Orbits of Linear Maps and Regular Languages

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Contents

Chamber hitting problem

Definition

An orbit $\mathrm{Orb}_\Phi\, x$ is $\{\Phi^k x : k \in \mathbb{Z}^+\}$, where $\Phi\colon \mathbb{Q}^d \to \mathbb{Q}^d$ is a linear map and $x \in \mathbb{Q}^d$.

Definition

A chamber $H_S = \{x \in \mathbb{Q}^d : \mathrm{sign}(h_i(x)) = s_i \text{ for } 1 \leqslant i \leqslant m\}$, where h_i are affine functions and $s \in \{\pm 1, 0\}^m$ is a sign pattern.

Chamber hitting problem (CHP)

INPUT: $\Phi, x_0, h_1, \ldots, h_m, s$.

OUTPUT: 'yes' if $\mathrm{Orb}_\Phi\, x_0 \cap H_s \neq \varnothing$ and 'no' otherwise.

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Special cases: the orbit problem

Orbit problem

INPUT: Φ, x, y.
OUTPUT: 'yes' if $y \in \mathrm{Orb}_{\Phi}\, x$ and
'no' otherwise.

In this case the chamber is $\{y\}$.

Theorem (Kannan, Lipton, 1986)

There exists a polynomial time algorithm for the orbit problem.

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Problems reducible to CHP

Skolem problem

INPUT: $a_1, \ldots, a_d; b_1, \ldots, b_d$.

x_n — a linear recurrent sequence

$$x_n = \sum_{i=1}^{d} a_i x_{n-i}, \ (n > d), \qquad x_n = b_n \ (1 \leqslant n \leqslant d)$$

OUTPUT: 'yes' if $x_n = 0$ for some n and
'no' otherwise.

Positivity problem

INPUT: $a_1, \ldots, a_d; b_1, \ldots, b_d$; x_n is LRS.

OUTPUT: 'yes' if $x_n > 0$ for all n and
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State of the art

Open questions

Is CHP decidable? Is Skolem problem decidable? Is positivity problem decidable?

Known decidability results for small d

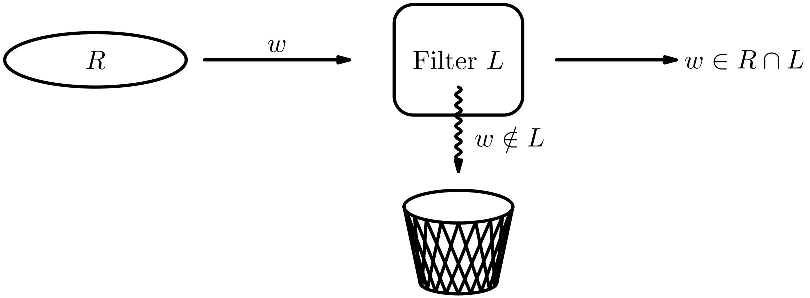
	$d = 2$	$d = 3$	$d = 4$	$d = 5$
Skolem	folklore	Vereshchagin, 1985		Halava et al., 2005
Pos. pr.	Halava et al., 2006	Laohakosol, Tangsupphathawat, 2009		
CHP	Sechin, 2011			

Regular realizability problems (RR)

A set $L \subset \Sigma^*$ is called a filter. Each filter determines a specific regular realizability problem:

L-realizability problem

INPUT: a description of a regular language R.
OUTPUT: 'yes' if $R \cap L \neq \varnothing$ and
'no' otherwise.



Permutation filter

Definition

$P_{\mathbb{B}} \subset \{\#, 0, 1\}^*$ consists of permutation words, i.e., words of the form

$$\#w_1\#w_2\# \dots w_N\#,$$

where

- $w_i \in \{0, 1\}^*$ are blocks,
- $|w_i| = n$, $i = 1, 2, \dots, N$ (n is the block rank),
- $N = 2^n$, $n \geqslant 1$,
- each binary word of length n is a block.

Examples

$$\#00\#11\#10\#01\# \in P_{\mathbb{B}}$$

$$\#10\#11\#00\#01\# \in P_{\mathbb{B}}$$

$$\#10\#01\#00\#11\# \in P_{\mathbb{B}}$$

Orbits vs Regular realizability

Theorem (Tarasov, Vyalyi, 2010)

CHP and $P_{\mathbb{B}}$-realizability problem are Turing equivalent.

From $P_{\mathbb{B}}$-realizability to CHP

Reduction starts from a $\mathbb{Q}$-linear extension of the transition monoid.

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From CHP to $P_{\mathbb{B}}$-realizability

The idea is to represent an arithmetic computation in a 'natural' form.

The main construction

- R_1, R_2 — regular languages.
- How to check that there exists an integer n such that

$$\mathrm{Card}(\{w : |w| = n \ \wedge \ w \in R_1\}) = \mathrm{Card}(\{w : |w| = n \ \wedge \ w \in R_2\})? \quad (\clubsuit)$$

- Regular expression

$$E = \#((R_1 \cap R_2)\#)^*((R_1 \setminus R_2)\#(R_2 \setminus R_1)\#)^*((\overline{R_1} \cap \overline{R_2})\#)^*$$

- $(\clubsuit)$ is equivalent to $E \cap P_{\mathbb{B}} \neq \varnothing$.

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More examples of relation between RR and linear algebra

undecidable	track product of the periodic and permutation filter	
unknown	permutation filter	
decidable	surjective filter	injective filter

Surjective filter

Definition

$S_{\mathbb{B}}$ consists of words of the form

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where

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Injective filter

Definition

$I_{\mathbb{B}}$ consists of words of the form

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where

- $w_i \in \{0,1\}^*$ are blocks,
- $|w_i| = n$, $i = 1, 2, \dots, N$, n is the block rank,
- $w_i \neq w_j$ for $i \neq j$.

Examples

$$\#00\#10\#01\# \in I_{\mathbb{B}}$$

$$\#101\#111\#001\#010\# \in I_{\mathbb{B}}$$

$$\#1000\#0110\#0000\#1111\# \in I_{\mathbb{B}}$$

Decidability results

Theorem

$I_{\mathbb{B}}$-realizability problem is decidable.
$S_{\mathbb{B}}$-realizability problem is decidable.

Proofs are based on converting $I_{\mathbb{B}}$-realizability problem (resp., $S_{\mathbb{B}}$-realizability problem) to a problem about orbits.

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An undecidable problem

Zero in the Upper Right Corner Problem (ZURC)

INPUT: $A_1, \ldots, A_N$ are $D \times D$ integer matrices.
OUTPUT: 'yes' if there exists a sequence $j_1, \ldots, j_\ell$ such that

$$(A_{j_1} A_{j_2} \ldots A_{j_\ell})_{1D} = 0 \quad \text{and}$$

'no' otherwise.

Theorem (Bell, Potapov, 2006)

The ZURC problem is undecidable for $N = 2$ and $D = 18$.

The ZURC problem is reduced to the regular realizability problem for the track product of periodic and permutation filters.

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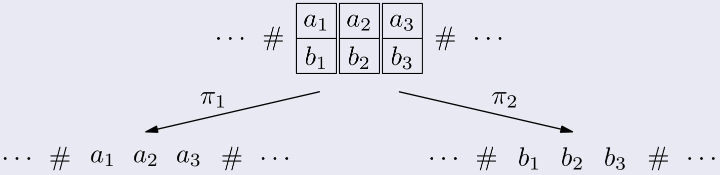
Track product

For languages $L_1 \subset (\{\#\} \cup \Sigma_1)^*$, $L_2 \subset (\{\#\} \cup \Sigma_2)^*$ the track product $L_1 \| L_2 \subset (\{\#\} \cup \Sigma_1 \times \Sigma_2)^*$.

Projections



Definition of $L_1 \| L_2$

$$L_1 \| L_2 = \{ w \in (\{\#\} \cup \Sigma_1 \times \Sigma_2)^* \mid \pi_1 w \in L_1;\ \pi_2 w \in L_2 \}$$

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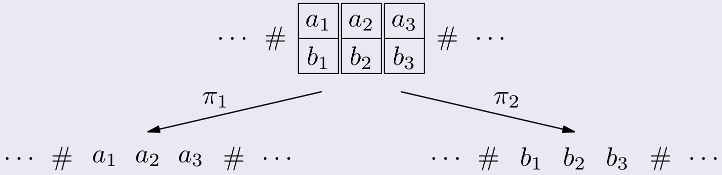

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Track product of periodic and permutation filters

Definitions

- Periodic filter $\mathrm{Per}_\Sigma \subset (\{\#\} \cup \Sigma)^*$ consists of words of the form
$$\#w\#w\#\ldots w\#,$$
where $w \in \{0,1\}^*$.
- Definition of the permutation filter P_Σ over the alphabet $\{\#\} \cup \Sigma$ is similar to the binary case.

Theorem

ZURC $\leq_m$ $(\mathrm{Per}_{\Sigma_1} \| P_{\Sigma_2})$*-regular realizability for* $|\Sigma_1| = 2$, $|\Sigma_2| = 648$.

Informally, the periodic part is to represent a sequence of matrices and the permutation part is to encode the condition that the URC entry is 0.

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