The strength of combinatorial principles can be accessed by their density*

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Abstract

This article builds upon the article [1] where the strength of infinitary ramseyan principles RT^2_2, $CanRT^2$, $RegRT^2$ and EM was approximated by the strength of their finite iterations. Here we do it for several other infinitary principles coming from WQO theory and reverse mathematics, not only from Ramsey Theory.

The method of densities was first envisaged by J. Paris (an unprovable statement "$\forall xy \exists z\ [x, z]$ is y-dense$(2, 3)$" was discovered before unprovability of PH) and later developed in [2] by H. Friedman, K. McAloon and S. Simpson.

We take this method yet further and apply it to questions of logical strength of second-order arithmetical statements.

1 Kruskal's theorem

Definition 1. A set X is called 0-dense if $|X| > \min X + 3$. A set X is called $(n+1)$-dense(k) if for every $F\colon X \to$ finite trees such that for all $i \in X$, $|F(i)| < i$, there is an n-dense subset $Y \subseteq X$ such that $F|_Y$ is weakly increasing with respect to inf-preserving embeddability of trees.

Let T be the theory $I\Sigma_1 + \cup_{n\in\omega}$" $\forall a \exists b\ [a, b]$ is n-dense".

Theorem 1. The set of Π_2 consequences of $\mathrm{WKL}_0 + \mathrm{KT}$ coincides with the set of Π_2 consequences of T.

Proof. The fact that $T \subseteq \mathrm{WKL}_0 + \mathrm{KT}$ is easy: each iteration of densities can be obtained from Infinite Kruskal Theorem.

Let us show that for every Π_2 formula $\Phi = \forall x\ \exists y\ \varphi(x, y)$, if $T + \neg\Phi$ is consistent then $\mathrm{WKL}_0 + \mathrm{KT} + \neg\Phi$ is also consistent.

Suppose $M \models T + \neg\Phi$ is countable and nonstandard, $a \in M$ and $M \models \forall y\ \neg\varphi(a, y)$. For every $n \in \mathbb{N}$, $M \models \exists b\ [a, b]$ is n-dense, hence there are $d > \mathbb{N}$ and $b > a$ such that $M \models [a, b]$ is d-dense.

Let $S_1, S_2, \ldots, S_i, \ldots i \in \omega$ be an external enumeration of all M-finite subsets of $[a, b]$ in which each set occurs infinitely-often and $F_1, F_2, \ldots, F_i, \ldots i \in \omega$ be

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an external enumeration of all M-finite sets of pairs $\langle c, T_c\rangle$, where $c \in M$ and T_c is an M-finite tree.

Let us build a sequence of sets $[a,b] = X_0 \supseteq Y_0 \supseteq X_1 \supseteq Y_1 \supseteq \cdots \supseteq X_i \supseteq Y_i \cdots$ $i \in \omega$ such that X_i is $(d-2i)$-dense, Y_i is $(d-2i-1)$-dense.

Suppose X_i has been defined and let us define Y_i. If $\operatorname{card} S_i > \min X_i$ then put $Y_i = X_i$. Otherwise define the following sequence of trees: for $x \in X_i$, let $G(x)$ be the linear order (hence a tree) of size $\min X_i - g(x)$, where $g(x)$ is the index in the increasing enumeration of S_i of the nearest to x element of S_i on the right if exists and 0 otherwise. I.e. each element $x \in X_i$ between s_m and s_{m+1} is assigned the linear order of length $\min X_i - m$. Let $Y_i \subseteq X_i$ be any $(d-2i-1)$-dense subset such that $G|_{Y_i}$ is weakly increasing. Clearly, $(Y_i \setminus \min Y_i) \cap S_i = \varnothing$.

Given Y_i, let us define X_{i+1} as follows. Let F be the first set of pairs in our enumeration whose domain contains Y_i and that has not yet been considered. Put X_{i+1} to be any $(d-2i-2)$-dense subset of Y_i such that $F|_{X_{i+1}}$ is weakly increasing.

Define $I = \sup_{i\in\omega} \min X_i$ with the second-order structure being the set of all sets $A \subseteq I$ such that for some coded $B \subset M$, $A = B \cap I$.

Let us show that $I \vDash \mathrm{WKL}_0 + \mathrm{KT} + \neg\Phi$. Clearly, $I \vDash \neg\Phi$ since I is an initial segment of M. We show that $I \vDash \mathrm{WKL}_0$ be demonstrating that I is semi-regular. For an M-finite set $S \subseteq [a,b]$, S occurs in the list as S_i for infinitely-many $i \in \omega$. If $\operatorname{card} S \in I$ then $\operatorname{card} S < \min X_i$ for some $i \in \omega$. Then at some stage $j \geq i$ of our construction, S is considered in building Y_j such that $(Y_j \setminus \min Y_j) \cap \varnothing$, so S is bounded in I.

Let us finally show that $I \vDash \mathrm{KT}$. Let $\langle T_i \mid i \in I\rangle$ be an M-coded sequence of I-finite trees and we wish to find a weakly increasing I-unbounded subsequence. Since I is closed under exponentiation, we can find another sequence $\langle T'_i \mid i \in I\rangle$ consisting of the same trees in the same order and such that $\operatorname{card}(T'_i) < i$ for all $i > j$ for some $j \in I$ (by inserting enough copies of the same tree). Now, by definition of the second-order structure on I and by overspill, there is an M-finite sequence F with $\operatorname{dom}(F) \subseteq [a,b]$ and such that for all $i \in I$, $F(i) = T'_i$. Define F arbitrarily on the rest of $[a,b]$ and notice that F occurs in our enumeration, so there is $i \in \omega$ such that $F|_{X_i}$ is weakly increasing. Since X_i is unbounded in I, $F|_{X_i} \cap I$ is the required subsequence. □

A similar theorem can be formulated for Kruskal theorem for plane trees, with similar proof.

2 Hilbert Basis theorem

Definition 2. Infinitary Hilbert Basis Theorem:

Definition 3. Finitary Hilbert Basis Theorem:
$\forall n\ \exists M\ \forall p_1, p_2, \ldots p_M \in K[x_1, \ldots x_n]$, if for all $i \leq M$, $\deg(p_i) \leq n + i$ then there is $j < M$ such that $(p_0, \ldots p_j) = (p_0, \ldots p_{j+1})$.

Definition 4. A set X is 0-hilbert-dense if $|X| > \min X + 3$. A set X is $(n+1)$-hilbert-dense if for any $F\colon X \to K[x_0, x_1]$ such that $\deg(F(x)) < 2^x$, there is an n-hilbert-dense subset $Y \subseteq X$ such that $F|_Y$ is nonincreasing.

3 Ascending subsequences principles

4 Infinitary Erdös-Moser Principle

Let EM be the statement "for every tournament on $\mathbb{N}$, there is a transitive subtournament". Let us approximate the strength of EM by its densities in order to prove the following theorem.

Theorem 2. Every provably recursive function of $\mathrm{EFA}+\mathrm{EM}$ is primitive recursive.

This theorem exposes a spectacular phenomenon: the strength of both CAC and EM does not exceed $I\Sigma_1$ but together they imply RT^2_2.

Definition 5. Any set is 0-dense. We say that X is $(n+1)$-dense if for any tournament T on X there is an n-dense subset $Y \subset X$ such that $2^{\min X} < \min Y$ and (Y, T) is transitive.

Lemma 3. Each function f_n defined as $f_n(a) = \min b$ such that $[a, b]$ is n-dense, is primitive recursive.

Proof. I shall try to find an argument tomorrow. A single tower may be not enough though. □

Now let us prove the theorem.

Proof.
By Lemma 3, the set $T = \{\forall a \exists b\ [a,b] \text{ is } n\text{-dense}\}\ \subseteq I\Sigma_1$. Consider a formula $\Phi = \forall x \exists y\ \varphi(x, y)$ and suppose there is a model of $I\Sigma_1 + \neg\Phi$. We shall build a model of $\mathrm{EFA}+\mathrm{EM}+\neg\Phi$.

Start off with a countable nonstandard model $M \models I\Sigma_1 + \forall y \neg\varphi(a, y)$ for $a \in M$. Since for every $n \in \omega$, $M \models \exists b\ [a, b]$ is n-dense, by Σ_1-overspill, there is $d \in M \setminus \mathbb{N}$, $b > a$ such that $M \models [a, b]$ is d-dense.

Let us externally enumerate all pairs (X, T), where X is an M-coded subset of $[a, b]$ and T is an M-coded tournament on X, so that each pair occurs infinitely-often. Using the density assumption, let us build a sequence of sets $[a, b] = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_i \supseteq \cdots\ i \in \omega$ as follows. At stage i, consider the first pair (X, T) in our enumeration such that $X \supseteq X_i$ that has not been considered and define X_{i+1} to be any $(d-i-1)$-dense subset of X_i such that $(X_{i+1}, T|_{X_{i+1}})$ is transitive and $2^{\min X_i} < \min X_{i+1}$.

Define $I = \sup_{i\in\omega} \min X_i$ with second order structure being the set of all intersections of M-coded sets with I. Let us show that I is the required model of $\mathrm{EFA}+\mathrm{EM}+\neg\Phi$. Indeed, $I \models I\Delta_0$ because Δ_0-formulas are absolute between M and I. Also, I is closed under exponentiation because for each $x \in I$, if $x < \min X_k$ then $2^x < 2^{\min X_k} < \min X_{k+1} \in I$.

Let us show that $I \models \mathrm{EM}$. Consider a tournament on I such that there is an M-coded tournament T^* on an M-finite set $X \subseteq [a, b]$ that contains I. The pair (X, T^*) occurs in our enumeration, so for some $i \in I$, $(X_i, T^*|_{X_i})$ is transitive. Since X_i is unbounded in I, $T^*|_{X_i}$ is the I-unbounded transitive subtournament of T that we are seeking. □

5 Well-quasi-orderings of pairs of natural numbers

Consider the following ordering on the pairs of natural numbers:

$$(m,n) \leq (m',n') \text{ if } m \leq m' \text{ and } n \leq n'.$$

Consider the followiing infinitary principle: "for every $F\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, there is an infinite subset $Y \subseteq \mathbb{N}$ such that $F|_Y$ is weakly increasing".

What is the strength of this principle? (I forgot, does it imply totality of all p.r. functions?)

Density treatment: X is $(n+1)$-wpodense if for all $F\colon X \to \mathbb{N}\times\mathbb{N}$, if $\forall x \in X$, $|F(x)| < 2^x$ then there is an n-wpodense subset $Y \subseteq X$ such that $F|_Y$ is weakly increasing.

6 Some remaining questions

1. Does Kruskal's Theorem imply RT^2_2?

2. Can we do the density argument for FS(2)?

3. What is the strength of $\mathrm{HT}(c)$, Hindman's Theorem with fixed number of colours c? Is $\mathrm{HT}(c)$ strictly weaker than HT?

4. Do the density analysis of HT.

5. For every ordinal α, do the density analysis of the principle $\mathrm{SWO}(\alpha)$: "$\forall F\colon \mathbb{N} \to \alpha\ \exists X \subseteq \mathbb{N}$, infinite, such that $F|_X$ is weakly increasing".

   X is $(n+1)$-swodense(α) if for all $F\colon X \to \alpha$ $(\forall x \in X\ \ |F(x)| \leq x)$ implies that there exists an n-swodense(α) subset $Y \subseteq X$ such that $F|_Y$ is weakly increasing.

6. A related family of recursion-theoretic principles. Does there exist a recursive function $F\colon \mathbb{N} \to \alpha$ for $\alpha \geq \omega_n$ (Σ^*, or trees in place of α) such that for every infinite Y s.t. $F|_Y$ is weakly increasing, $0^{(n)} \leq_T \leq Y$???

7. Can the density analysis re-prove that Kruskal for binary trees with two lables gives Γ_0 and Kruskal for binary trees gives ε_0?

References

[1] Bovykin, A., Weiermann, A. (2006). The strength of infinitary ramseyan statements can be accessed by their density. *Submitted.*

[2] Friedman, H., McAloon, K., Simpson, S. (1982). A finite combinatorial principle which is equivalent to the 1-consistency of predicative analysis. In:*Logic Symposium I (Patras 1980)*, North-Holland, pp.197-220.