

## The aims of the subject

- Find examples of mathematical statements unprovable in strongest axiomatic systems. (Examples from the past:
  - examples springing from the indicator method of J.Paris (e.g. the Paris-Harrington Principle);
  - unprovability of finite versions of Kruskal's theorem (H.Friedman) and the graph minor theorem (Friedman-Robertson-Seymour);
  - new ZFC-unprovable statements by Friedman;
  - results by A.Weiermann, L.Carlucci, myself and other authors.
- Reverse Mathematics (given a mathematical theorem, determine its logical strength).

$\text{RT}_2^2$  is the following statement in the language of second-order arithmetic:

“for every colouring  $F$  of 2-subsets of  $\mathbb{N}$  into 2 colours, there is an infinite subset whose 2-subsets are all of the same colour”.

The strength of  $\text{RT}_2^2$  is an open problem.

### **Density**

A set  $X$  is 0-dense(2, 2) if  $|X| > 3$ ,  $X$  is  $(n + 1)$ -dense(2, 2) if for every  $f: [X]^2 \rightarrow 2$  there is an  $n$ -dense(2, 2) set  $H$  such that  $f|_{[H]^2} = \text{const}$  and  $|H| > \min H$ .

Consider a theory  $T = I\Delta_0 + \bigcup_{n \in \omega} \forall a \exists b [a, b]$  is  $n$ -dense(2, 2).

**Theorem** (joint with A. Weiermann)

The set of  $\Pi_2$ -consequences of  $WKL_0 + RT_2^2$  coincides with the set of  $\Pi_2$ -consequences of  $T$ .

This result may shed light on the strength of  $RT_2^2$  and on whether  $RT_2^2$  proves totality of the Ackermann function.

**METHOD:** inside a nonstandard model of  $T$ , we build an initial segment satisfying  $WKL_0 + RT_2^2$  (nonstandard chunks of first-order definable sets play the role of future infinite sets).

Canonical Ramsey Theorem for pairs ( $\text{CRT}^2$ ) is the following statement: “for any function  $f: [\mathbb{N}]^2 \rightarrow \mathbb{N}$ , there is an infinite set  $X \subseteq \mathbb{N}$  such that one of the following four cases occurs:

- (1)  $X$  is  $f$ -homogeneous;
- (2)  $f$  is injective on  $X$ ;
- (3)  $f$  depends only on the first coordinate on  $X$ ;
- (4)  $f$  depends only on the second coordinate on  $X$ ”.

Regressive Ramsey Theorem for pairs ( $\text{RegRT}^2$ ) is the following statement: “for any function  $f: [\mathbb{N}]^2 \rightarrow \mathbb{N}$  such that  $f(x, y) \leq x$ , there is an infinite set  $X$  such that on  $X$ ,  $f$  depends only on the first coordinate”.

With an analogous definition of  $T$ , we can prove the following theorem

**Theorem** (joint with A.Weiermann)

The  $\Pi_2$  consequences of  $T$  are the same as of  $\text{WKL}_0 + \text{CRT}^2$ .

Using the model-theoretic machinery developed by Paris and Kirby (**strong initial segments in models of Peano Arithmetic**), it is possible to prove:

- the first-order consequences of  $\text{RegRT}^2$  coincide with Peano Arithmetic (the recursion-theoretic approach was developed by P.Clote in the 1980s);
- the first-order consequences of  $\text{CRT}^2$  coincide with Peano Arithmetic (the recursion-theoretic proof was first obtained by J.Mileti in 2004).

METHOD: showing that a semi-regular initial segment is strong if and only if it satisfies  $\text{CRT}^2$ , using a definable ultrapower construction inside a countable model of arithmetic.

Friedman's sine-principle (see FOM internet forum):

“for any  $k$  there exists  $N$  such that whenever  $x_1, x_2, \dots, x_N$  are rational numbers, there exist  $p_1 < p_2 < \dots < p_{k+2}$  such that

$$|\sin(x_{p_1} \cdot x_{p_2} \cdot \dots \cdot x_{p_k}) - \sin(x_{p_1} \cdot x_{p_3} \cdot \dots \cdot x_{p_{k+1}})| < 4^{-p_1}$$

$$|\sin(x_{p_2} \cdot x_{p_3} \cdot \dots \cdot x_{p_{k+1}}) - \sin(x_{p_2} \cdot x_{p_4} \cdot \dots \cdot x_{p_{k+2}})| < 4^{-p_2}$$

is unprovable in PA.

My sharp versions:

Let  $SP^n$  be: “for all  $m$  there is  $N$  such that for any sequence of rational numbers  $A = \langle a_1, a_2, \dots, a_N \rangle$ , there is a subset  $H \subset A$  of size  $m$  such that for any two  $n$ -element subsets

$a_{i_1} < a_{i_2} < \dots < a_{i_n}$  and  $a_{j_1} < a_{j_2} < \dots < a_{j_n}$ , we have

$$|\sin(a_{i_1} \cdot a_{i_2} \cdot \dots \cdot a_{i_n}) - \sin(a_{j_1} \cdot a_{j_2} \cdot \dots \cdot a_{j_n})| < r^{\log^{n-1}(i_1)},$$

where  $r \approx 0.5960716\dots$

For  $n > 1$ ,  $SP^{n+1}$  is not provable in  $I\Sigma_n$ .

There is also an  $I\Sigma_1$  version.

METHOD: the proof uses reduction of Kanamori-McAloon Principle and the Rhin-Viola theorem on irrationality measure of  $\pi$ .

A related series of results about dynamical systems will appear shortly.

## **Braids**

Positive braids are ordered as  $\omega^{\omega^{\omega}}$ , hence giving rise to several  $I\Sigma_2$ -unprovable statements.

METHODS: the proofs use 2-extendible initial segments and indiscernible elements in a model of  $I\Delta_0 + \text{exp}$ .

There is an opportunity to use Artin's Braid Theorem to obtain unprovable statements with physical meaning.

## Threshold results

The study of exact unprovability results ('threshold results') started with Weiermann's investigations in 2000 and 2001 of the gap between provable and unprovable instances of the Paris-Harrington principle and of Kruskal's theorem.

Example: let  $\text{PH}_f$  be the statement "for all  $n, m, c$ , there is  $N$  such that for all colourings  $f: [N]^n \rightarrow c$ , there is an  $f$ -homogeneous subset  $H$  of size at least  $m$  such that  $|H| > f(\min H)$ ". Weiermann proved that for  $f(x) = \log^k(x)$ ,  $\text{PH}_f$  is PA-unprovable but  $\text{PH}_{\log^*}$  is provable in  $I\Delta_0 + \text{exp}$ .

My model-theoretic approach to threshold results can be found on my webpage <http://logic.pdmi.ras.ru/~andrey>.

The current problem is to find the gap between provability and unprovability in the graph minor theorem. Let  $\text{GMT}_f$  be the statement “for all  $K$  there is  $N$  such that whenever  $G_1, G_2, \dots, G_N$  are simple graphs such that  $|G_i| < K + f(i)$  then there are  $i < j$  such that  $G_i$  is a minor of  $G_j$ ”.

The current situation is: the statement  $\text{GMT}_{\sqrt{2} \cdot \sqrt{\log}}$  is provable in  $I\Delta_0 + \text{exp}$ , but the statement  $\text{GMT}_{c \cdot \log}$  is unprovable in PA.

METHOD:

- provable instances by Polya’s theorem and the asymptotic pigeonhole principle;
- unprovable instances: by combining the Friedman-Robertson-Seymour translation of trees into graphs with Weiermann’s threshold for Kruskal’s theorem.

thank you