Independence results in concrete mathematics

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This project proposal describes my research plans for the near future. It is devoted to the program in Foundations of Mathematics which dates back to Gödel's discovery of Incompleteness. The main goals can be grouped as the eight concrete tasks below. Ramsey-theoretic, well-order-theoretic and well-quasi-order-theoretic considerations have always been the usual sources of independence results, since the pioneering discoveries of J. Paris and H. Friedman. We are going to obtain independence results in these and other parts of 'concrete' mathematics: Ramsey theory (Tasks 4, 5, 6, 7, 8), well-quasi-order theory (Tasks 2 and 3), analytic combinatorics (Task 3), braid theory (Task 1), number theory (Tasks 4 and 7) and the theory of chaotic dynamical systems (Task 5).

In each of the eight tasks, there are already some preliminary results and it is somehow clear how to obtain final theorems.

Objective	Preliminary results and how to proceed
Task 1 Braids Establish logical strength of statements about braids, such as "for every infinite sequence of positive braids, there is an infinite increasing subsequence". Prove $I\Sigma_2$-unprovability of some miniaturisations, e.g., "for every K there is N such that for any sequence $B_1, B_2, \ldots, B_N$ of positive braids such that $\|B_i\| < K + i$, there are $i < j \leq N$ such that $B_i \prec B_j$".	These conjectures are motivated by Dehornoy's ordering $\prec$ of positive braids as ω^{ω^ω} and Burckel's ordering of n-strand positive braids as $\omega^{\omega^{n-2}}$. It is clear how $I\Sigma_2$ proves well-orderedness of braids with a fixed number of strands. Some unprovability results immediately follow from $I\Sigma_2$-unprovability of transfinite induction up to ω^{ω^ω}, some other results need extra braid-theoretic work.
Task 2 Kruskal and GMT Develop model theory for Kruskal's theorem and Graph Minor Theorem.	It is an easy observation in [6] that in a model $M \vDash I\Sigma_1$, if a cut I satisfies the infinite Kruskal theorem (IKT) then I is regular. In [6], we have a sketch that I is also strong, using characterisation of strong initial segments from [12]. The proof uses ultrafilters as in [12] and automatically re-proves that IKT implies ACA_0. It also works if you restrict IKT to binary trees. The next problem is to show how an initial segment satisfying PA can be built out of certain non-standard instances of Finite Kruskal Theorem.

Objective	Preliminary results and how to proceed
Task 3 Phase transition for GMT For every primitive recursive number a, introduce the statement GM_a as: "for all K there is N such that for any sequence of graphs $G_1, G_2, \ldots, G_N$ such that $\lvert G_i\rvert < K + a \cdot \sqrt{\log i}$, there are $i < j$ such that G_i is a minor of G_j". We are planning to prove that there is a real number r such that for every primitive recursive real number a, we have: if $a \leq r$ then GM_a is $I\Sigma_1$-provable but if $a > r$ then GM_a is PA-unprovable. There is a similar conjecture about unprovability threshold for Kruskal with gap condition which can be solved at the same time. A successful completion of this task may shed some light on the long-standing open problem of the actual logical strength of the Graph Minor Theorem.	This conjecture is inspired by Weiermann's Theorem [16]: let KT_a be the statement "for all k there exists N such that if $\{T_i\}_{i=1}^N$ is a sequence of finite trees such and for all $i \leq N$, $\lvert T_i\rvert \leq k + a \cdot \log(i)$ then there are $i < j \leq N$ such that T_i is inf-preserving embeddable into T_j". Weiermann proved that for a certain (presumably transcendental) number r, we have: if $a \leq r$ then KT_a is $I\Sigma_1$-provable but if $a > r$ then KT_a is PA-unprovable. The first unprovability result of Finite Kruskal Theorem was introduced and proved by H. Friedman in [8], for graph minor theorem see [10]. Provability of $\mathrm{KT}_{\frac{1}{2}}$ and unprovability of KT_4 were established by M. Loebl and J. Matoušek [13]. I am planning to prove the Graph Minor version model-theoretically. The plan is to borrow the technique from Task 2 to build initial segments of models out of nonstandard instances of Finite Kruskal Theorem and Graph Minor Theorem. Step 1 is to do the proof for $I\Sigma_1$. Step 2 is to include analytic combinatorics considerations to separate provable instances from unprovable ones. (This involves analysing the behaviour of generating functions on the complex plane.) Step 3 is to notice how the unprovability proof transfers beyond $I\Sigma_1$, at least to PA.
Task 4 Zeta-function Establish analogues of Friedman's sine-principle (see [3]) for Riemann zeta-function. A final theorem may look like: for any $n \geq 2$, the statement "for all m, there is N such that for any set $A = \{a_1, a_2, \ldots, a_N\}$ of rational complex numbers, there is $H \subseteq A$ of size m such that for any two n-element subsets $a_{i_1} < a_{i_2} < \cdots < a_{i_n}$ and $a_{i_1} < a_{k_2} < \cdots a_{k_n}$ in H, we have $\lvert\zeta(a_{i_1} \cdot a_{i_2} \cdots a_{i_n}) - \zeta(a_{i_1} \cdot a_{k_2} \cdots a_{k_n})\rvert < \frac{1}{i_1}$" is unprovable in $I\Sigma_{n-1}$.	In my recent paper [3], we gave unprovability proofs for sharper versions of Friedman's sine-principle, using Rhin-Viola theorem on transcendence degree of π. Here, we plan to prove similar theorems in another context. The proof (if successful) will rely on a lemma similar to Lemma 2 in [3]: for any $\varepsilon > 0$ and any dimension n, any number K and any function $g\colon [K]^n \to [-1, 1]$, there is a set of rational numbers $A = \{a_1, a_2, \ldots, a_K\}$ such that for any $i_1 < i_2 < \cdots < i_n \leq K$, we have $$\lvert g(i_1, i_2, \ldots, i_n) - \sin(a_{i_1} \cdot a_{i_2} \cdots a_{i_n})\rvert < \varepsilon.$$ The abundance of zeroes of zeta-function on the critical line hints at the possibility to prove an analogous lemma.
Task 5 Dynamical system Prove analogues and generalisations of Friedman's sine-principle for chaotic dynamical systems.	One possible shape of a final theorem may be: "there exists a diffeomorphism of a smooth compact manifold which has a countable set of periodic points such that existence of a chaotic invariant set is not provable in Peano Arithmetic". This line is currently pursued jointly with Sergey Kryzhevich (a dynamical systems expert). We shall build examples that are meaningful and interesting for the chaos theory community. There are several other attractive formulations to end up with. One of them uses Sharkovsky's order, another one replaces sine by the logistic map. Logistic map (with large parameter) exhibits very chaotic behaviour which will probably allow me to prove a lemma as in Task 4, so, building upon [3], this independence result seems to be within easy reach.

Objective	Preliminary results and how to proceed
Task 6 Density Approximation of strength of Hindman's and Milliken's theorems by their densities. This will also affect the strength of related theorems, e.g. Auslander-Ellis theorem about dynamical systems.	The strength of RT^2_2 has been recently shown to be approximated by their densities in [4] (i.e., $\mathrm{WKL}_0 + \mathrm{RT}^2_2$ has the same Π^2_0-consequences as $I\Delta_0 + \cup_{n\in\omega}\forall a\exists b\ [a,b]$ is n-dense$(2,2)$). Analogous constructions were designed for Canonical Ramsey Theorem and Regressive Ramsey Theorem (also in [4]). Recently, the approach to strength of infinitary statements via densities was extended (in [5]) from pure Ramsey Theory to cover also Kruskal's theorem, Hilbert Basis theorem and a few other examples. Preliminary results in this Task have been obtained jointly with A. Weiermann [4, 5].
Task 7 Indiscernibles Explore further possibilities with Paris' indiscernibles to produce new unprovability results, hopefully of number-theoretic nature.	As a very rudimentary example, take the usual indiscernibles used in PA-independence proofs (as in [14]) and notice that for $i < j$, the ith and the jth indiscernible elements c_i and c_j will have the same congruence classes modulo all primes up to the c_{i-1}th prime. This observation, combined with usual pigeonhole arguments, has been used to weaken the largeness condition in versions of PH or regressivity condition in versions of KM, see e.g. [1]. However, in the context of PH and KM, this rudimentary argument is weaker than Weiermann's threshold results for PH (see [17] or, for model-theoretic approach, see my paper [2], for sharpest threshold results, see Carlucci-Lee-Weiermann Theorem). We shall use *sieve theory* (learning from [7]) to produce a not-so-trivial argument which will bear number-theoretic consequences.
Task 8 Infinitary Ramsey Theory Establish logical strength of several infinitary statements from modern Ramsey Theory (see Todorcevic's book [15]): e.g. Pudlak-Rödl theorem, Gowers' theorems, some statements about blocks and barriers, about strategically Ramsey sets and about oscillation stability.	In [4], I gave model-theoretic proofs of logical strength of Canonical Ramsey theorem for pairs and Regressive Ramsey theorem for pairs (thus model-theoretically re-proving existing recursion-theoretic results). The three ways to establish logical strength of new infinitary ramseyan principles of [15] are: first to look for (mutual) implications with principles of known strength (e.g., RT, RT^1_n, RT^3) and unknown strength (e.g. RT^2_2, CAC), secondly to try model-theoretic constructions (as in [4] or [12]), thirdly, try density approach as in Task 6.

Let me finally mention some ideas which I shall try to realise in the longer run.

1. It is usually pointless to speculate which classical open problems in mathematics *should* turn out to be unprovable without expert knowledge of those open problems. However, for one such open problem this may be possible. This is Schinzel's **Hypothesis H**, which begs to become an independence result. It says (see [7]): "for any finite collection of irreducible polynomials $F_1(x), F_2(x), \ldots, F_n(x)$ with integer coefficients and such that $\prod_{i\leq n} F_i$ has no fixed prime divisor, there exist infinitely-many integers m such that for all $i \leq n$, $F_i(m)$ are prime". This conjecture is extremely strong and its formulation already provides some necessary ingredients for independence proofs (e.g. the enumeration of all Δ_0-formulas can be extracted).

2. The historical prototypes of PH and the earlier PA-unprovable statements [14] are **large cardinal axioms**. In the case of arithmetic, closedness properties postulated by large cardinal axioms correspond to closedness properties of initial segments of models of arithmetic. It would be good to borrow ideas from modern theory of large cardinals to introduce new kinds of initial segments (other than semi-regular, regular, strong, n-extendible and n-Ramsey) which will give new independence results in arithmetic. Hopefully, even relatively direct arithmetical translations of large cardinals will work.

3. Find a PA-unprovable statement in terms of existence of a winning strategy in **a game of Noughts and Crosses** (on an unbounded many-dimensional board). We will attempt to do it via finding an unprovable version of the Hales-Jewett theorem or by directly formulating the rules in a way that ensures that for a certain game, the resulting "generalised diagonal" can be converted into our desired set of indiscernibles. Recall that first examples of indicators were also built in terms of games between two players (see [14, 12]).
4. Learning to build **models of strong theories** (e.g. ZF or extensions of ZF as in [9], [11]) directly, 'by hands'. (Clearly, this will be a construction of a countable oriented graph.) The additional principles which we shall need to employ in order to make the direct construction work will be the unprovable statements we are seeking.

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Curriculum Vitae

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Junior Researcher	Research Assistant
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Academic Degrees

Ph.D., awarded on December 13th, 2000, University of Birmingham, UK
Title: *On order-types of models of arithmetic*. Supervisor: Dr. Richard Kaye.
M.Phil. Qual. January 1998, University of Birmingham, UK
1993 -1996: Undergraduate studies in St. Petersburg State University.

Visiting positions

Fall 2000: *Visiting researcher*, Institut Mittag-Leffler, Sweden.
February 2001: *Visitor*, University of Helsinki, Finland.
March-December 2001: NATO-PC Advanced Fellowship, *Postdoctoral Fellow*, Istanbul Bilgi University, Turkey.
February 2002-July 2002: *Visitor*, University of Helsinki.
NWO-funded research visits to Utrecht University in 2004 and 2005.

Academic positions

Since September 2002: *Junior Researcher*, St. Petersburg department of Steklov Mathematical Institute, St. Petersburg. (Currently on leave.)
July 2004 -now: *Research Assistant*, Department of Computer Science, University of Liverpool, England.

Conferences organised

Co-organising A.A. Markov Centenary Conference in St. Petersburg (2003), organising (together with A. Vershik and Yi Zhang) the series of meetings “Methods of Logic in Mathematics” (St. Petersburg, 2004 and 2005), organising this year (together with A. Weiermann and L. Carlucci) a workshop “Logic, Combinatorics and Independence results” in Oberwolfach (26 November – 2 December 2006).

Talks

One-hour talks in London (Set Theory and its Neighbours meeting, 2000), at Institut Mittag-Leffler Seminar (2000), in Istanbul (Model Theory and Algebra Conference, 2001), at Bosphorus University (2001), Helsinki Logic Seminar (2001, 2002), St. Petersburg Logic Seminar (2002, 2003, 2004, 2006), Utrecht Logic Lunch (2004), Liverpool Logic Seminar (2004), a talk in Pec pod Snezkou Fall School (2005), contributed talks at LC2003 and LC2004. Invited speaker at the "Methods of Logic in Mathematics 2005" conference in St. Petersburg. I shall be a Proof Theory session speaker at Logic Colloquium 2006.

Publications

- On order-types of models of arithmetic and a connection with arithmetic saturation. (2004). *Lobachevskii Journal of Mathematics*, **16**, pp. 3-15.
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- Resplendent linear orders. (2002). Joint with Richard Kaye. *Unpublished*. Downloadable from `http://logic.pdmi.ras.ru/~andrey/research.html`.
- Model-theoretic treatment of threshold results for PH. (2005). Downloadable at `http://logic.pdmi.ras.ru/~andrey/research.html`.
- Unprovability threshold of Friedman's sine-principle. (2006). Downloadable at `http://logic.pdmi.ras.ru/~andrey/research.html`.

In preparation:

- The strength of infinitary statements can be accessed by their densities. (2006). Joint with Andreas Weiermann.
- Model theory of Kruskal's theorem. (2006).

Also, I recently co-authored two articles in theoretical computer science.