

Long games on braids

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Abstract

This note is an exposition of a connection between the well-ordering of braids and ramseyan unprovable statements PH^2 and PH^3 , the Paris-Harrington principles for pairs and triples. We introduce two games played on positive braids. The first game is played on 3-strand positive braids and its termination time is Ackermannian in the input. We give a combinatorial and a model-theoretic proof (using semi-regular cuts in models of arithmetic). The second game is played on arbitrary positive braids and its termination is unprovable in $\text{I}\Sigma_2$, the two-quantifier induction arithmetic. We provide proofs of the result using ordinals and the method of indicators (building a 2-extendible cut in a model of arithmetic). The results were inspired by Dehornoy's well-ordering of positive braids of order-type ω^{ω^ω} .

The n -strand braid group B_n is a group with the following presentation:

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1}; \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| \geq 2, \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ for } |i-j| = 1 \rangle.$$

A braid is called positive if it has a representation without σ_i^{-1} for any i . We denote the set of positive n -strand braids by B_n^+ .

Many phenomena about braids were explained by the discovery by Dehornoy in 1992 of a left-invariant linear ordering of positive braids (see [7]). Laver later showed that Dehornoy's ordering is a well-ordering when restricted to positive braids [14]. Burckel showed that the order-type is ω^{ω^ω} (the reason for this order-type of braids essentially coming from Higman's Lemma). This well-ordering led us to our unprovable statements about braids in this article.

For each of our theorems we provide two proofs: model-theoretic and ordinal-theoretic to cater for mathematicians' different tastes and intuitions. The model-theoretic part of the article was inspired by the treatment of α -large sets by Paris in [15], the ordinal-theoretic side follows the approach uses standard methods such as fundamental sequences and fast-growing hierarchies. Many experts will see connections with Paris-Kirby hydra games from [13], Beklemishev's Worm Principle [2, 6] and reductions of α -large sets. Indeed, our theorems were modelled after theorems about α -large sets but adjusted to produce the most natural games on braids that reduce the braids with respect to the ordering of braids.

At the dawn of logic, early logicians, most notably K. Gödel and R. Goodstein wrote about ordinal descent through ω^{ω^ω} with justification why this is an acceptable mathematical principle. This ordinal was perceived as the biggest ordinal that allows for a convincing verbal "justification" of why the corresponding

*The authors were supported by grant number 613080000 of the NWO to participate in a workshop in Utrecht University in July 2006.

ordinal descent principle is true. The fact that all existing concrete mathematics can be conducted in $\text{I}\Sigma_2$ can be seen as one reason why the well-ordering of positive braids looked so mysterious. The bijection between braids and ordinals can be done constructively but the proof of well-orderedness necessarily requires means stronger than those formalizable in $\text{I}\Sigma_2$. Our first game terminates in Ackermannian time. Termination of our second game is equivalent to PH^3 , the Paris-Harrington Principle for triples which is equivalent to 1-consistency of $\text{I}\Sigma_2$.

This article is intended for two audiences: logicians (who will see yet another connection between logic and concrete combinatorial mathematics, a connection related to hydra-games or worm games in a very natural mathematical setting) and braid-theorists (to see logical and complexity-theoretic effects of the well-ordering of braids).

Originally, we planned this article to start with unprovable statements about braids and then turn to reformulations of these statements into shapes that have some amount of physical and geometrical meaning, using the abundance of connections of braids with physics, geometry and topology. But later we decided that this is a topic of its own and dealt only with combinatorial treatment of braid games.

From the point of view of logic, the theorems of this paper are not very difficult: the methods we are using date back to the 1970s and 1980s. However, we see its importance in connecting these methods with such concrete and important mathematical objects as braids and hope that they will in the future extend further to unprovable statements (of combinatorial character) in topology, geometry and physics.

We describe our long processes in a deterministic language but a minor alteration (the first player chooses a nontrivial block of crossings and removes one letter, then the second player creates a new block in its place) turns this process into a game. We opted for a deterministic presentation in order to simplify presentation without missing out on the essence.

There have been many examples of unprovable arithmetical statements in the last 30 years. A lot of these examples, especially the ones at levels of low strength, should not be viewed as final conclusive results but rather as steps of our developing understanding of arithmetical unprovability and of more and more emerging connections with important classes of concrete mathematical objects. Eventually, this long line of developments should lead to ultimate independence results of the future: first-order arithmetical statements φ such that both φ and $\neg\varphi$ are equally beautiful and plausible (possibly describing opposite pictures of the mathematical universe) but unprovable in axiomatic systems of different huge (and possibly incomparable) strengths. Mathematical logic is still gearing up for this ultimate breakthrough.

1 Preliminaries

A positive braid word with n -strands is a word in the alphabet $\{1, \dots, n-1\}$, the set of all words being denoted by Σ_n^* . A positive braid is a (finite) equivalence class of braid words. We shall write a positive braid word in its general form as $b_n^{m_n} b_{n-1}^{m_{n-1}} \dots b_1^{m_1} b_0^{m_0}$, where $n \geq 0$, $m_n, \dots, m_1 \geq 1$ and for all i , $b_i^{m_i}$ and $b_{i+1}^{m_{i+1}}$ are blocks of distinct letters. By a non-trivial block we mean a block of length ≥ 3 . By the first non-trivial block in a word w we mean the rightmost

non-trivial block in w if it exists and the leftmost block in w otherwise.

We denote by $>_D$ Dehornoy's left-invariant well-order of B_∞^+ . Burckel defined a bijection between braid words and uniform trees (trees with all leaves at the same level) and then used the fact that uniform trees are well-ordered under the short-lexicographic order $<_{\text{ShortLex}}$ to prove that the order type of $(B_n^+, >_D)$ is $\omega^{\omega^{n-2}}$. A notion of *irreducibility* for words is essential in Burckel's work. Burckel showed that for all n

1. every braid $b \in B_n^+$ has an irreducible representative $n_b \in \Sigma_n^*$;
2. the ordinal of n_b is the smallest ordinal assignable to b ;
3. $a >_D b$ if and only if $n_a >_B n_b$.

The notion of irreducibility is very simple for braids in B_3^+ but involved in the general case. Burckel's analysis has given us the key insights to design our long games. Preserving irreducibility at each step while keeping the games as simple as possible has been a major concern.

2 Easy case: a game on 3-strand braids whose termination time is Ackermannian

Nowadays, nobody can be surprised at the sight of a function of Ackermannian growth (and thus $\text{I}\Sigma_1$ -unprovability of its totality). However, we include full treatment of this case because of its simplicity and because it is an important step in understanding the general case.

Let us first define a game on braid words and then observe that, once we start with an irreducible word for a braid then the rest of the game is played on irreducible words. Hence, the game is truly a game on braids, not just on words.

Rules of Game 1

For each word $a = b_n^{m_n} b_{n-1}^{m_{n-1}} \dots b_1^{m_1} 1^{m_0}$ in the alphabet $\{1, 2\}$ and $n \in \mathbb{N}$ we define the reduct $a[n]$ of a as follows.

- If $a = a_0 1$, then $a[n] = a_0$.
- Otherwise, let $b_i^{m_i}$ be the first non-trivial block of a . Put

$$a[n] = b_n^{m_n} \dots b_i^{m_i-1} b_{i-1}^{m_{i-1}+n} \dots 2^{m_1} 1^{m_0}.$$

For $k \leq \ell$ we denote $w[k][k+1] \dots [\ell]$ by $w[k : \ell]$. The following assertion expressing termination of Game 1 will be our first unprovable statement: "for every braid a in B_3^+ and any natural number x , there is y such that $n_a[x : y]$ is the zero braid".

Termination proof

We show that Game 1 always terminates with the empty word, using Burckel's map ord between words in $\{1, 2\}$ and ordinals below ω^ω in Cantor. The word $b_n^{m_n} b_{n-1}^{m_{n-1}} \dots 2^{m_1} 1^{m_0}$, where b_i is 1 if i is even and 2 if i is odd, $m_n, \dots, m_1 \geq 1$, maps to the ordinal

$$\omega^n \cdot m_n + \omega^{n-1} \cdot (m_{n-1} - 1) + \dots + \omega \cdot (m_1 - 1) + m_0.$$

Let $e_0 = \epsilon$, $e_1 = 2$, $e_2 = 12$, $e_{n+1} = 1e_n$ if n is even, and $2e_n$ if n is odd. Clearly, $\text{ord}(e_n) = \omega^n$ for $n > 0$. Notice also (using the definition of the mapping ord) that if

$$a = \underbrace{b_n^{m_n} b_{n-1}^{m_{n-1}} \dots b_{i+1}^{m_{i+1}}}_c \underbrace{b_i^{m_i} \dots 2^{m_1} 1^{m_0}}_d$$

then $\text{ord}(a) = \text{ord}(ce_i) + \text{ord}(d')$, where $d' = d[b_i^{m_i}/b_i^{m_i-1}]$.

Proposition 1. For any word a in the alphabet $\{1, 2\}$ and any $k \in \mathbb{N}$, we have $\text{ord}(a) > \text{ord}(a[k])$.

Proof. If $a = c1$ then the conclusion is straightforward since $\text{ord}(a) = \text{ord}(c) + 1$. If the last letter of a is 2 then we shall distinguish two cases. Case 1: a contains a non-trivial block. Let $a = db_i^{m_i} b_{i-1}^{m_{i-1}} \dots 2^{m_1}$, with $m_i \geq 3$ and $m_{i-1}, \dots, m_1 \leq 2$ and possibly empty d . The ordinal of a is

$$\text{ord}(de_i) + \omega^i \cdot (m_i - 1) + \dots + \omega \cdot (m_1 - 1)$$

but the ordinal of $a[k]$ is $\text{ord}(de_i) + \omega^i \cdot (m_i - 2) + \omega^{i-1} \cdot (m_i + k) + \dots + \omega \cdot (m_1 - 1)$, which is smaller. Case 2 (a contains no non-trivial block) is similar: the leading term in the Cantor normal form of $\text{ord}(a)$ is $\omega^n \cdot m_n$ but the leading term in $\text{ord}(a[k])$ is $\omega^n \cdot (m_n - 1)$. \square

Game 1 is a game on braids

We now show that the rules of Game 1 preserve irreducibility of words. This implies that Game 1 – when started on an irreducible word – is a game on braids and not only on braid words: at each step, we go from a braid word to a braid word for a braid that is smaller with respect to Dehornoy's ordering of braids.

Definition 2 (Burckel, [4]). A word $b = b_n^{m_n} b_{n-1}^{m_{n-1}} \dots 2^{m_1} 1^{m_0}$ in the alphabet $\{1, 2\}$ is irreducible if $m_{n-1}, \dots, m_2 \geq 2$.

For example, 212 is reducible since $212 = 2^{m_3} 1^{m_2} 2^{m_1} 1^{m_0}$, with $m_2 = 1$ but 121 is irreducible since $121 = 1^{m_2} 2^{m_1} 1^{m_0}$.

Proposition 3. For any word a in the alphabet $\{1, 2\}$, if a is irreducible then $a[k]$ is irreducible for all $k \in \mathbb{N}$.

Proof. Let a be an irreducible word. Case 1: $a = a'1$. Then $a[k] = a'$ which is again an irreducible word. Case 2: a does not end in 1. Let $b_i^{m_i}$ be the first non-trivial block of a . Since a is irreducible, it has the form $a = \dots b_i^{m_i} b_{i-1}^{m_{i-1}} \dots 2^{m_1}$, with $m_{i-1}, \dots, m_2 = 2$. Then $a[k]$ is $\dots b_i^{m_i-1} (b_{i-1})^{m_{i-1}+k} \dots 2^{m_1}$ if $i > 1$ and $\dots 2^{m_1-1} 1^k$ if $i = 1$. Obviously, irreducibility is preserved since, the non-leftmost blocks of length ≤ 2 are trivial and $m_i \geq 3$ if $b_i^{m_i}$ is not the leftmost block in a . \square

Thus, we have that if a is irreducible, then $\text{ord}(a) > \text{ord}(a[k])$ implies $\alpha >_D \alpha'$, where $\alpha, \alpha' \in B_3^+$ are such that $a \in \alpha$ and $a[k] \in \alpha'$. This is so by Burckel's result and by irreducibility of $a[k]$.

Length of Game 1

For a word a in the alphabet $\{1, 2\}$, let $h_a(x) = \min\{y \mid a[x][x+1] \dots [y] \text{ is the empty word}\}$.

Proposition 4. The function $n \mapsto h_{(1122)^n}(n)$ is Ackermannian.

Proof. We can directly compare $n \mapsto h_{(1122)^n}(n)$ with the following version of the Ackermann function. Put $F_1(x) = 2^x + 1$, $F_{n+1}(x) = F_n^x(x)$ and let us show that $h_{(1122)^n}(x) \geq F_n(x)$. It is easy to check by hands that $h_{(1122)}(x) > 2^x$. Let us now show that

$$h_{(1122)^{n+1}}(x) \geq h_{(1122)^n}^x(x).$$

Indeed, $(1122)^{n+1}[x] = 12^{x+2}(1122)^n$. Then in the block 2^{x+2} , the destruction of the first letter 2 needs at least $h_{(1122)^n}(x+1)$ steps and destruction of the i th letter 2 needs at least $h_{(1122)^n}^i(x+1)$ steps. \square

Model-theoretic proof

In a model $M \models \text{I}\Delta_0 + \text{exp}$, an initial segment I is called semi-regular if for every subset S coded in M , if $|S| \in I$ then S is bounded in I . By indicator theory [15] we know that if a Δ_0 -formula $\varphi(x, y)$ is such that whenever $M \models \varphi(a, b)$ then there is a semi-regular cut between a and b then $\text{I}\Sigma_1$ does not prove $\forall x \exists y \varphi(x, y)$. Let us now give a model-theoretic proof that termination of our game is unprovable in $\text{I}\Sigma_1$.

Proof. Let us show that if $M \models \text{I}\Sigma_1$ is countable and nonstandard, $d > \mathbb{N}$ and $(1122)^d[a][a+1] \dots [b]$ is the zero braid then there is a semi-regular initial segment between a and b . Let $\langle S_i \mid i \in \omega \rangle$ be an enumeration of all M -coded subsets of M of size less than b such that each set appears infinitely-often. Put $a_0 = a, b_0 = b$ and suppose that a_n and b_n have been already defined, such that $(1122)^{d-n}[a_n][a_n+1] \dots [b_n]$ is the zero braid. If $|S_n| > a_n - 2$ then put $a_{n+1} = a_n$ and $b_{n+1} = b_n$. Otherwise, since $h_{(1122)^{d-n}}(a_n) \geq h_{(1122)^{d-n-1}}^{a_n}(a_n)$, each of the a_n segments $J_i = [h_{(1122)^{d-n-1}}^i(a_n), h_{(1122)^{d-n-1}}^{i+1}(a_n)]$ for $i \leq 0 < a_n$, kills the braid $(1122)^{d-n-1}$. By pigeonhole principle, there is $i > 0$ such that $J_i \cap S_n = \emptyset$. Since a_n are increasing, we can put $I = \sup_{n \in \omega} a_n$ and notice that I is semi-regular. \square

Another model-theoretic way to prove our theorem would be to show that there is a regular initial segment between a and b . Remember the fact from [15] that regular and semi-regular cuts are symbiotic (occurring or not occurring simultaneously between any two points in a model). Yet another model-theoretic approach would be to build a set of indiscernibles between a and b and then a model of $\text{I}\Sigma_1$ out of it, as was done for example in [1] for α -large sets or originally done for PH^2 , the Paris-Harrington principle for pairs. As is typical at this low level of logical strength, model-theoretic proofs don't give us extra information. Notice that in the model-theoretic proof above, we had to borrow the crucial inequality from the combinatorial proof of Ackermannianity. The same situation has so far occurred in the treatment of all other unprovable statements at this level. (However, the situation is radically different in cases of high logical strength.)

3 A game on braids whose termination is unprovable in $I\Sigma_2$

In principle, it is nowadays easy to define games on words whose termination is unprovable in $I\Sigma_2$ or even in PA. The difficulty here lies in relating the game to the existing well-ordering of positive braids. The notion of irreducibility for braid words is involved and depends on Burckel's bijection between words and uniform trees. This makes it a non-trivial task to design a game of the right complexity that is stated in terms of braids (not ordinals or trees) and decreases the braid along $<_D$.

Waves and Crossings

If $a < b$, we let $w_{[a\uparrow b]} := a(a+1)^2 \dots (b-1)^2 b^2 (b-1)^2 \dots (a+1)^2 a$, and $w_{[b\downarrow a]} := b(b-1)^2 \dots (a+1)^2 a^2 (a+1)^2 \dots (b-1)^2 b$. We call these words *waves* from a to b and from b to a respectively. For example, $w_{[3\uparrow 8]} = 34455667788776655443$, $w_{[6\downarrow 1]} = 65544332211223344556$, $w_{[4\uparrow 4]} = 44 = w_{[4\downarrow 4]}$, etc. $(w_{[3\uparrow 5]})^3 = 344554433445544334455443$. Note that waves are always irreducible words, since no braid relation can be applied. Thus, by Burckel's result, the ordinal of a wave w is the smallest ordinal assignable to the braid represented by w .

Let $w = *ba*$. We define \hat{b} , the *first crossing* of b in w as follows: (1) if $a > b$, then \hat{b} is the first occurrence of a letter $c \leq b$ in $a*$; (2) if $a < b$, then \hat{b} is the first occurrence of a letter $c \geq b$ in $a*$; (3) if $a* = \epsilon$ or there is no occurrence as in 1, 2, then $\hat{b} = \epsilon$. For example, in 3124, 4 is the first crossing of 3. In 25342, the rightmost 2 is the first crossing of the leftmost 2.

Rules of Game 2

Given a word w let us define its reduct $w[k]$ for every $k > 0$ as follows.

- If the last letter of w is 1 then $a[w]$ is obtained from w by deleting this 1.
- Otherwise, let $b_i^{m_i}$ be the first non-trivial block in w . Let w be as follows:

$$w = *b_i^{m_i} w_0 \hat{b}_i^*,$$

where \hat{b}_i is the first crossing of b_i in w (if $\hat{b}_i = \epsilon$, then $w = *b_i^{m_i} w_0$). Let m be the minimal letter in w_0 and M be the maximal letter in w_0 .

Then, $w[k]$ is defined by the following substitution:

$$*b_i^{m_i} w_0 \leftrightarrow \begin{cases} b_i^{m_i-1} (b_i + 1)^2 (w_{[(b_i+1)\uparrow M]})^k (b_i + 1) & \text{if } \epsilon \neq w_0 = aw_1, a > b_i, \\ b_i^{m_i-1} (b_i - 1)^2 (w_{[(b_i-1)\downarrow m]})^k (b_i - 1) & \text{if } \epsilon \neq w_0 = aw_1, a < b_i. \\ b_i^{m_i-1} (b_i - 1)^2 (w_{[(b_i-1)\downarrow 1]})^k (b_i - 1) & \text{if } w_0 = \epsilon. \end{cases}$$

We note that Game 2 essentially coincides with Game 1 on elements of Σ_3^* .

Termination

This is most easily ascertained using Burckel's bijection between Σ_n^* and (uniform) trees of height n .

Definition. [Burckel, [4]] An *address* is a finite sequence of positive integers. The empty address is denoted by Λ . A *tree T of height n* is a set of addresses of the form

$$\{\Lambda\} \cup \{kw : w \in T_k\} \cup \dots \cup \{2w : w \in T_2\} \cup \dots \cup \{1w : w \in T_1\},$$

with $n > 1$, $k \geq 1$ and T_k, \dots, T_1 trees of height $n - 1$.

Definition. [Burckel, [4]] Let T be a tree of height n . Let w be an address in T . The *domain* of w in T is a sequence of consecutive letters. The domain of Λ is $(n - 1, \dots, 2, 1)$. If the domain of an address x is (c_i, \dots, c_2, c_1) then *the domain of the address xk* is $(c_{i-1}, \dots, c_2, c_1)$ if k is odd and (c_2, c_3, \dots, c_i) if k is even. If xk is a leaf in T then the *word of xk* in T is the unique letter of the domain of the address x , except for the rightmost leaf 1^{n-1} which has empty word. The *word of w* in T is obtained by the concatenation of the words of all leaves under the address w . The *word of the tree T* is the word of its root.

Trees are compared by the short-lexicographic ordering $<_{\text{ShortLex}}$ defined as follows: $T_1 <_{\text{ShortLex}} T_2$ if and only if T_1 is thinner than T_2 at the root or their subtrees are ordered lexicographically from left to right. It is easy to see that $<_{\text{ShortLex}}$ is a well-ordering of order-type $\omega^{\omega^{n-2}}$ on the uniform trees of height n . A block aa in a word w corresponds to two adjacent leaves at the end of some branch in the tree of w . The rules of Game 2 cancel one of these leaves, leaving the structure of the tree on the left of this leaf intact. It can be easily checked that when one branch is cancelled under a node e by the rules of Game 2 at height h then a number of subtrees is added under some node e' to the right of e and at height $h' < h$. Thus, the tree assigned to the reduct $w[k]$ is smaller than the tree of w in the short-lexicographic well-ordering.

Preservation of irreducibility

In the general case, Game 2 is a game on braids *in the weak sense* that w and $w[k]$ always represent different braids, but we do not claim that it preserves irreducibility in general. Nevertheless, it is a game *in the strong sense*, i.e., such that $w >_D w[k] >_D w[k][k+1] >_D \dots$ when played on a special class of braid words. For the sake of our unprovability proof, it suffices to notice that all words $w[k]$ occurring in a game that starts with a single-letter braid word are irreducible. This implies that Game 2, when started on braid words of the form n , is a game on braids in the strong sense. Notice that the single-letter braid words are cofinal in B_∞^+ with respect to $<_D$.

Ordinal assignment

Let $\{H_\alpha : \alpha < \omega^\omega\}$ be the Hardy Hierarchy defined with respect to the standard system of fundamental sequences $(\gamma + \omega^{\delta+1}[k] = \gamma + \omega^\delta \cdot k; \gamma + \omega^\lambda[k] = \gamma + \omega^\lambda[k])$ if λ is a limit; $\alpha + 1[k] = \alpha$ and $0[k] = 0$.

We show that the function $n \mapsto h_n(n)$ grows as $H_{\omega^{\omega^{n-2}}}$, for all $n > 2$. For $n = 3$, this is implied by the coincidence of Game 1 and Game 2 on elements of

Σ_3^* : note that $3 \hookrightarrow_k 22(2112)^k 2$ in Game 2. It is easy to see that $h_{(2211)^{n+1}}(x) \geq h_{(1122)^n}(x)$, so that $h_3(n)$ grows as the Ackermann function, i.e., as H_{ω^ω} .

For the general case we will refer to the following ordinal assignment o of ordinals $< \omega^{\omega^{n-1}}$ to uniform trees of height n defined by Burckel in [5]. The assignment is defined in terms of trees.

Definition. [Burckel, [5]] For any tree T of height n and width k ,

$$o(T) = \begin{cases} 0 & \text{if } n = 0, \\ k - 1 & \text{if } n = 1, \\ o(T_1) & \text{if } k = 1, \\ \delta^{k-1}(1 + o(T_k)) + \delta^{k-2}o(T_{k-1}) + \cdots + \delta^1 o(T_2) + o(T_1) & \text{if } n, k \geq 2, \end{cases}$$

where $\delta = \omega^{\omega^{n-2}}$.

For $w \in \Sigma_n^*$, we denote by $ord(w)$ the ordinal $o(T_w)$ where T_w is the uniform tree corresponding to w in Burckel's bijection. We will show that, in general, that $ord(n[k : \ell]) > ord(\omega^{\omega^{n-2}})[k : \ell]$ (where $[\cdot]$ in the right term is the standard systems of fundamental sequences). By definition of h and H , and since $ord(n) = \omega^{\omega^{n-2}}$, this is sufficient.¹

From now on we only consider braid words w of the form $n[k : \ell]$, i.e., those braid words that occur in some reduction sequence of Game 2 starting with a single letter braid n .

We introduce the notion of *normal decomposition* of a braid word as follows. Let w be a word in Σ_n^* . Then w can be written uniquely as

$$w = w_t a^{s_t} w_{t-1} \dots w_2 1^{s_2} w_1 n^{s_1} w_0 1^{s_0},$$

with $s_0 \geq 0$, $s_{i+1} > 0$, $a = n$ if t is odd and $a = 1$ if t is even, $w_i \in \{1, \dots, (n-1)\}^*$ if i is odd, and $w_i \in \{2, \dots, n\}^*$ if i is even. We call the above expression the normal decomposition of w . This normal form corresponds to the decomposition of the tree associated to w in direct subtrees, numbered from 0 to t from right to left. The subwords $w_i a_i^{s_i}$ with $b_i = 1, n$ are called *the principal components* of w and correspond to the direct subtrees of the tree assigned to w . By applying Burckel's ordinal assignment, we obtain the following equation, if the normal decomposition of w is $w_t a_t^{s_t} w_{t-1} \dots w_2 a_2^{s_2} w_1 a_1^{s_1} w_0 a_0^{s_0}$.

$$ord(w) = \omega^{\omega^{n-2}t} (1 + ord(w_t a_t^{s_t})) + \sum_{1 \leq i \leq t} \omega^{\omega^{n-2}(t-i)} (ord(w_{t-i} a_{t-i}^{s_{t-i}})). \quad (1)$$

We can generalize the notion to any word w be a word in $\{m, \dots, M\}$ (with both m and M occurring in w and $m < M$). Then w can be written uniquely as

$$w = w_t a^{s_t} w_{t-1} \dots w_2 m^{s_2} w_1 M^{s_1} w_0 m^{s_0},$$

with $s_0 \geq 0$, $s_{i+1} > 0$, $a = M$ if t is odd and $a = m$ if t is even, and $w_i \in \{m, \dots, (M-1)\}^*$ if i is even and $w_i \in \{m+1, \dots, M\}^*$ if i is odd. Thus, we

¹The expert reader will recognize that a difficulty here arises because the system of fundamental sequences derived from Game 2 - $\alpha[k] = ord(w[k])$ if $ord(w) = \alpha$ - does not satisfy some of the properties of standard systems that allow for smooth comparison of the corresponding Hardy Hierarchies (e.g., property (B5) in [3]).

can recursively decompose a word w starting to decompose with m the minimal letter in w and M the maximal letter in M , and then decomposing each principal component in the same way.

Using the ordinal assignment and the normal decomposition we can show the following proposition.

Proposition 5. The function h_{nn} majorizes $H_{\omega^{\omega^{n-2}}}$.

As a corollary we obtain the $\text{I}\Sigma_2$ -unprovability of the sentence asserting the termination of Game 2.

Model-theoretic proof

The usual model-theoretic way of showing $\text{I}\Sigma_2$ -unprovability of a statement of the form $\forall x \exists y \varphi(x, y)$ is by demonstrating that for any countable nonstandard model $M \models \text{I}\Sigma_1 \wedge \varphi(a, b)$, there exists a 2-extendible initial segment J between a and b . An initial segment $J \subseteq M$ is called 2-extendible if there are elementary extensions $M \prec K_1 \prec K_2$ such that J is an initial segment of K_1 and K_2 and there is a point of K_1 between J and $M \setminus J$ and a point of K_2 between J and $K_1 \setminus J$. Our proof imitates the model-theoretic proof of termination of reductions of $< \omega^{\omega^\omega}$ -large sets. We learnt the methods of building n -extendible initial segments from the ultimate textbook on indicator theory [15], and are closely following these ideas in the proof below.

Proof. Suppose $M \models \text{I}\Delta_0 + \text{exp}$ is countable and nonstandard and $[a, b]$ destroys the single-letter braid d , $d > \mathbb{N}$.

First notice that whenever a segment $[c, e]$ destroys the braid with ordinal ω^α for $\alpha \leq \omega^{d-2}$ then for any partition of $[c, e]$ into c segments, there is a member of this partition that destroys the braid with ordinal $\omega^{\alpha^{[c]}}$. This follows from ordinal-theoretic analysis above that relates descent through fundamental sequences with descent through braid ordering.

We shall build an ultrafilter U on the M -definable subsets of $[a, b]^2$ such that if $J = \{c \mid [a, c]^2 \notin U\}$, we have:

1. if $f: [a, b]^2 \rightarrow c$ and $c \in J$ then $f^{-1}(e) \in U$ for some $e < c$;
2. if $g: [a, b] \rightarrow M$ then either $\{\langle x, y \rangle \mid x < g(y)\} \in U$ or there is $c \in J$ such that $\{\langle x, y \rangle \mid g(y) = c\} \in U$.

Notice that if such an ultrafilter can be defined then J is a 2-extendible cut we are seeking. Indeed, let

$$K_1 = \{f: [a, b]^2 \rightarrow M \mid f \text{ is coded and depends only on the second coordinate}\}/U,$$

$$K_2 = \{f: [a, b]^2 \rightarrow M \mid f \text{ is } M\text{-coded}\}/U.$$

By conditions on the ultrafilter, J is an initial segment of both K_1 and K_2 . Also, $M \prec K_1 \prec K_2$ by a version of Los Theorem. Let $f(x, y) = y$ and $g(x, y) = x$ and notice that $J < [f] < M \setminus J$ and $J < [g] < K_1 \setminus J$. Hence J is 2-extendible.

Let us now build such an ultrafilter U . Following [15], fix a number $D = 2_{10}(b)$ and define for any $x < y < D$, an M -finite set $Q(x, y)$ as $\{51z \mid z < \frac{x}{3}\} \cup \{2^{4z} \mid z < \frac{x}{3}\} \cup \{z \mid M \models \exists w \theta(w, \bar{e}) \rightarrow z = w \text{ for some formula } \theta(w, \bar{e}) \text{ with } \ulcorner \theta(w, \bar{e}) \urcorner < \frac{x}{3}\}$. Clearly, $|Q(x, y)| < x$.

Set $a_0 = a$, $b_0 = b$ and suppose for $i \in M$, an interval $[a_i, b_i]$ has been defined such that $[a_i, b_i]$ destroys the braid with ordinal $\omega^{\omega^d[a_0][a_1]\dots[a_{i-1}]}$ and $(a_i, b_i) \cap Q(a_{i-1}, b) = \emptyset$. Since $|Q(a_i, b)| < a_i$, the points of $Q(a_i, b)$ partition $[a_i, b_i]$ into at most a_i intervals, hence one of them destroys the braid with ordinal $\omega^{\omega^d[a_0]\dots[a_{i-1}][a_i]}$. Set $[a_{i+1}, b_{i+1}]$ be this interval. After k steps ($k \in M$), this process terminates and the resulting set $\{a_i\}_{i=0}^k$ destroys the braid $(1122)^d$.

Do the same process for the set $\{a_i\}_{i=0}^k$ and define intervals $[c_i, e_i]$ for $i \leq s \in M \setminus \mathbb{N}$, where c_i and e_i are among $\{a_i\}_{i=0}^k$ and $[c_i, e_i]$ destroys the braid with ordinal $\omega^d[c_0][c_1]\dots[c_{i-1}]$ and $(c_i, e_i) \cap Q(c_{i-1}, \langle b, a_k \rangle) = \emptyset$.

Put $I = \sup_{i \in \omega} c_i$, $M_1 = \bigcup_{a_i \in I} Q(a_i, b)$, $M_2 = \bigcup_{i \in \omega} Q(c_i, \langle b, a_k \rangle)$ and notice that I is closed under exponentiation and $I \subset M_1 \prec M_2 \prec D^2$ and $I < a_k < M_1 \setminus I$, $I < c_s < M_2 \setminus I$. Let us now recover an ultrafilter U and thus also our initial segment J (a symbiotic twin of I) from I , M_1 and M_2 .

Let us enumerate all M -coded functions $f: [a, b]^2 \rightarrow b$ and $g: [a, b] \rightarrow b$ in a sequence of length ω . Set $X_0 = [a, b]^2$ and suppose that for $i \in \omega$, a set X_{i-1} , unbounded in I has been defined ($\{x \mid \text{there is } y \in I: \langle x, y \rangle \in X_{i-1}\}$ is unbounded in I).

Suppose the i th function is $f: [a, b]^2 \rightarrow c$. If $c > I$ then put $X_i = X_{i-1} \cap [a, c]^2$. If $c \in I$ then let $I < d_1 < M_1 \setminus I$ and $I < d_2 < M_2 \setminus I$ be such that $\langle d_2, d_1 \rangle \in X_{i-1}$ (such pair exists by overspill). Put $c^* = f(d_2, d_1)$ and notice that the set $\{\langle x, y \rangle \in X_{i-1} \mid f(x, y) = c^*\}$ is unbounded in I by a standard argument from [12] using overspill and the fact that $M_1 \prec M_2 \prec D$.

If the i th function is $g: [a, b] \rightarrow c$ then set $X_i = \{\langle x, y \rangle \in X_{i-1} \mid g(y) = g(d_1)\}$ if $g(d_1) \in I$ and $X_i = \{\langle x, y \rangle \in X_{i-1} \mid g(y) > x\}$ if $g(d_2) > d_1$. Unboundedness is proved by the same argument as for f . Now, let U be any ultrafilter on the M -coded sets that contains X_i for all $i \in \omega$. \square

Indicator theory arguments (including the argument above) can be recognized as arithmetical not-so-distant relatives of unprovability proofs for large cardinals. Curiously, it was the large cardinals that originally led to the discovery of the left-invariant linear ordering on braids. So we completed a full circle here. Another way to show IS_2 -unprovability of our statement would be to build a set of indiscernibles between a and b and construct a model of IS_2 out of it. To convert the above proof into a proof that uses indiscernibles is an exercise in model theory of arithmetic.

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²here we consider D as a structure in the language with relation symbols $+$, \times , $<$

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