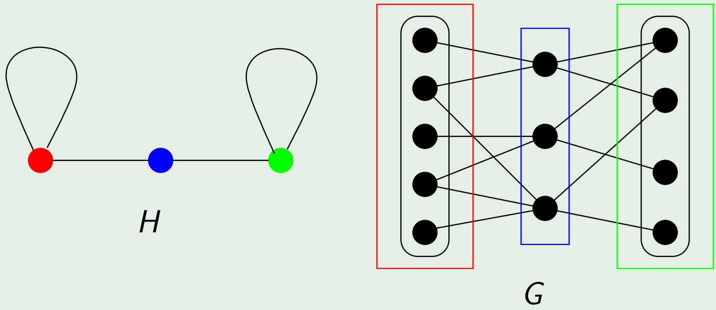
- Let H be an induced subgraph of a graph G. A homomorphism f from a graph G to H is a *retraction* from G to H if $f(h) = h$ for all $h \in V_H$.
- A homomorphism from a graph G to a graph H is called *edge-surjective* or a *compaction* if for any edge $xy \in E_H$ with $x \neq y$ there exists an edge $uv \in E_G$ with $f(u) = x$ and $f(v) = y$.

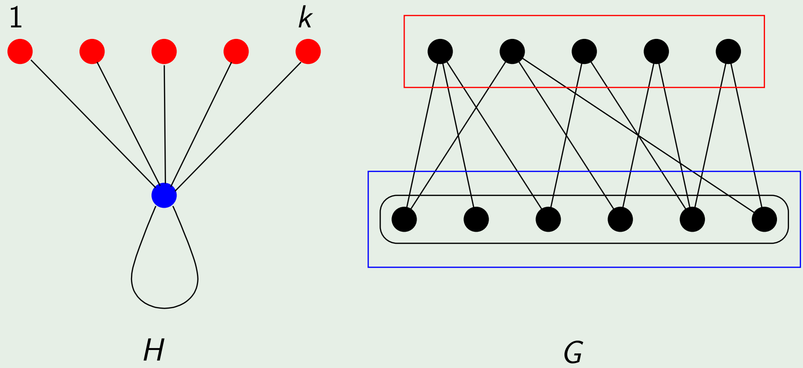
Problems that can be expressed via surjective homomorphisms

Stable Cutset



Problems that can be expressed via surjective homomorphisms

Independent Set



Our results

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- We give a complete classification of the computational complexity of the SURJECTIVE H-COLORING problem when H is a partially reflexive tree.
- We analyze the running time of the polynomial-time solvable cases and for connected n-vertex graphs, we find a running time of $n^{k+O(1)}$, where k is the number of leaves of H, and observe that this running time is asymptotically optimal.
- But we prove that for these cases, SURJECTIVE H-COLORING parameterized by $|V_H|$ is FPT on graphs of bounded expansion.

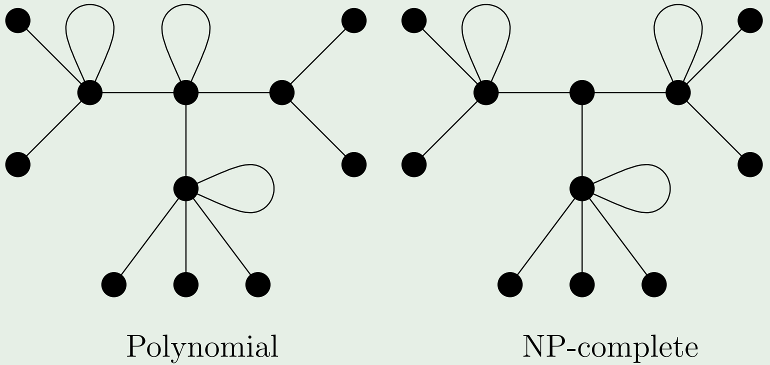
Classification of computational complexity

Theorem

For any fixed partially reflexive tree H, the SURJECTIVE H-COLORING *problem is polynomial time solvable if the set of reflexive vertices induces a connected subgraph (we call such graphs loop-connected), and NP-complete otherwise.*

Classification of computational complexity

Polynomial and NP-complete cases



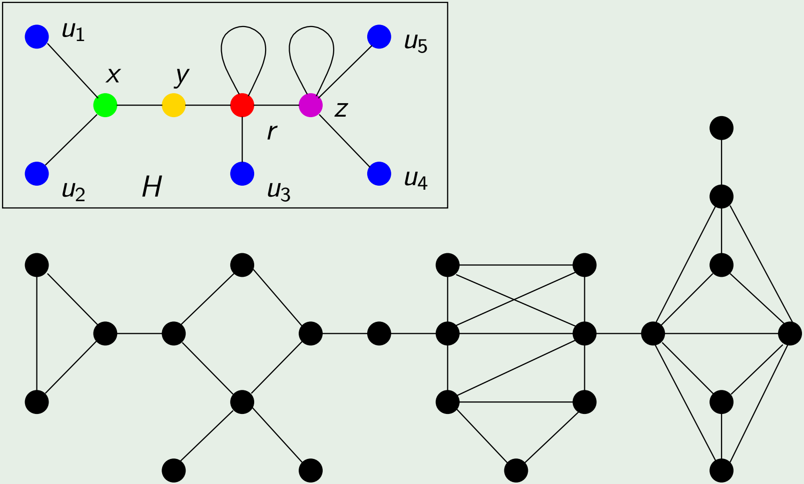
Polynomial cases

Theorem

Let H be a loop-connected tree with k leaves. For n-vertex connected graphs, Surjective *H*-Coloring *can be solved in time $n^{k+O(1)}$.*

Sketch of the proof

Special homomorphisms



Sketch of the proof

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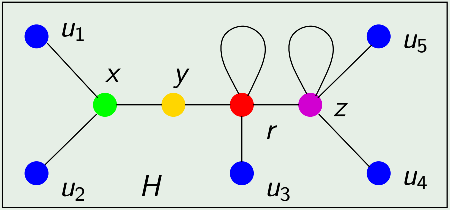

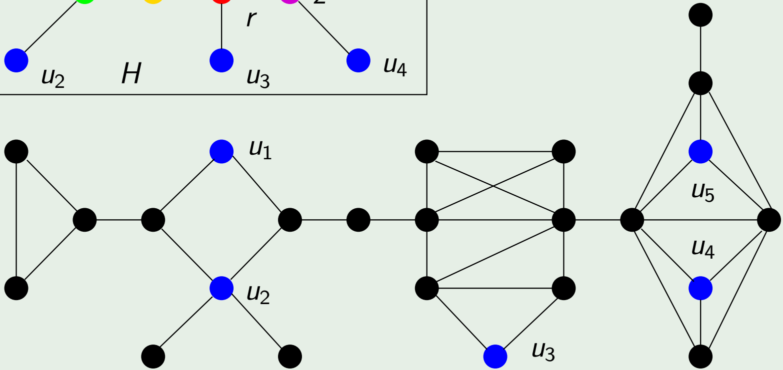

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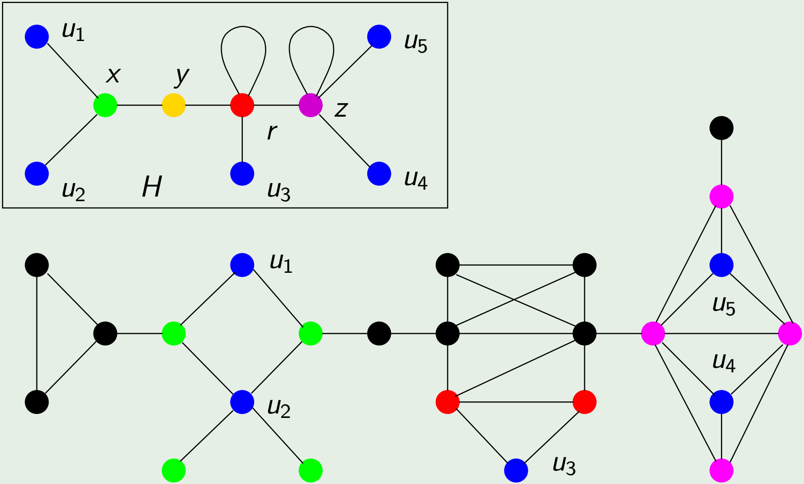

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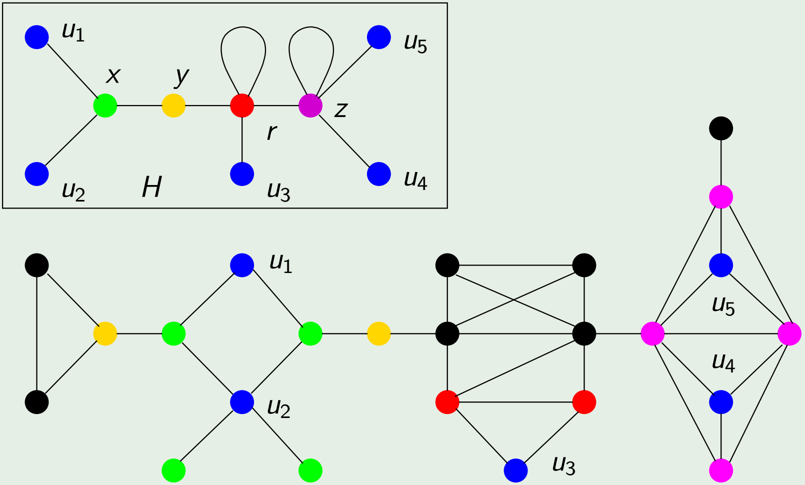

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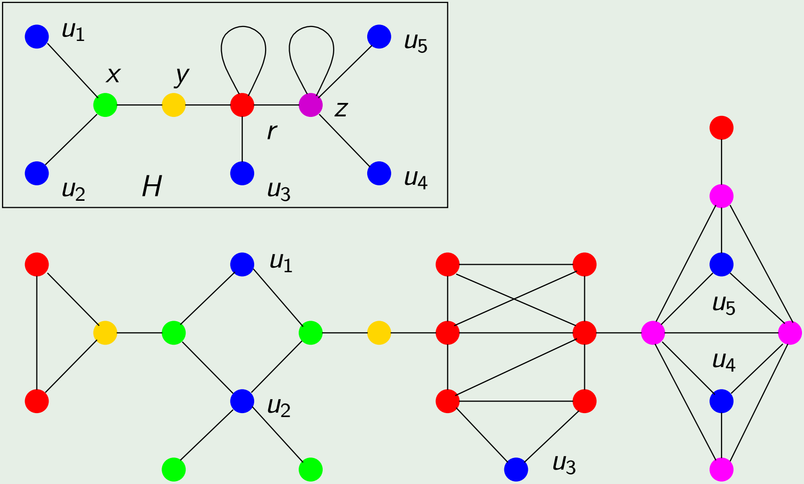


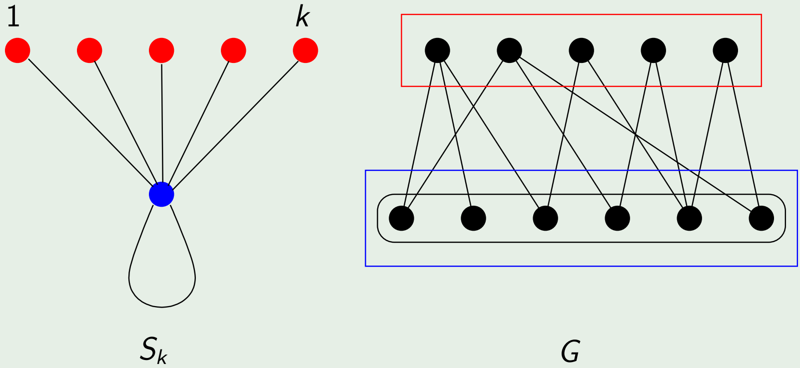
Parameterized complexity

Theorem

Let H be a loop-connected tree with k leaves. For n-vertex connected graphs, Surjective *H-*Coloring *can be solved in time $n^{k+O(1)}$, i.e. this problem is in XP when parameterized by the number of leaves.*

Parameterized complexity

Independent Set



Parameterized complexity

Proposition

SURJECTIVE S_k-COLORING *is W[1]-complete when parameterized by* k.

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Proposition

Surjective S_k-Coloring *cannot be solved in* $f(k) \cdot n^{o(k)}$ *time on n-vertex graphs, unless the Exponential Time Hypothesis collapses.*

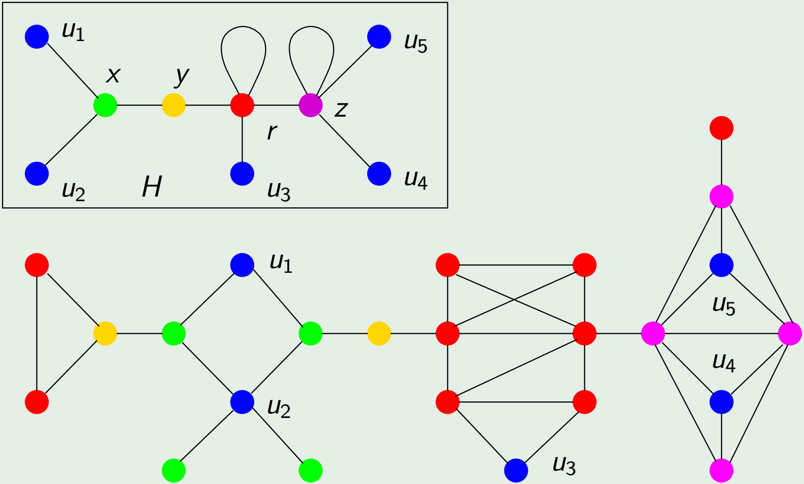
Parameterized complexity

Theorem

Let H be a loop-connected tree. Then SURJECTIVE *H*-COLORING *is FPT for any graphs of bounded expansion when parameterized by $|V_H|$.*

Idea of the proof

Special homomorphisms



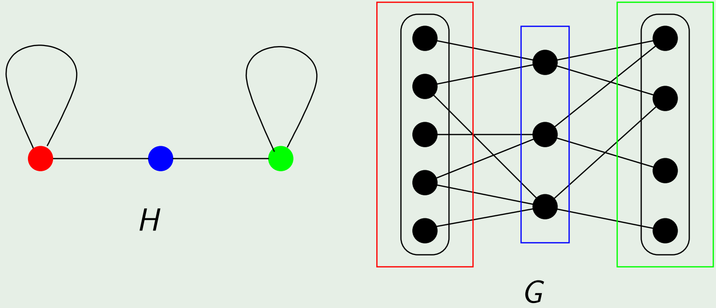
NP-complete cases

Theorem

For any fixed partially reflexive tree H that is not loop-connected, the SURJECTIVE H-COLORING *problem is NP-complete.*

Idea of the proof

Stable Cutset



Open problems

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- What can be said about special case when H is a partially reflexive cycle?
- Are H-COMPACTION, H-RETRACTION and SURJECTIVE H-COLORING polynomially equivalent to each other for each target graph H?

Thank You!