On optimal heuristic randomized semidecision procedures, with application to proof complexity

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Acceptors and proof systems

- $\mathcal{A}$ is an acceptor for language L if
  - $\forall x \in L\ \mathcal{A}(x) = 1$,
  - $\forall x \notin L\ \mathcal{A}(x)$ does not stop.
- [Cook, Reckhow, 70s] A proof system for language L is a polynomial-time surjective mapping $\Pi\colon \{0,1\}^* \to L$.
  - w is called a Π-proof of $f(w)$.
- Proof system from acceptor:
  - $\Pi_{\mathcal{A}}$: [protocol of $\mathcal{A}(x)$] $\mapsto x$.
- An automatizable proof system is a pair $(\Pi, \mathcal{B})$:
  - $\forall x \in L\ \mathcal{B}(x)$ outputs a Π-proof of x in time $\leq \mathrm{poly}$(size of the shortest Π-proof of x).
  - $\forall x \notin L\ \mathcal{B}(x)$ does not stop.
- $(\Pi_{\mathcal{A}}, \widetilde{\mathcal{A}})$ is an automatizable proof system.

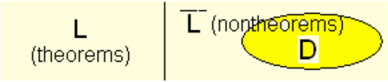
Propositional proof systems

- Propositional proof systems: proof systems for the language of Boolean tautologies **TAUT**.
- Every algorithm for **TAUT** yields a proof system, but not vice versa.
- $\mathbf{NP} = \mathbf{coNP}$ iff there is a proof system that has a polynomial-size proof for every tautology.
- $\mathbf{P} = \mathbf{NP}$ iff there is an automatizable proof system that has a polynomial-size proof for every tautology.

Simulation and Optimality

- A proof system Π_1 p-simulates a proof system Π_2 if $\exists$ polynomial-time computable function f that maps Π_2-proofs to Π_1-proofs.
- An acceptor $\mathcal{A}_1$ simulates an acceptor $\mathcal{A}_2$ if $\forall x \in L$ running time of $\mathcal{A}_1(x) \leq poly(\text{running time of } \mathcal{A}_2(x))$.
- [Krajíček, Pudlák, 1989] $\exists$ p-optimal proof system for **TAUT** $\iff$ $\exists$ optimal acceptor for **TAUT**.
- [Messner, 1999] For every paddable language L, $\exists$ p-optimal proof system for $L \iff \exists$ optimal acceptor for L.
- [Cook, Krajíček, 2007] $\exists$ p-optimal proof system with 1 bit of nonuniform advice.

Heuristic proof systems and acceptors



- Distributional proving problem: (L, D) where D is polynomial-time samplable distribution on $\overline{L}$.
- Heuristic proof system for (L, D) is a randomized algorithm $\Pi(x, w, d)$:
  - Running time of $\Pi(x, w, d)$ is $poly(|x|, |w|, d)$.
  - (Completeness) $\forall x \in L \; \forall d \in \mathbb{N} \; \exists w \;\; \Pr\{\Pi(x, w, d) = 1\} > \frac{1}{2}$.
  - (Soundness) $\Pr_{x \leftarrow D_n}\{\exists w \;\; \Pr\{\Pi(x, w, d) = 1\} > \frac{1}{4}\} < \frac{1}{d}$.
- Heuristic acceptor for (L, D) is a randomized algorithm $\mathcal{A}(x, d)$:
  - $\forall x \in L \; \forall d \in \mathbb{N} \; \mathcal{A}(x, d) = 1$.
  - $\Pr_{x \leftarrow D_n}\{\Pr\{\mathcal{A}(x, d) \text{ stops}\} > \frac{1}{4}\} < \frac{1}{d}\}$.
- Median running time:
  - $\min\{t \mid \Pr\{A(x, d) \text{ runs in} \leq t \text{ steps}\} \geq \frac{1}{2}\}$.

Optimal heuristic acceptor

Theorem. For every r.e. language L and p-samplable D with support in $\overline{L}$ there exists an acceptor $U(x, d)$ for (L, D) that has optimal (up to $poly(|x|, d)$) median time.

Construction (sketch):

Optimal heuristic acceptor $U(x, d)$:

- In parallel for all $1 \leq i \leq \log_* n$:
  1. Execute $A_i(x, d')$; Let T_i be its running time.
  2. Verify the correctness of A_i:
     Repeat for many times:
     - $r \leftarrow D_n$,
     - If $A_i^{\leq T_i}(r, d') = 1$ too often,
       then put a black point;

     and verify that the number of black points is small.
  3. Return "1".
- Execute the semidecision procedure for r.e. language L.

Further research

- Some recent observations (unpublished)
  - An optimal heuristic automatizable proof system (under weak enough notion of automatizability).
  - $\exists$ one-way functions $\implies$ $\exists$ p-samplable distribution D on $\overline{\textbf{TAUT}}$ such that every heuristic acceptor for $(\textbf{TAUT}, D)$ is not polynomial bounded.
  - A universal distribution on $\overline{\textbf{TAUT}}$ that dominates distributions on $\overline{\textbf{TAUT}}$ that are provably correct or certify their results.
- Open questions
  - Construct an optimal heuristic proof system.
  - Extend the equivalence between p-optimal proof systems and optimal acceptors to the heuristic case.