Compressed Membership in Automata with Compressed Labels

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- all domains, where massive string data arise and have to be processed, e.g. bioinformatics
- large (and highly compressible) strings often occur as intermediate data structures (e.g. in computational group theory, program analysis, verification).

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One may have $|\mathrm{val}(\mathbb{A})| = 2^n$.

Thus, an SLP $\mathbb{A}$ can be seen as a compressed representation of $\mathrm{val}(\mathbb{A})$.

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\begin{aligned}
A_3 &= A_2A_1 = ab \\
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- From LZ77(w) one can compute in polynomial time an SLP $\mathbb{A}$ with $\mathrm{val}(\mathbb{A}) = w$.

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The following problem can be solved in polynomial time:

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Plandowski's algorithm uses combinatorics on words, in particular the Periodicity-Lemma of Fine and Wilf.

Improvements of Plandowski's result

Gasieniec, Karpinski, Miyazaki, Plandowski, Rytter, Shinohara, Takeda (mid 90's)

The following problem can be solved in polynomial time
(fully compressed pattern matching):

INPUT: SLPs $\mathbb{P}, \mathbb{T}$
QUESTION: Is val($\mathbb{P}$) a factor of val($\mathbb{T}$), i.e., $\exists x, y : \text{val}(\mathbb{T}) = x \, \text{val}(\mathbb{P}) \, y$?

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The best known algorithm has a running time of $O(|\mathbb{P}| \cdot |\mathbb{T}|^2)$ (Lifshits 2006).

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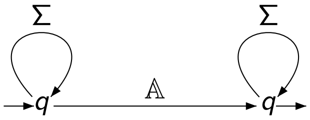

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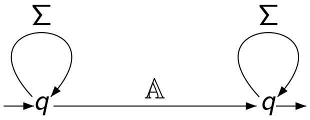

accepts all words that have val($\mathbb{A}$) as a factor.

The size $|\mathcal{A}|$ of the compressed automaton $\mathcal{A}$ (with the set Δ of transition triples) is

$$|\mathcal{A}| = \sum_{(p,\mathbb{A},q)\in\Delta} |\mathbb{A}|.$$

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Plandowski and Rytter conjectured that compressed membership for compressed automata is NP-complete (for every alphabet size).

Some combinatorics on words

The period of the word $w \in \Sigma^*$ is the smallest number p such that
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For a compressed automaton $\mathcal{A}$, let:

$$\begin{aligned} \text{order}(\mathcal{A}) &= \max\{\text{order}(\text{val}(\mathbb{A})) \mid \mathbb{A} \text{ occurs as a label in } \mathcal{A}\} \\ \text{period}(\mathcal{A}) &= \max\{\text{period}(\text{val}(\mathbb{A})) \mid \mathbb{A} \text{ occurs as a label in } \mathcal{A}\} \end{aligned}$$

First main result

Theorem 1

For a compressed automaton $\mathcal{A}$ and an SLP $\mathbb{B}$, we can check $\mathsf{val}(\mathbb{B}) \in L(\mathcal{A})$ in time polynomial in (i) $|\mathcal{A}|$, (ii) $|\mathbb{B}|$, and (iii) $\mathsf{order}(\mathcal{A})$.

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We use the following combinatorial fact:

Let $u, v \in \Sigma^*$ and let p be a position in v. Then there exist at most $\mathsf{order}(u)$ many occurrences of u in v that "touch" position p.

Second main result

Theorem 2

For a compressed automaton $\mathcal{A}$ and an SLP $\mathbb{B}$, we can check $\mathsf{val}(\mathbb{B}) \in L(\mathcal{A})$ *nondeterministically in time polynomial in (i)* $|\mathcal{A}|$*, (ii)* $|\mathbb{B}|$*, and (iii)* $\mathsf{period}(\mathcal{A})$*.*

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The theorem is proven by a reduction to the case of a unary alphabet.

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If $\mathrm{val}(\mathbb{B}) \in L(\mathcal{A})$ for an SLP $\mathbb{B}$ and a compressed automaton $\mathcal{A}$, then there exists an accepting run of $\mathcal{A}$ on $\mathrm{val}(\mathbb{B})$ (viewed as a word over the set of transition triples of $\mathcal{A}$), which can be generated by an SLP of size $\mathrm{poly}(|\mathbb{B}|, |\mathcal{A}|)$.

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Conjecture 2 implies Conjecture 1:

- Guess nondeterministically an SLP $\mathbb{C}$ of polynomial size.
- Check (in deterministic polynomial time), whether $\mathrm{val}(\mathbb{C})$ is an accepting run of $\mathcal{A}$ on $\mathrm{val}(\mathbb{B})$.