A Multiple-Conclusion Calculus for First-Order Gödel Logic

Arnon Avron Ori Lahav

Tel Aviv University

CSR June 2011

Gödel Logic

- In 1933, Gödel introduced a sequence $\{G_n\}$ of n-valued matrices and used them to show some important properties of intuitionistic logic.
- In 1959, Dummett embedded all the $G_n s$ in an infinite-valued matrix G_ω.
- G_ω is equivalent to $G_{[0,1]}$, a natural matrix for the truth-values $[0, 1]$.
- The logic of $G_{[0,1]}$ is called Gödel logic.
- Gödel logic is perhaps the most important intermediate logic.
- Nowadays, Gödel logic is also recognized as one of the three most basic fuzzy logics.

Many-Valued Semantics

- A *structure* M consists of:
  - Non-empty domain D
  - An interpretation I:
    - $I[c] \in D$ for every constant
    - $I[f] \in D^n \to D$ for every n-ary function
    - $I[p] \in D^n \to [0, 1]$ for every n-ary predicate

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- An *M-evaluation* is a function e:
  - Assigning an element of D for every free variable
  - Naturally extended to all terms (according to I)

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- $\| \bullet \|_e^M$ is defined as follows:
  - $\|p(t_1, \ldots, t_n)\|_e^M = I[p][e[t_1], \ldots, e[t_n]]$
  - $\|\bot\|_e^M = 0$

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  - $\|\psi_1 \vee \psi_2\|_e^M = \max\{\|\psi_1\|_e^M, \|\psi_2\|_e^M\}$
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- M is a model of a formula if φ if $\|\varphi\|_e^M = 1$ for every e

Kripke-Style Semantics

A *frame* is a tuple $\mathcal{W} = \langle W, \leq, D, I, \{I_w\}_{w \in W} \rangle$ where:

- W - nonempty set of worlds
- $\leq$ - linear order on W
- D - non-empty (constant) domain
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- I - interpretation of constants and functions
- I_w - is a predicate interpretation for every $w \in W$:
  - $I_w[p] \subseteq D^n$ for every n-ary predicate p
  - Persistence: $I_u[p] \subseteq I_w[p]$ if $u \leq w$

Kripke-Style Semantics

- Satisfaction relation for a frame, a world, and an evaluation of the free variables:
  - $\mathcal{W}, w, e \models p(t_1, \ldots, t_n)$ iff $\langle e[t_1], \ldots, e[t_n] \rangle \in I_w[p]$
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  - $\mathcal{W}, w, e \models \psi_1 \vee \psi_2$ iff $\mathcal{W}, w, e \models \psi_1$ or $\mathcal{W}, w, e \models \psi_2$
  - $\mathcal{W}, w, e \models \psi_1 \wedge \psi_2$ iff $\mathcal{W}, w, e \models \psi_1$ and $\mathcal{W}, w, e \models \psi_2$

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  - $\mathcal{W}, w, e \models \forall x \psi$ iff $\mathcal{W}, w, e_{[x:=d]} \models \psi$ for every $d \in D$
  - $\mathcal{W}, w, e \models \exists x \psi$ iff $\mathcal{W}, w, e_{[x:=d]} \models \psi$ for some $d \in D$

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- $\mathcal{W}$ is a model of a formula if φ if $\mathcal{W}, w, e \models \varphi$ for every w and e

Proof Theory

- Sonobe 1975 - first cut-free Gentzen-type sequent calculus
- Other calculi have been proposed later by Corsi, Avellone et al., Dyckhoff and others
- All of them use some ad-hoc rules of a nonstandard form
- A. 1991 - hypersequent calculus with standard rules: **HG**
- Extended to first-order Gödel logic by Baaz and Zach in 2000: **HIF**

Hypersequents

- A *sequent* (Gentzen 1934) is an object of the form $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m$
  - Intuition: $\varphi_1 \wedge \ldots \wedge \varphi_n \supset \psi_1 \vee \ldots \vee \psi_m$
  - Single-conclusion sequent: $m \leq 1$

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  - Single-conclusion sequent: $m \leq 1$
- A *hypersequent* is an object of the form $s_1 \mid \ldots \mid s_n$ where the s_i's are sequents
  - Intuition: $s_1 \vee \ldots \vee s_n$
  - Single-conclusion hypersequent consists of single-conclusion sequents only

HG Single-Conclusion Hypersequent System

Structural Rules

$$\varphi \Rightarrow \varphi \qquad (EW)\ \frac{H}{H \mid \Gamma \Rightarrow E}$$

$$(IW \Rightarrow)\ \frac{H \mid \Gamma \Rightarrow E}{H \mid \Gamma, \psi \Rightarrow E} \qquad (\Rightarrow IW)\ \frac{H \mid \Gamma \Rightarrow}{H \mid \Gamma \Rightarrow \psi}$$

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$$(cut)\ \frac{H_1 \mid \Gamma_1 \Rightarrow \varphi \quad H_2 \mid \Gamma_2, \varphi \Rightarrow E}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2 \Rightarrow E}$$

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$$(com)\ \frac{H_1 \mid \Gamma_1, \Gamma_1' \Rightarrow E_1 \quad H_2 \mid \Gamma_2, \Gamma_2' \Rightarrow E_2}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2' \Rightarrow E_1 \mid \Gamma_2, \Gamma_1' \Rightarrow E_2}$$

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$$(\vee \Rightarrow) \qquad \frac{H_1 \mid \Gamma_1, \psi_1 \Rightarrow E \quad H_2 \mid \Gamma_2, \psi_2 \Rightarrow E}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2, \psi_1 \vee \psi_2 \Rightarrow E}$$

$$(\Rightarrow \vee_1) \quad \frac{H \mid \Gamma \Rightarrow \psi_1}{H \mid \Gamma \Rightarrow \psi_1 \vee \psi_2} \qquad (\Rightarrow \vee_2) \quad \frac{H \mid \Gamma \Rightarrow \psi_2}{H \mid \Gamma \Rightarrow \psi_1 \vee \psi_2}$$

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$$(\supset\Rightarrow) \quad \frac{H_1 \mid \Gamma_1 \Rightarrow \psi_1 \quad H_2 \mid \Gamma_2, \psi_2 \Rightarrow E}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2, \psi_1 \supset \psi_2 \Rightarrow E}$$

$$(\Rightarrow\supset) \quad \frac{H \mid \Gamma, \psi_1 \Rightarrow \psi_2}{H \mid \Gamma \Rightarrow \psi_1 \supset \psi_2}$$

HIF Single-Conclusion Hypersequent System

$$(\forall \Rightarrow) \quad \frac{H \mid \Gamma, \varphi\{t/x\} \Rightarrow E}{H \mid \Gamma, \forall x \varphi \Rightarrow E}$$

$$(\Rightarrow \forall) \quad \frac{H \mid \Gamma \Rightarrow \varphi\{y/x\}}{H \mid \Gamma \Rightarrow \forall x \varphi}$$

where y doesn't occur free in any component of the conclusion.

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- similar rules for $\exists$

Cut-Admissibility

$$(cut) \quad \frac{H_1 \mid \Gamma_1 \Rightarrow \varphi \qquad H_2 \mid \Gamma_2, \varphi \Rightarrow E}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2 \Rightarrow E}$$

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- One of the most important properties of a (hyper)sequent calculus, provides the key for proof-search
- Traditional syntactic cut-admissibility proofs are notoriously prone to errors, especially (but certainly not only) in the case of hypersequent systems
  - the first proof of cut-elimination for **HIF** was erroneous
- A semantic proof is usually more reliable and easier to check
- A semantic proof usually provides also a proof of completeness as well as strong cut-admissibility

$$\varphi \Rightarrow \varphi \qquad (EW)\ \frac{H}{H \mid \Gamma \Rightarrow \Delta}$$

$$(IW \Rightarrow)\ \frac{H \mid \Gamma \Rightarrow \Delta}{H \mid \Gamma, \psi \Rightarrow \Delta} \qquad (\Rightarrow IW)\ \frac{H \mid \Gamma \Rightarrow \Delta}{H \mid \Gamma \Rightarrow \Delta, \psi}$$

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MCG Multiple-Conclusion Hypersequent System

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$$(split)\ \frac{H \mid \Gamma \Rightarrow \Delta_1, \Delta_2}{H \mid \Gamma \Rightarrow \Delta_1 \mid \Gamma \Rightarrow \Delta_2}$$

MCG Multiple-Conclusion Hypersequent System

Logical Rules

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$$(\forall\Rightarrow)\quad \frac{H \mid \Gamma, \varphi\{t/x\} \Rightarrow \Delta}{H \mid \Gamma, \forall x\varphi \Rightarrow \Delta} \qquad (\Rightarrow\forall)\quad \frac{H \mid \Gamma \Rightarrow \Delta, \varphi\{y/x\}}{H \mid \Gamma \Rightarrow \Delta, \forall x\varphi}$$

Results

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- As a corollary, we obtain the same results for **HIF**, the original single-conclusion calculus

Thank you!