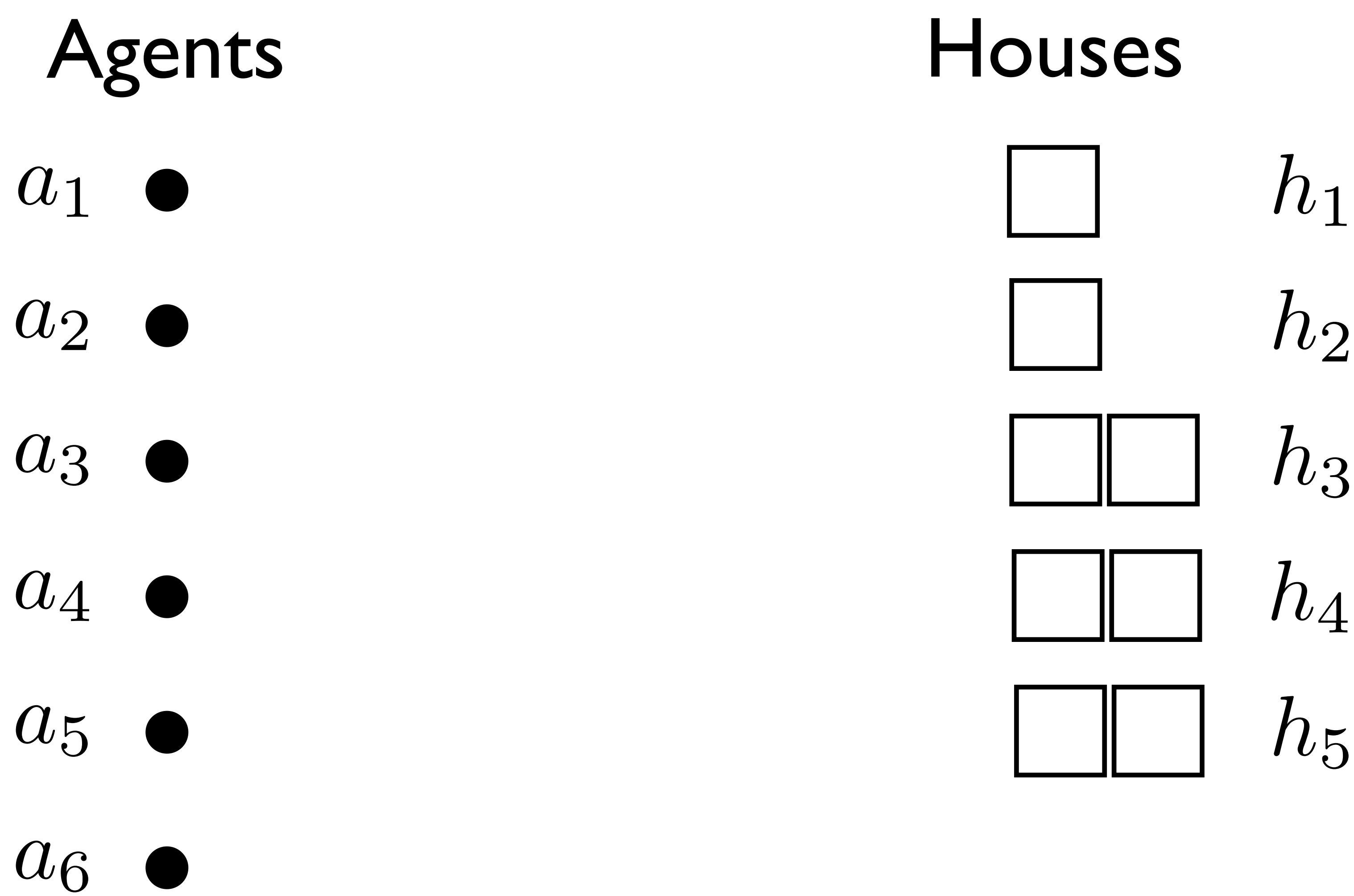
Counting Popular Matchings in House Allocation Problems

Rupam Acharyya, Sourav Chakraborty and Nitesh Jha
Chennai Mathematical Institute
India

CSR 2014

The Problem



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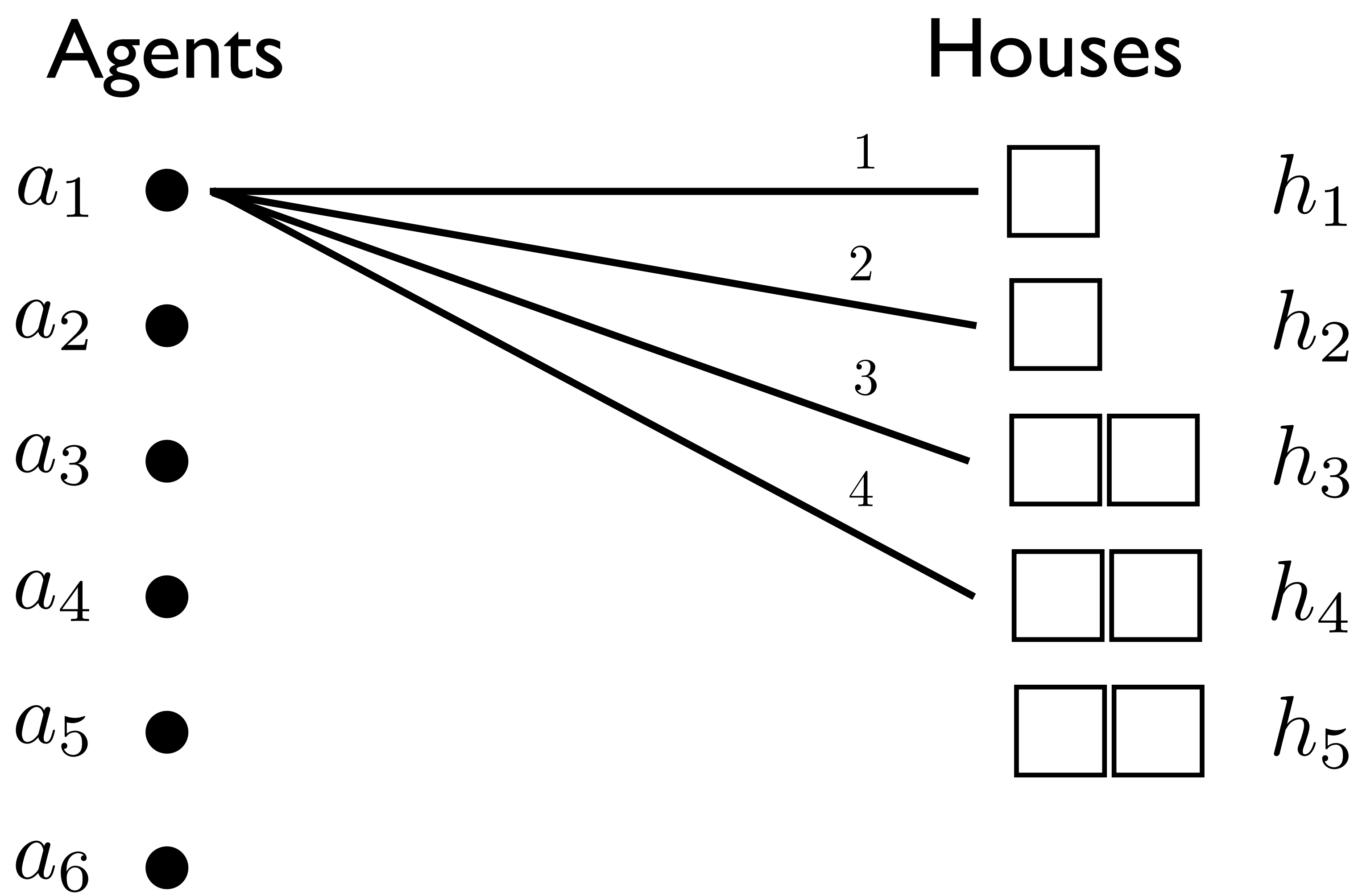

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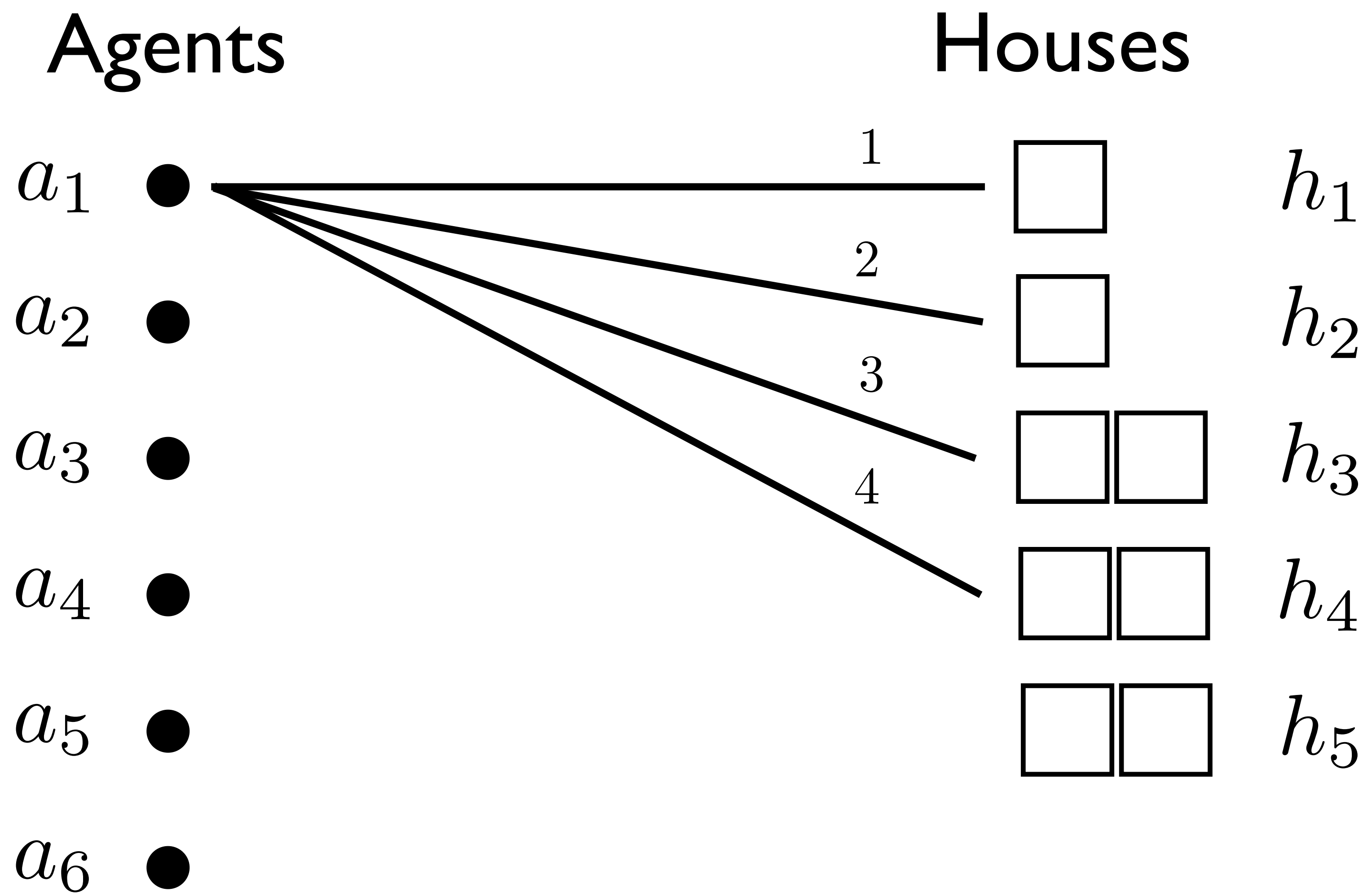

$$a_1 \quad : \quad h_1 \quad h_2 \quad h_3 \quad h_4$$

The Problem Instance

Preferences

a_1	h_1	h_2	h_3	h_4
a_2	h_1	h_3	h_2	
a_3	h_2	h_1	h_3	
a_4	h_2	h_4	h_3	
a_5	h_5	h_1		
a_6	h_5	h_2		

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a_5	h_5	h_1		
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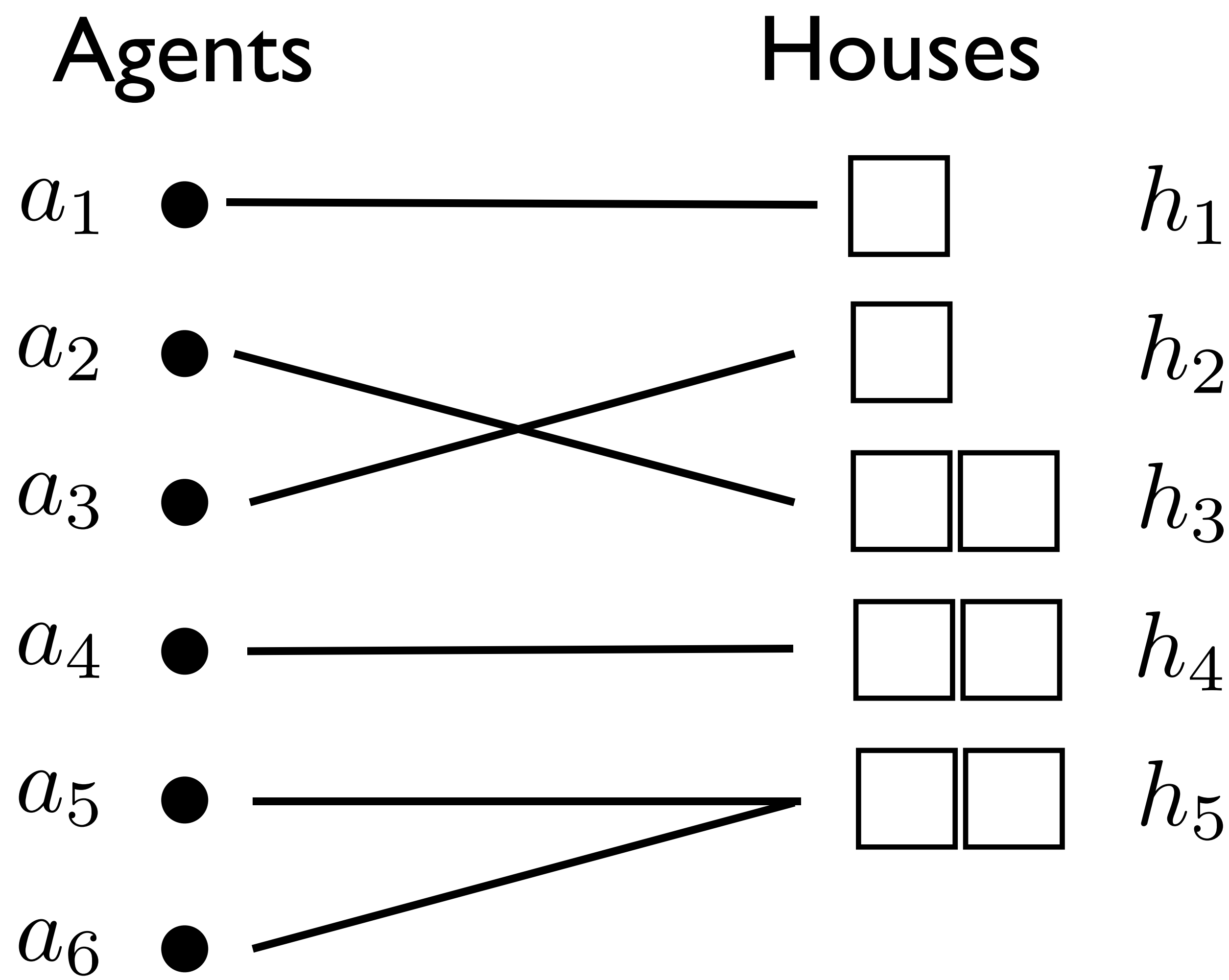
Capacities

h_1	1
h_2	1
h_3	2
h_4	2
h_5	2

A Matching



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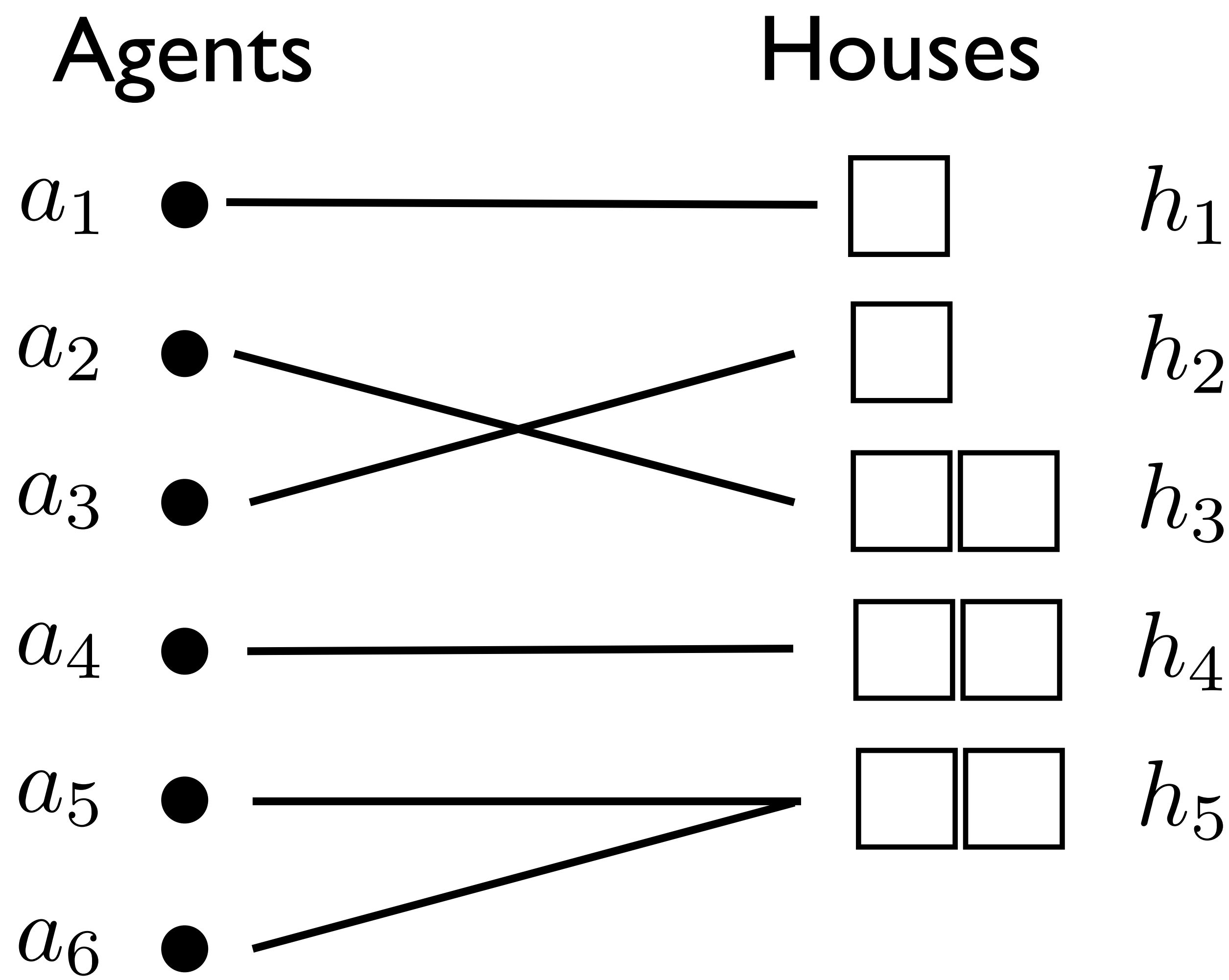

Is it a good matching?

Comparing two matchings

M_1		M_2	
a_1	h_1	a_1	h_4
a_2	h_3	a_2	h_1
a_3	h_2	a_3	h_3
a_4	h_4	a_4	h_4
a_5	h_5	a_5	h_5
a_6	h_5	a_6	h_5

Voting

M_1		M_2	
a_1	h_1	a_1	h_4
a_2	h_3	a_2	h_1
a_3	h_2	a_3	h_3
a_4	h_4	a_4	h_4
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a_6	h_5	a_6	h_5

$a_1 \mid h_1 \; h_2 \; h_3 \; h_4$

a_1

M_1

Voting

M_1		M_2	
a_1	h_1	a_1	h_4
a_2	h_3	a_2	h_1
a_3	h_2	a_3	h_3
a_4	h_4	a_4	h_4
a_5	h_5	a_5	h_5
a_6	h_5	a_6	h_5

a_1	a_2	a_3	a_4	a_5	a_6
M_1	M_2	M_1	*	*	*

“more popular than”

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ M_1 & M_2 & M_1 & * & * & * \end{array}$$

$$M_1 \text{ more popular than } M_2$$

Popular Matching - Definition

M is popular if $\nexists M'$ more popular than M

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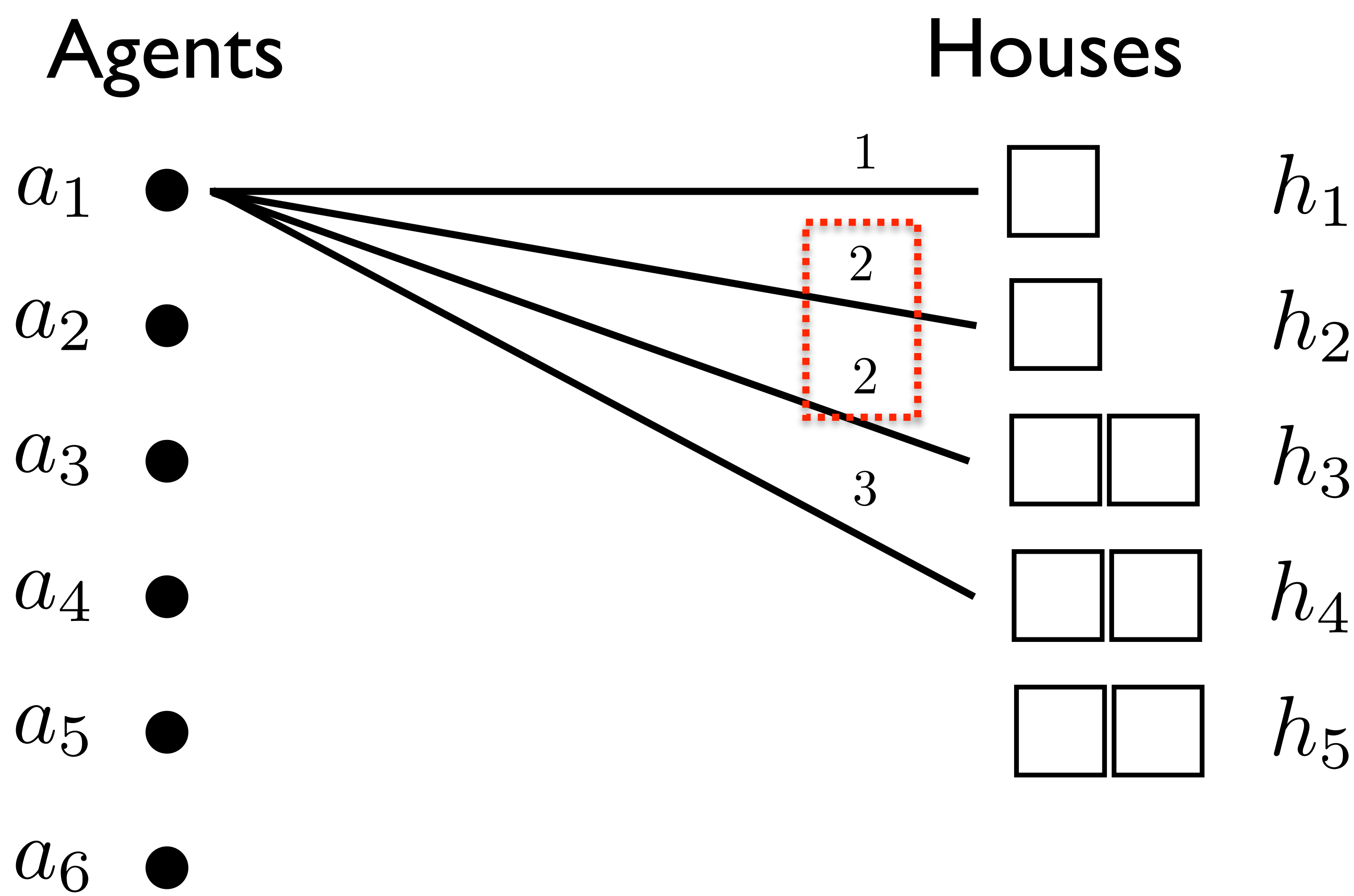
Questions

Does there always exist one? No.

Can there exist more than one? Yes.

Instances with Ties



$$a_1 \; : \quad h_1 \; \underline{(h_2 \quad h_3)} \; h_4$$

History

[Abraham et al. 2005] gave a poly-time algorithm to output a popular matching

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Studied Extensively in Different Settings

- Random Popular Matching [Mahdian 2006]
- Different Optimality Criteria [KNN2010, KMN2011]
- Preferences on both sides [Kavitha 2012]
- Games on Popular Matching [Nasre 2013]

Counting Popular Matchings

	Hardness	Approximation

Counting Popular Matchings

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no ties, capacities = 1	poly-time	Exact Count in O(n) [McDermid and Irving 2011]

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Counting Popular Matchings

	Hardness	Approximation
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ties, capacities = 1	#P-hard [Nasre 2013]	FPRAS [Acharyya, Chakraborty, J]
no ties, integer capacities	#P-hard [Acharyya, Chakraborty, J]	OPEN

Fully Polynomial Randomised Polynomial Scheme

For problem $f : \Sigma^* \to \mathrm{N}$, input $x \in \Sigma^*$, tolerance $\epsilon > 0$, error probability $\delta > 0$ outputs N such that for all x:

$$P[(1-\epsilon)f(x) \leq N \leq (1+\epsilon)f(x)] \geq \delta$$

in time $\mathrm{poly}(|x|, 1/\epsilon, \log 1/\delta)$.

Our Focus

Strict Ordering, Integer Capacities

Preferences

a_1	h_1	h_2	h_3	h_4
a_2	h_1	h_3	h_2	
a_3	h_2	h_1	h_3	
a_4	h_2	h_4	h_3	

Capacities

h_1	1
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Capacities

h_1	1
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f-houses $= \{h_1, h_2\}$

Preferences

a_1	h_1	$\cancel{h_2}$	h_3	h_4
a_2	h_1	h_3	h_2	
a_3	h_2	h_1	h_3	
a_4	h_2	h_4	h_3	

Capacities

h_1	1
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a_4	h_2	h_4	h_3	

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f-houses $= \{h_1, h_2\}$

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a_2	h_1	h_3	h_2	
a_3	h_2	$\cancel{h_1}$	h_3	
a_4	h_2	h_4	h_3	

Capacities

h_1	1
h_2	1
h_3	2
h_4	2

f-houses $= \{h_1, h_2\}$

s-houses $= \{h_3, h_4\}$

Popular Matching Characterisation

In every popular matching:

- every agent gets either its f-house or s-house

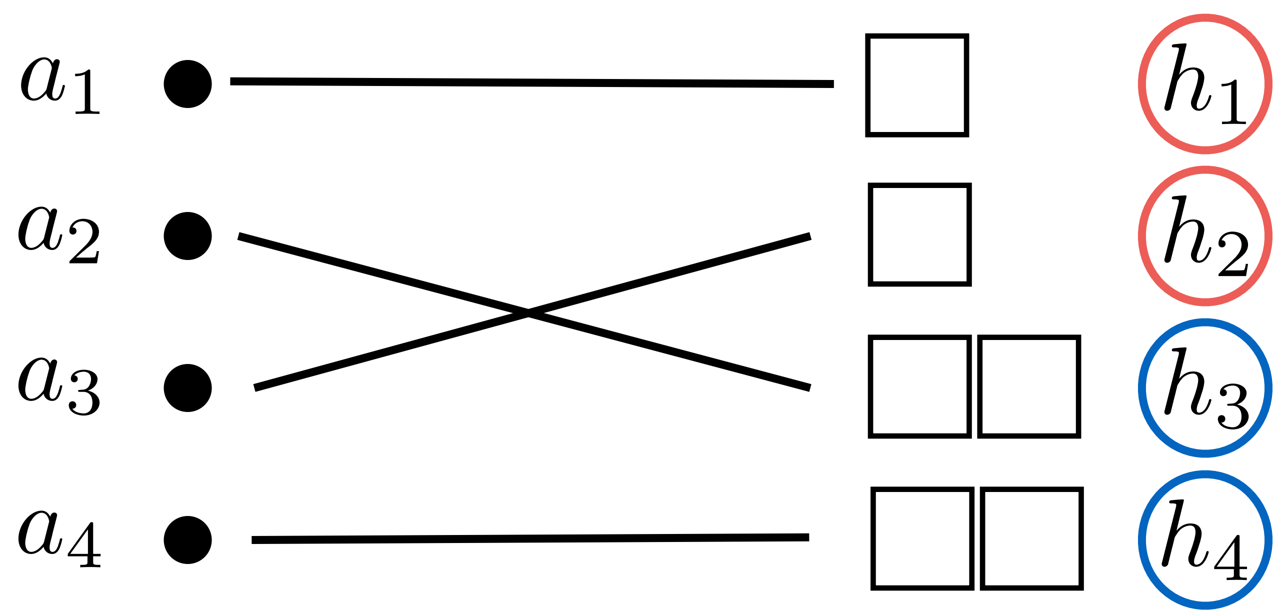
Popular Matching Characterisation

In every popular matching:

- every agent gets either its f-house or s-house
- every f-house is used to maximum capacity

a_1	h_1	$\cancel{h_2}$	h_3	h_4
a_2	h_1	h_3	h_2	
a_3	h_2	$\cancel{h_1}$	h_3	
a_4	h_2	h_4	h_3	

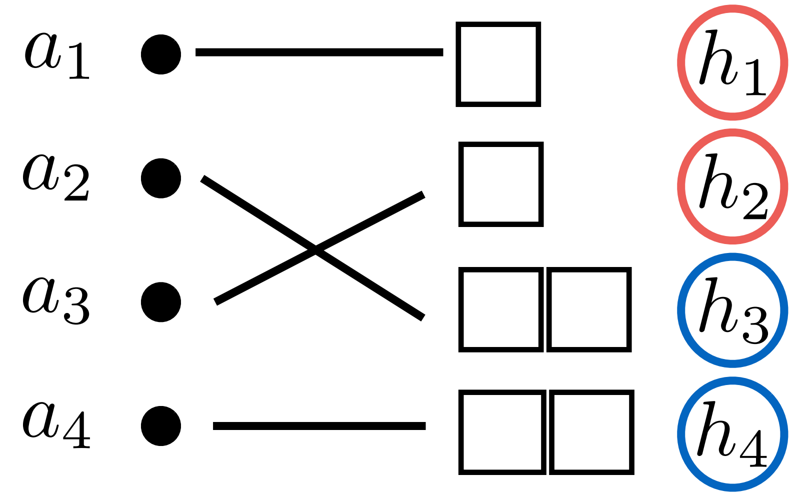
h_1	1
h_2	1
h_3	2
h_4	2



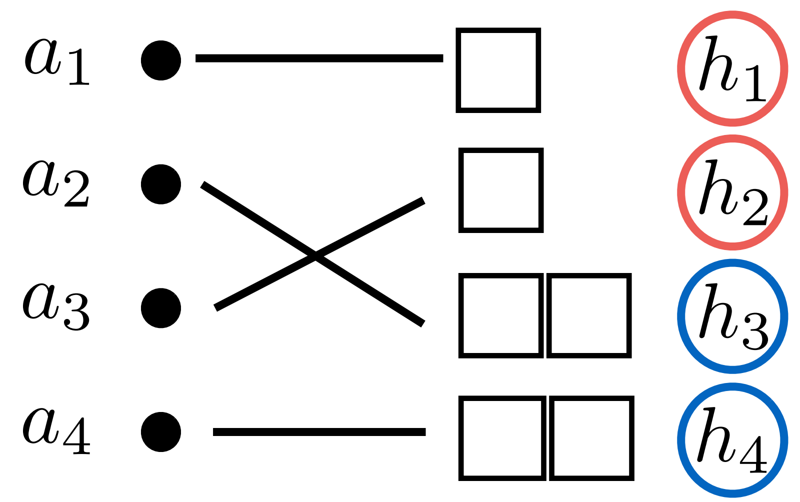
Switching Graph Characterisation

Another way to look at popular matchings!

a_1	h_1	$\cancel{h_2}$	h_3	h_4
a_2	h_1	h_3	h_2	
a_3	h_2	$\cancel{h_1}$	h_3	
a_4	h_2	h_4	h_3	



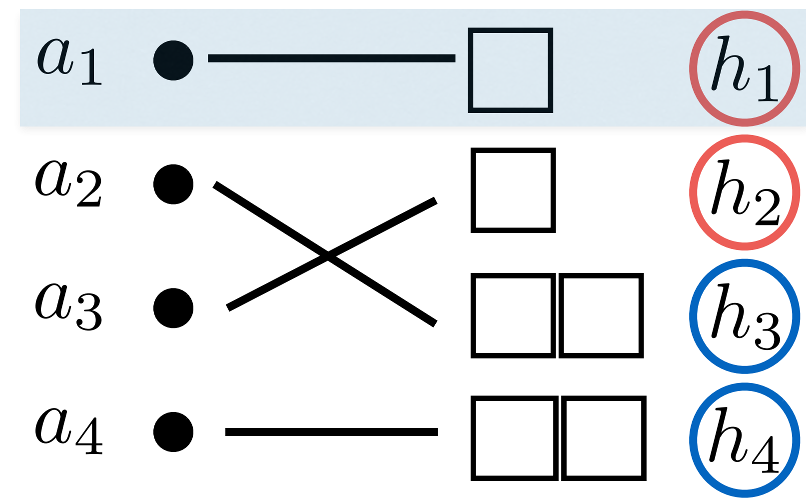
a_1	h_1	$\cancel{h_2}$	h_3	h_4
a_2	h_1	h_3	h_2	
a_3	h_2	$\cancel{h_1}$	h_3	
a_4	h_2	h_4	h_3	

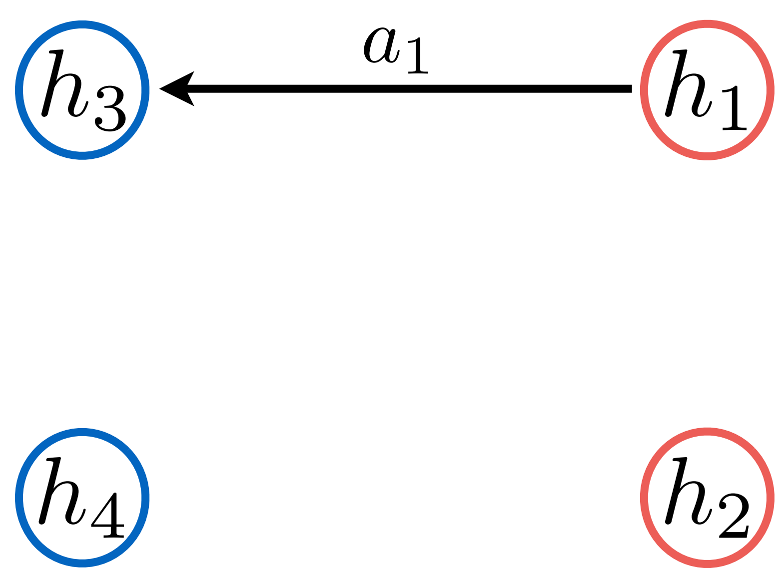


h_3 h_1

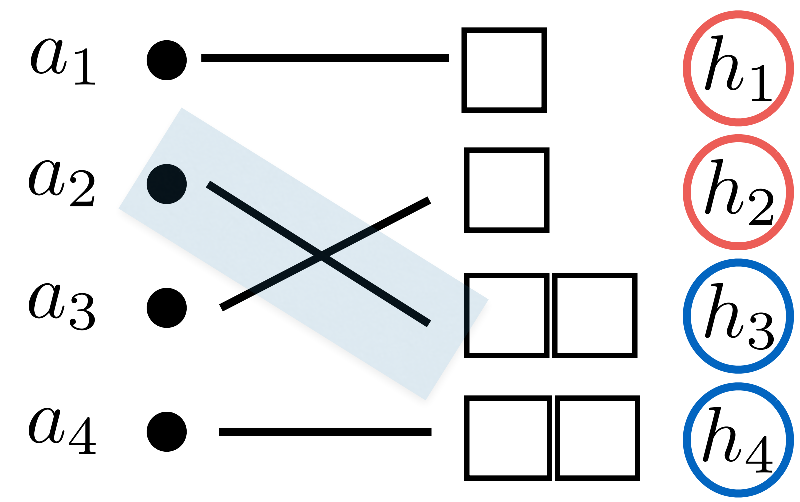
h_4 h_2

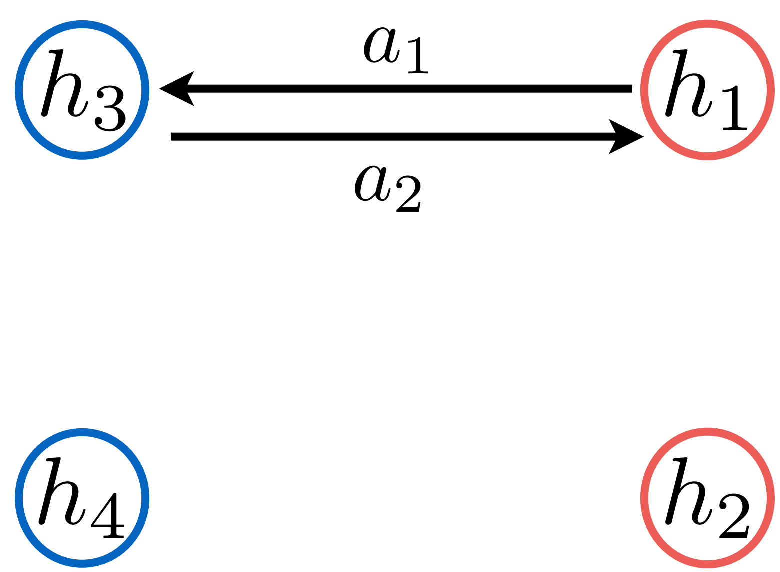
a_1	h_1	$\cancel{h_2}$	h_3	h_4
a_2	h_1	h_3	h_2	
a_3	h_2	$\cancel{h_1}$	h_3	
a_4	h_2	h_4	h_3	





a_1	h_1	$\cancel{h_2}$	h_3	h_4
a_2	h_1	h_3	h_2	
a_3	h_2	$\cancel{h_1}$	h_3	
a_4	h_2	h_4	h_3	





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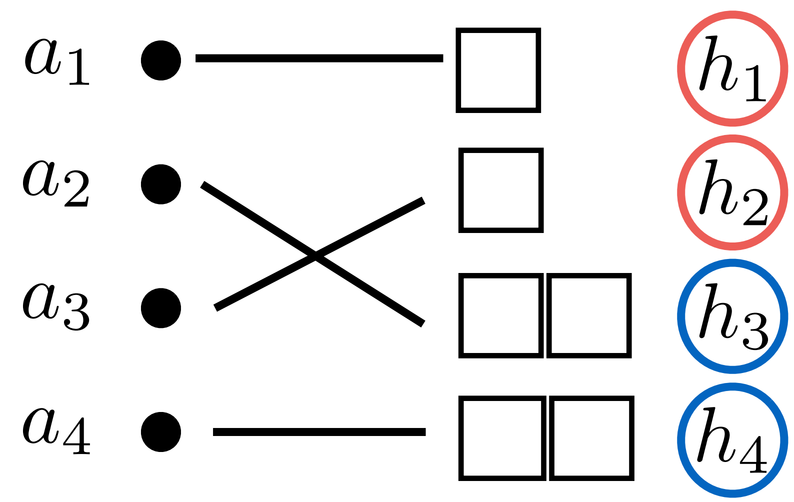
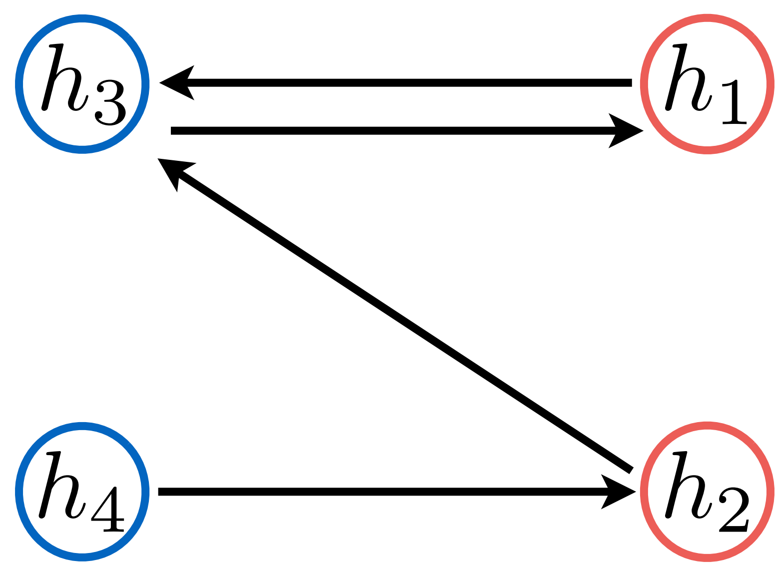
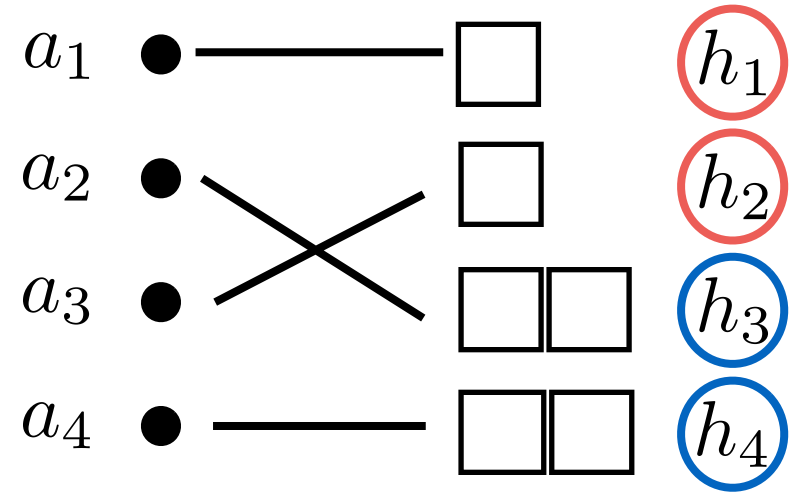
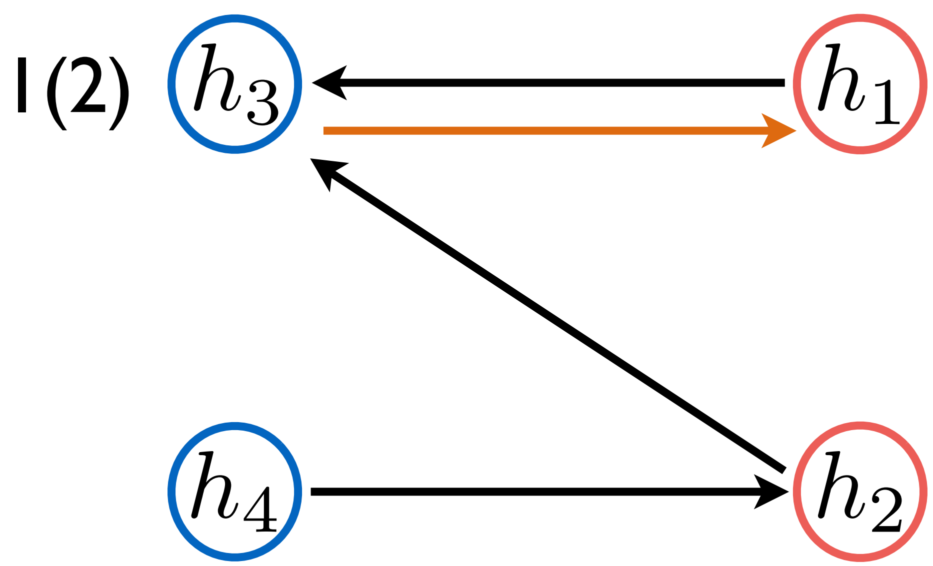




h_1	1
h_2	1
h_3	2
h_4	2





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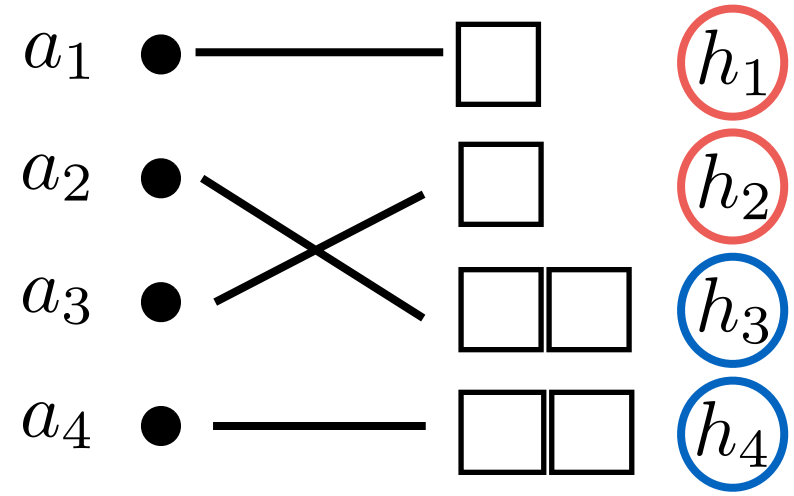
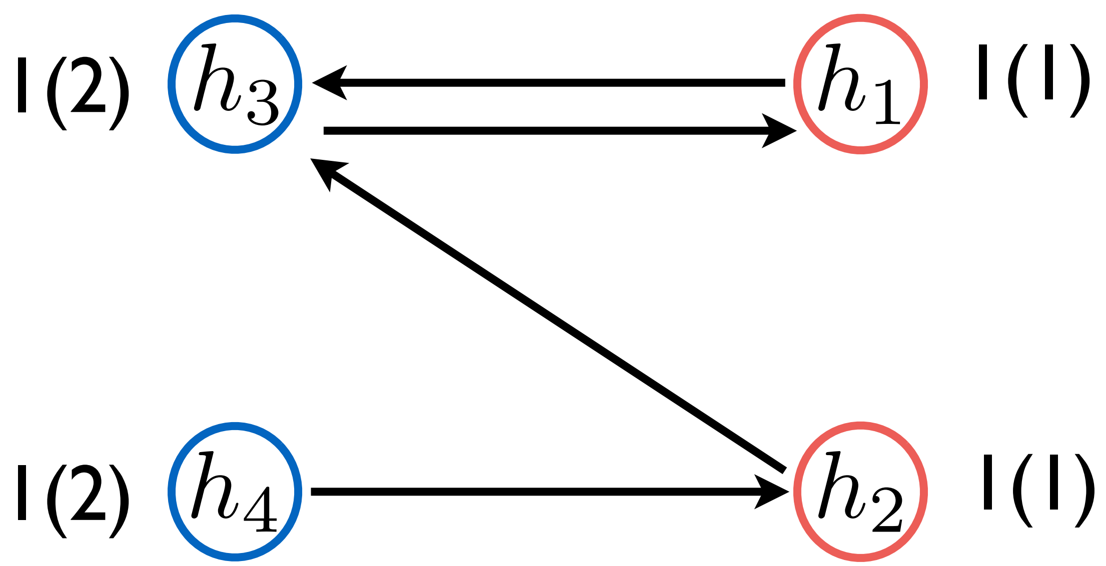
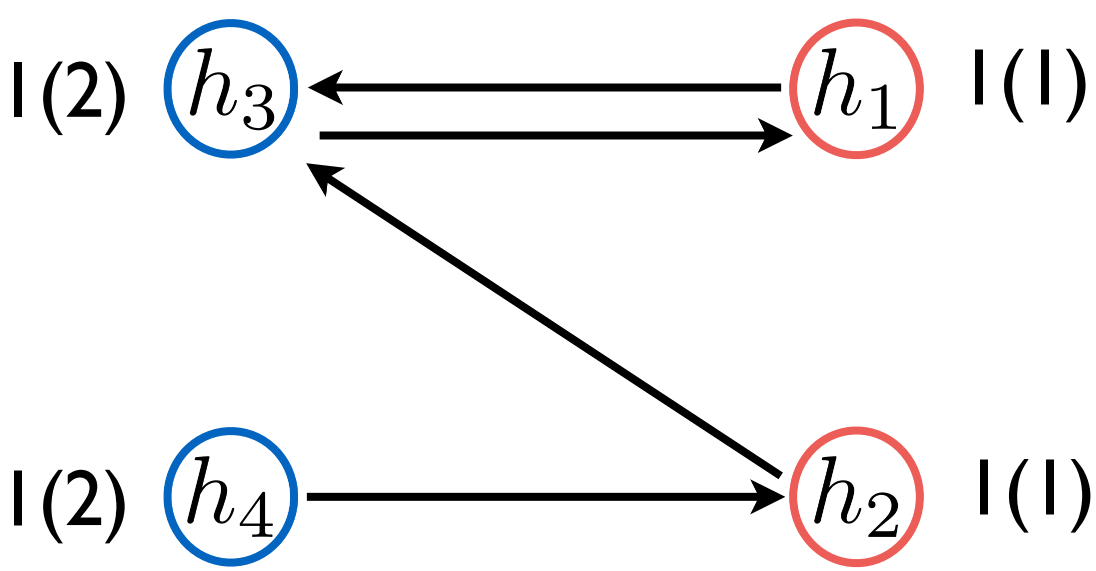
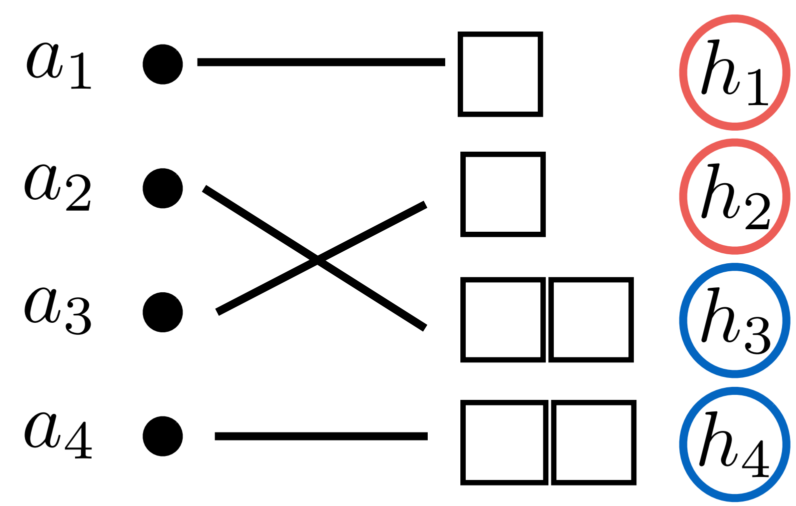




Moving from one Popular Matching to Another

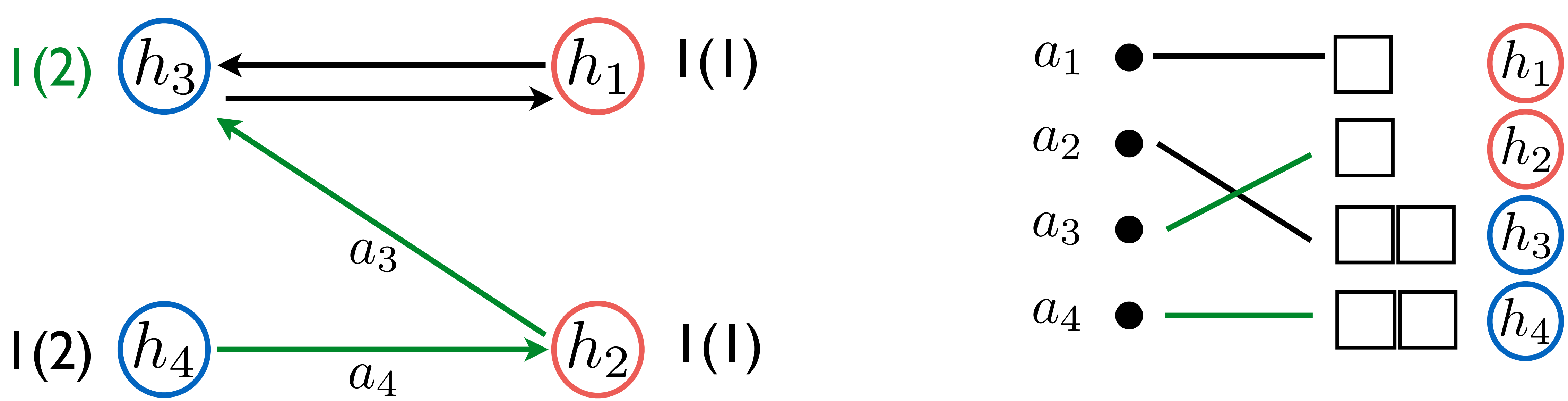




switching graph → popular matching

Moving from one Popular Matching to Another

Trick : Path Reversal



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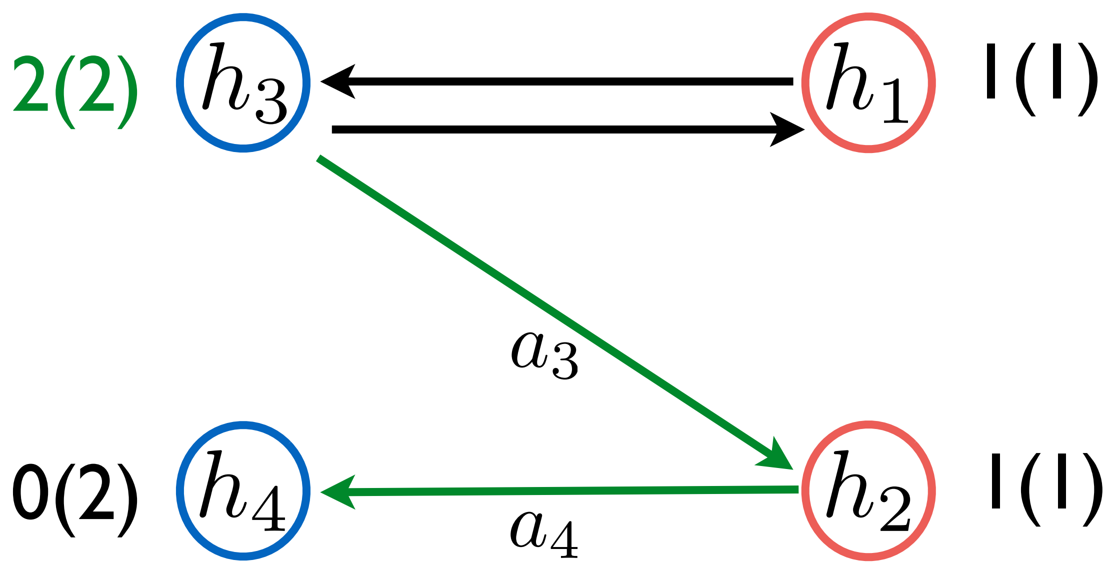

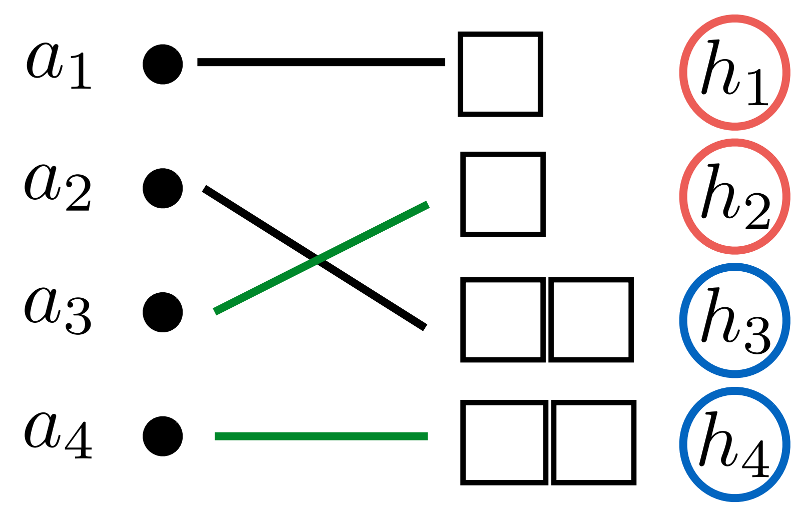

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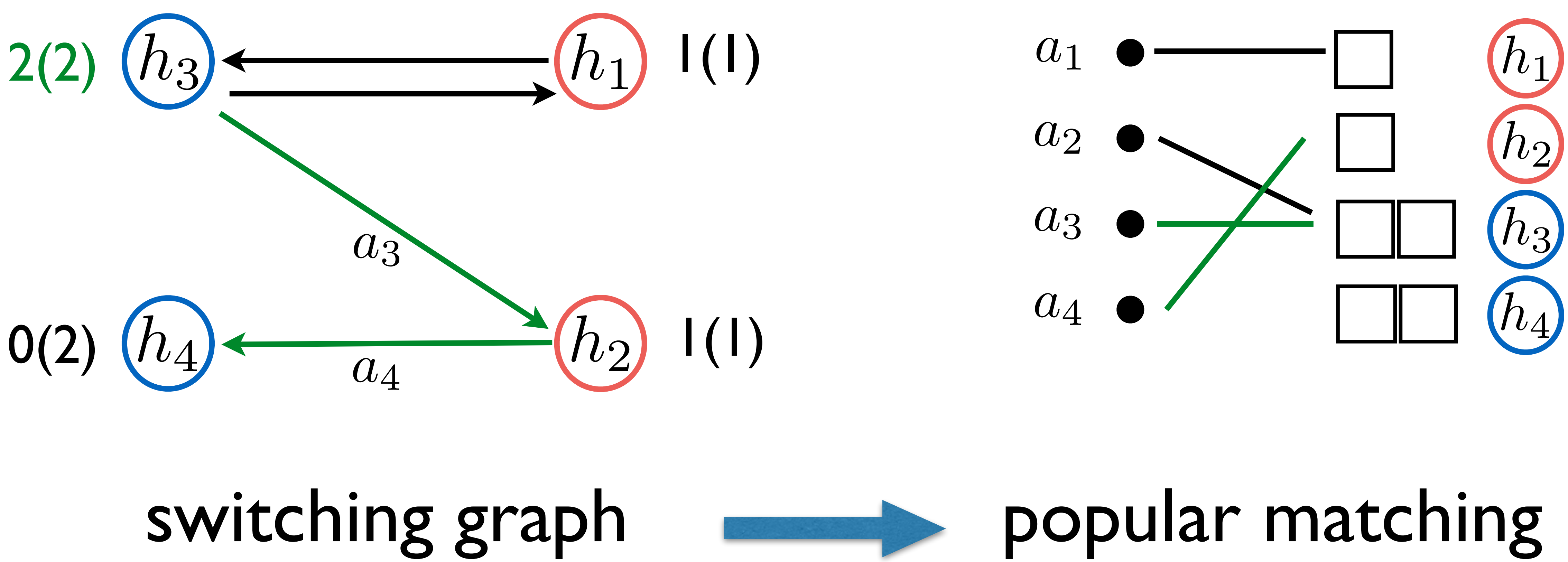

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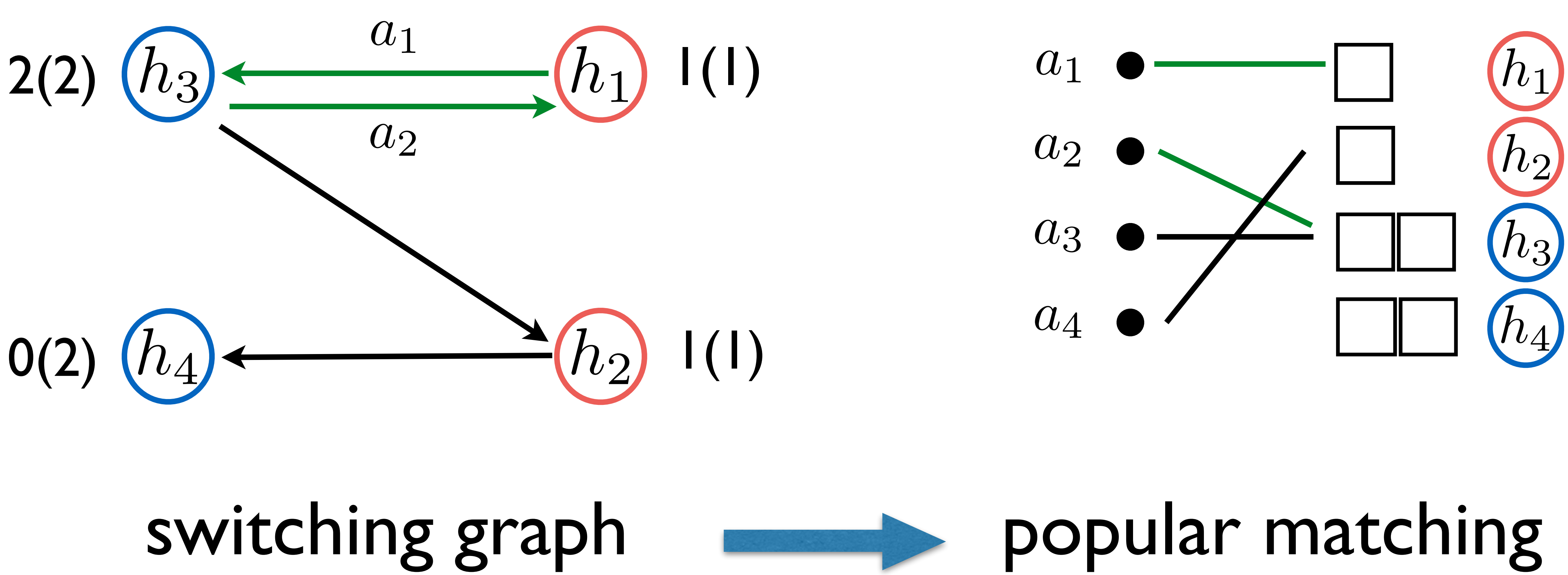
Trick : Cycle Reversal



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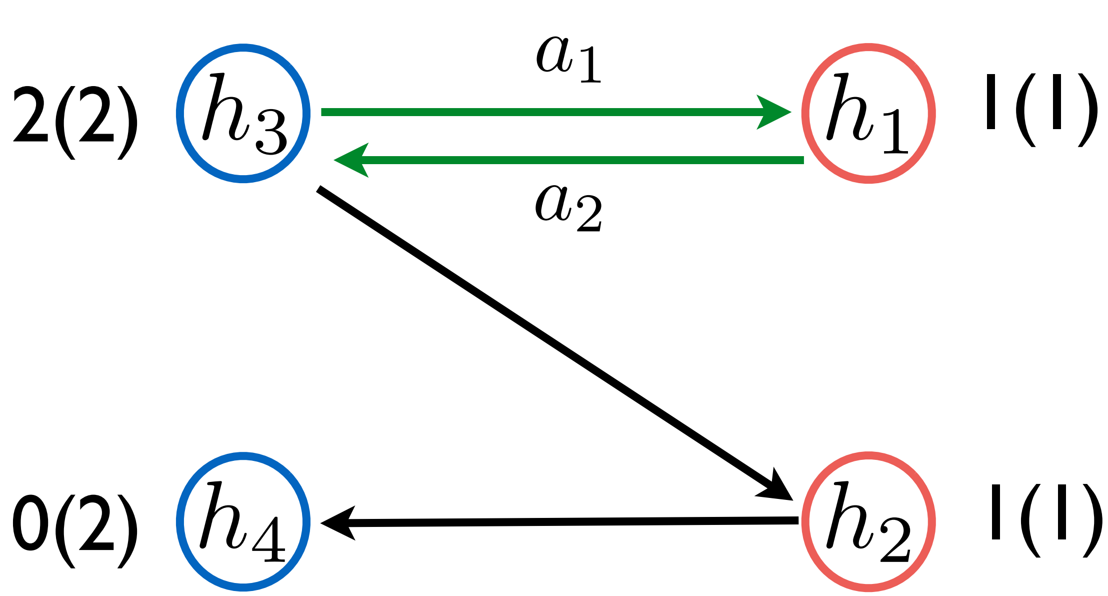

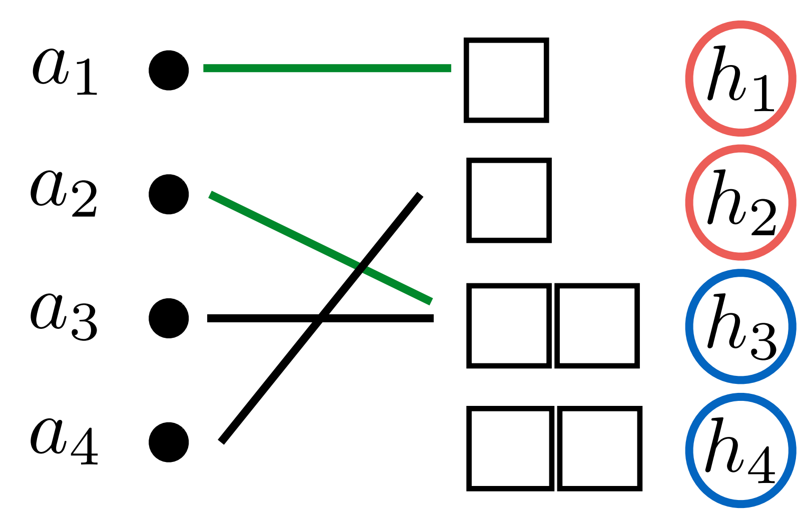

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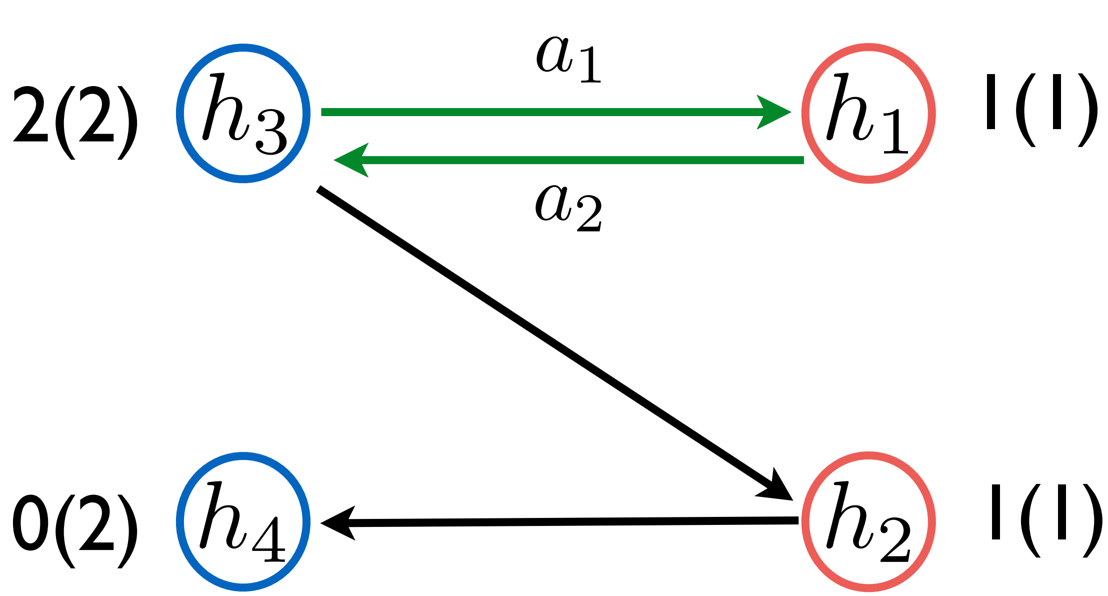

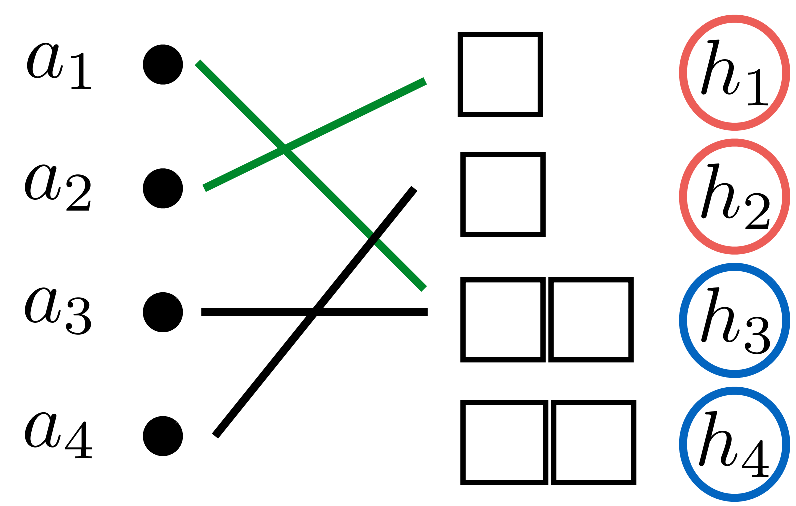

switching graph → popular matching

Moving from one Popular Matching to Another

edge-disjoint union of switching paths and switching cycles could be reversed

Moving from one Popular Matching to Another

edge-disjoint union of switching paths and switching cycles could be reversed

Careful: number of switching paths ending at a house should be less than its remaining capacity

Theorem *If G_M is the switching graph of the CHA instance G with respect to a popular matching M, then*
(i) every switching move on G_M generates another popular matching, and
(ii) every popular matching of G can be generated by a switching move on M.

#P-Hardness of Counting Popular Matchings

matchings in bipartite graph → # popular matchings in no-ties, integer capacity instance

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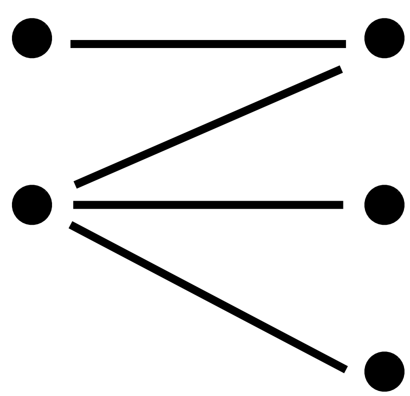

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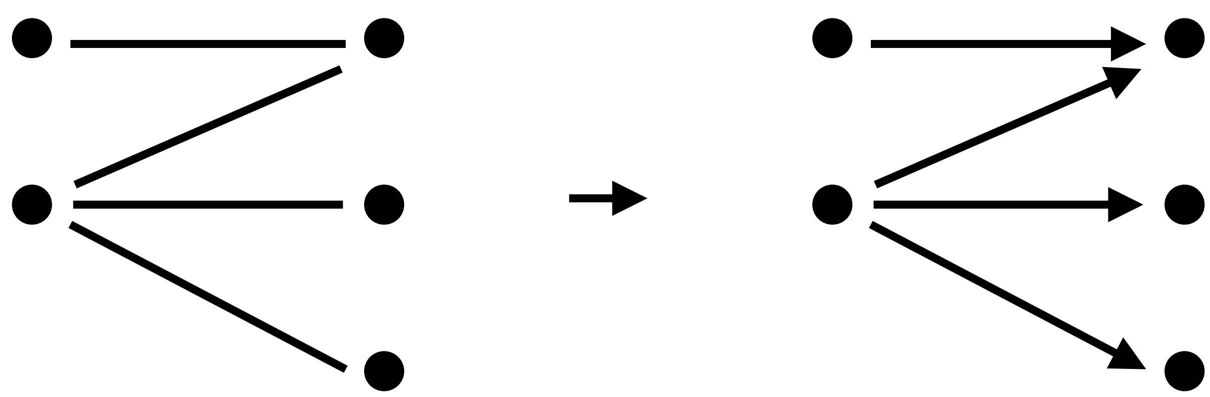

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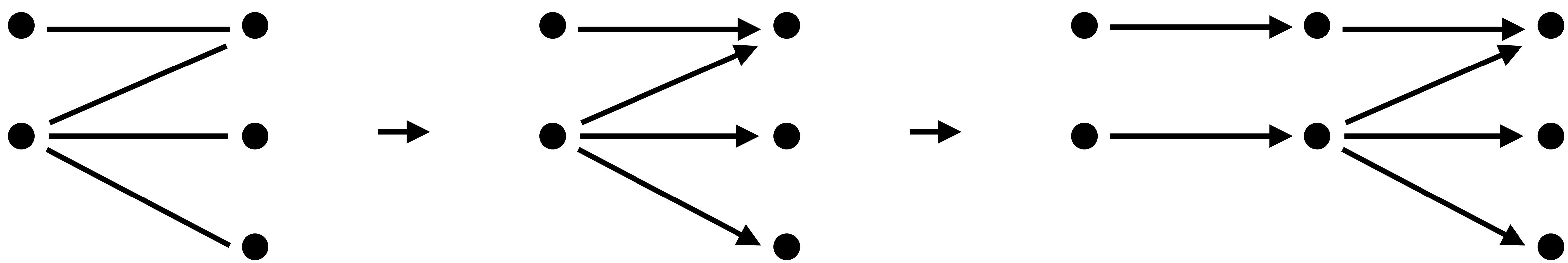

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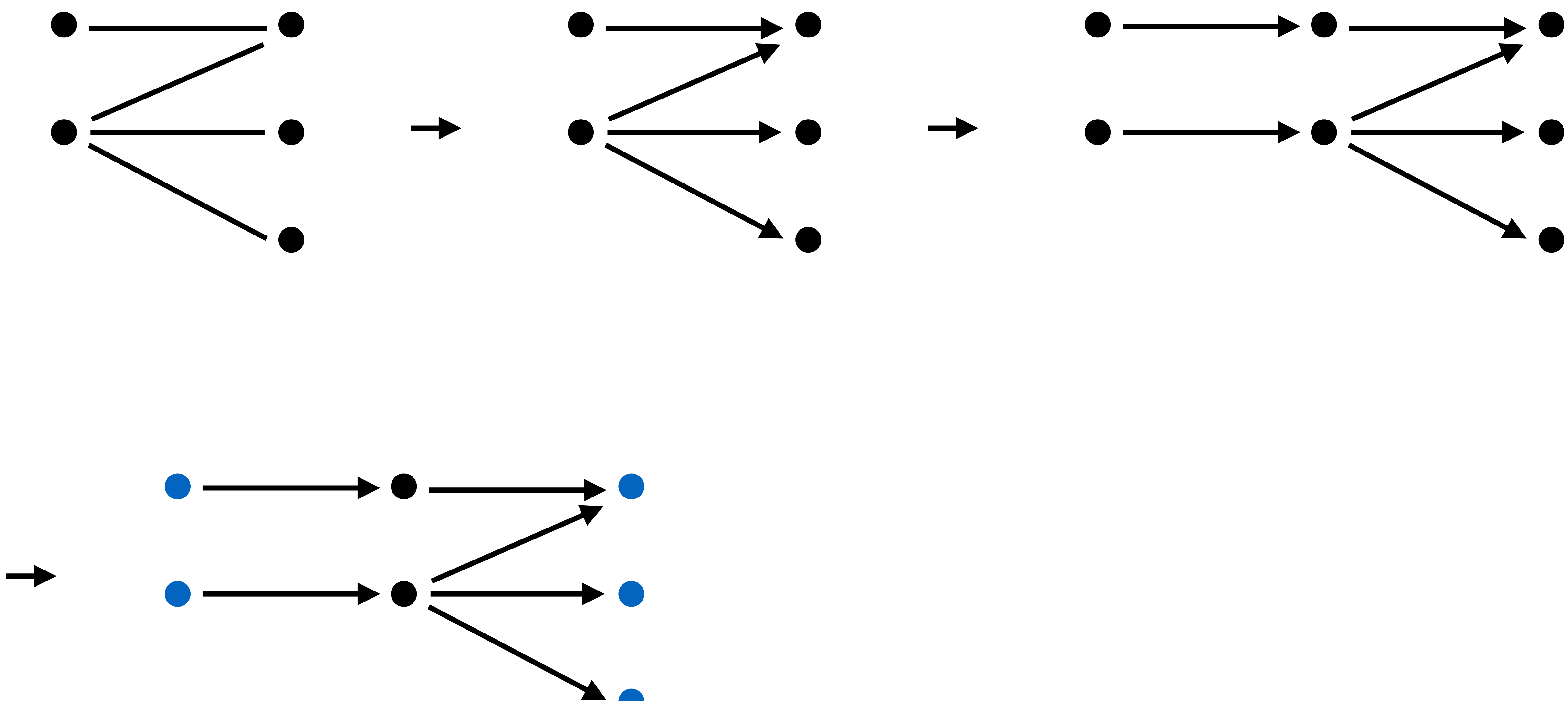

s-houses

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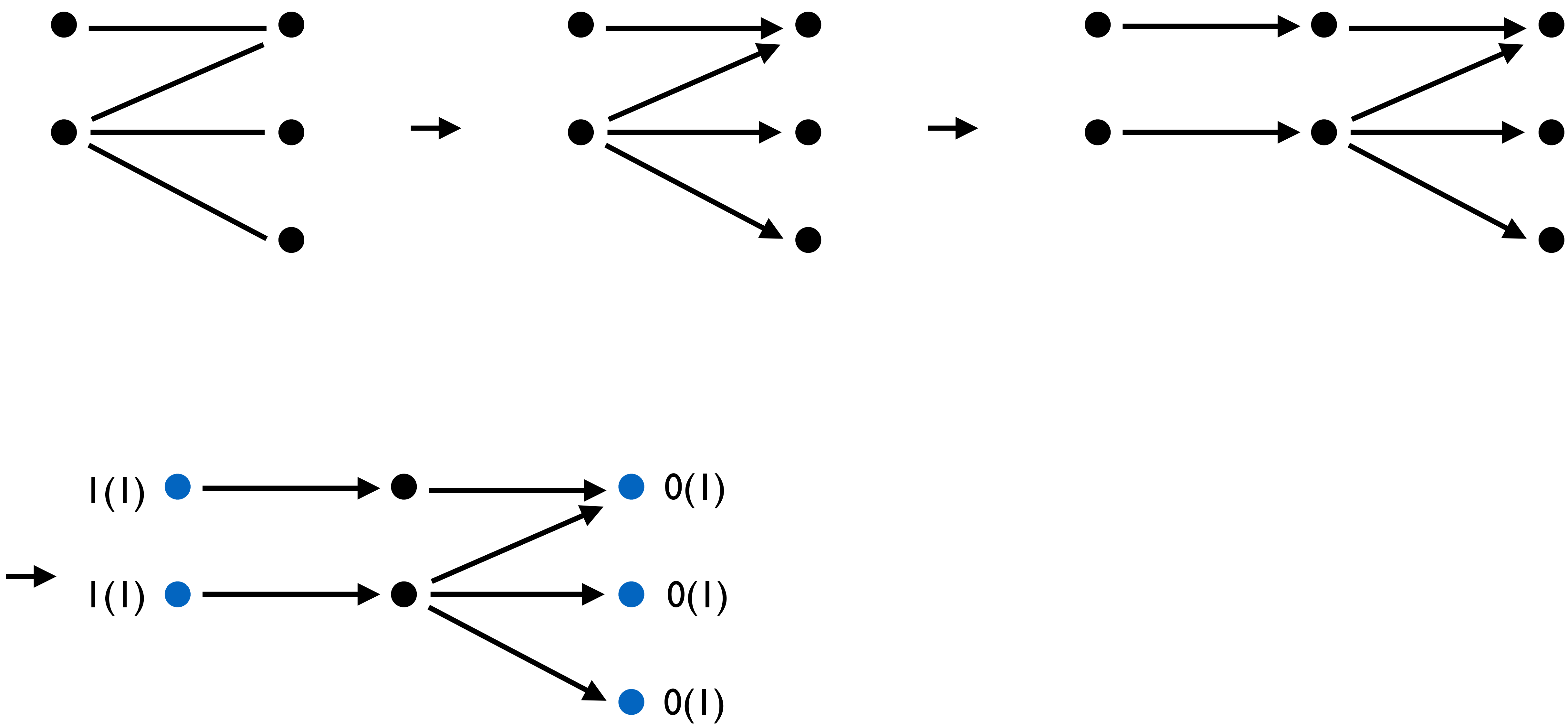

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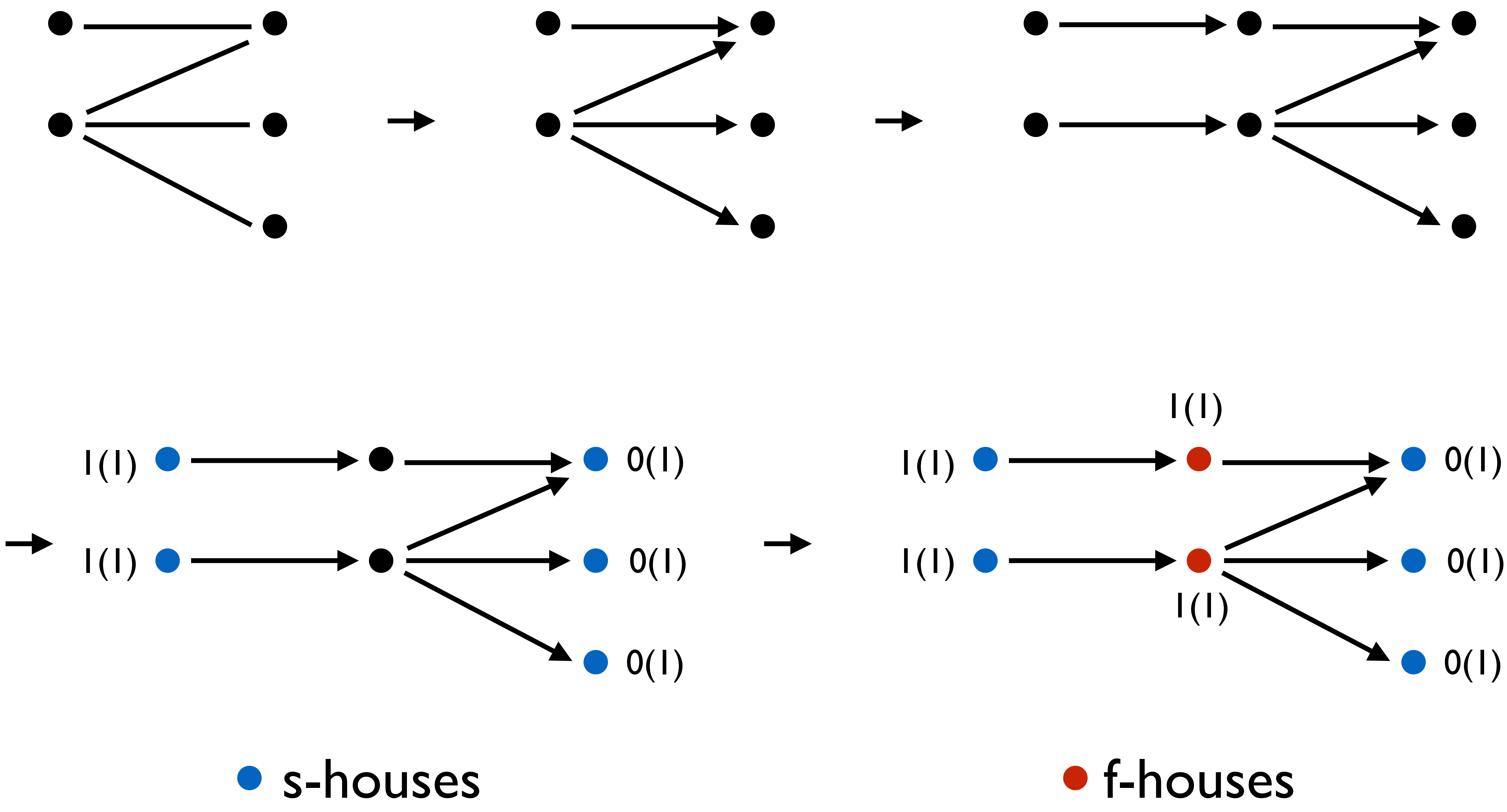

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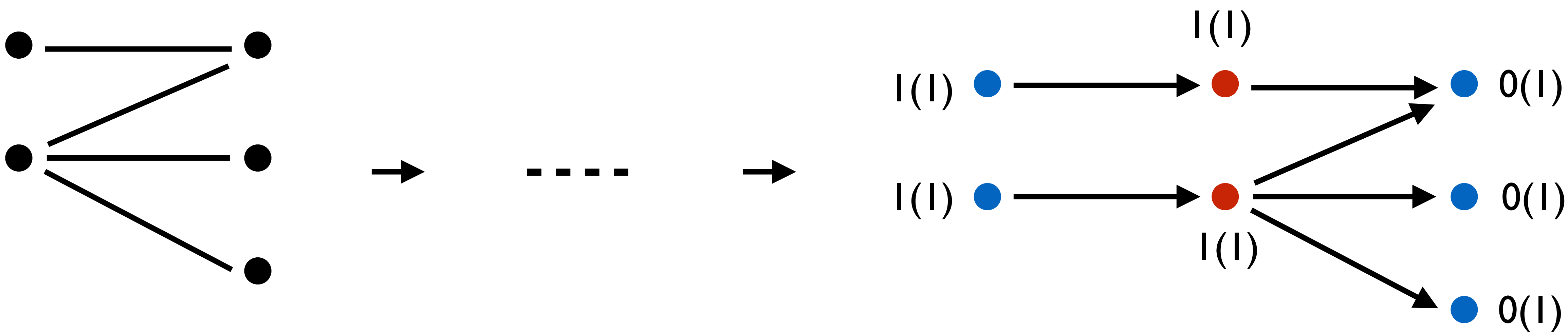

Matching

Switching set

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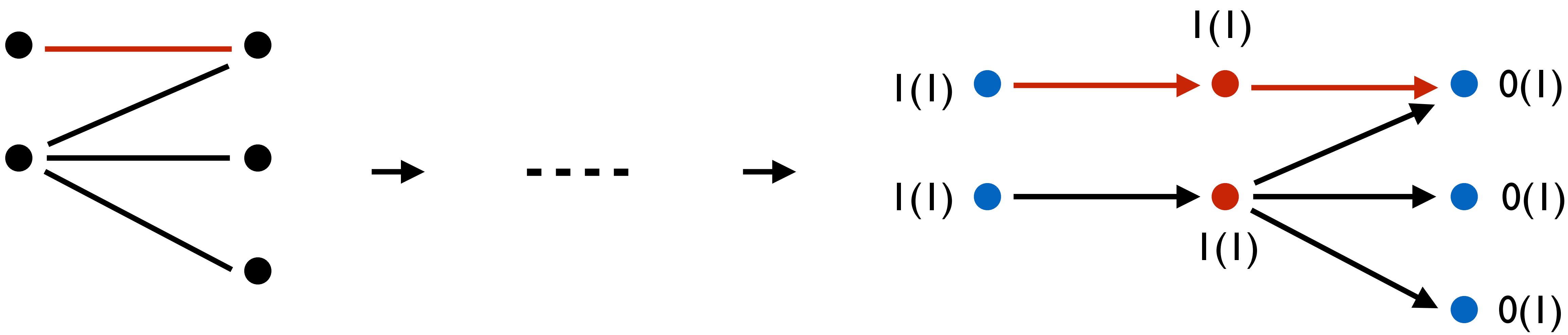

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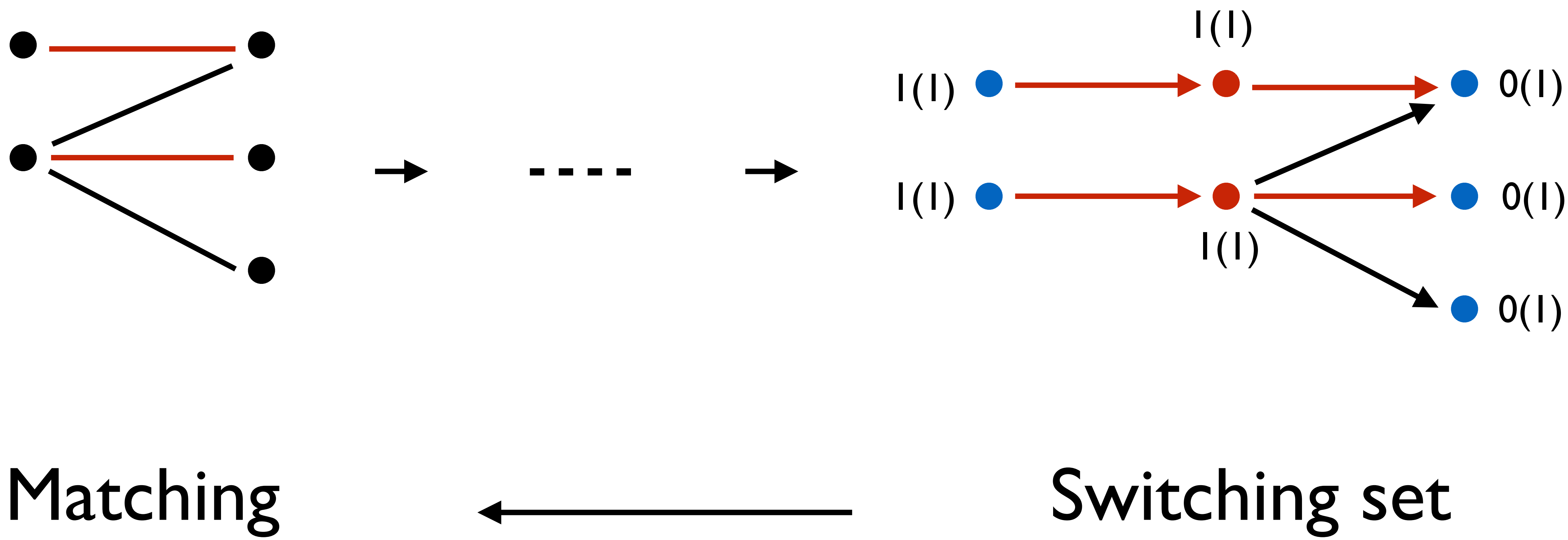

Summary

We Give

- A ‘Switching Graph’ Characterisation for no-ties, capacitated case and prove #P-hardness of counting
- We give an FPRAS for ties case

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Open

Algorithm for Counting in no-ties, capacitated case