Graph Theory

0268 - Discrete Mathematics

September 2017

1 Instructions

Solve as many problems as you can by September 18. Be ready to explain and present your solutions to other students.

2 Problems

Problem 1 Let G be a simple graph with n vertices and m edges.

A Show that if $m > \binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$ then G is connected.

B For $n > 1$, find a disconnected simple graph G with $m = \binom{n-1}{2}$

C Show that if minimum degree $\delta_G \stackrel{\text{def}}{=} \min_{v \in V} d_v > \frac{n-2}{2}$, then G is connected.

D For even n even, find a disconnected $\frac{n-2}{2}$-regular simple graph.

Problem 2 King (in chess) did a tour around 8×8 board visiting every square exactly once and returned to its initial position. Prove that he did even number of diagonal moves.

Problem 3 There is a volleyball net with 200×25 cells. Every second Max cuts one of the links in the net. What is the longest possible time for Max to keep the net in one piece, that is so that the net does not break into a few separate pieces?

Problem 4 In a country every pair of cities is connected by railway or airplane route. Is it sufficient to leave only one type of transportation (either railway, or airplane), so that all cities in the country remain connected?

Problem 5 In group A everyone knows at least half of the people from group B. In group B everyone knows at most half of the people from A. Show that $|A|$ and $|B|$ are even.

Problem 6 In a chess position there are a queen, 2 rooks, and 2 knights placed so that every piece attacks exactly one other piece out of the remaining 4 pieces and also is being attacked by exactly one other piece. Prove that the queen attacks a rook diagonally (i.e., as if Queen is a bishop).

Problem 7 In Wonderland there are a few cities some of which are connected by flights operated by k different air companies. There is exactly one flight going out of every city operated by each of the k air companies. One can fly from each city to any other city in Wonderland, potentially using connection flights of different air companies. One day $k-1$ different flights got canceled, but not more than 1 flight per a single company. Show that it is still possible to get from every city to every other city.

Problem 8 There are 17 students in a class such that within any 3 students at least 2 know each other. Prove that at least one person in a class knows 8 people.

Problem 9 A connected graph has 100 vertices. Show that there is a path (which may visit some of the vertices multiple times) of length at most 196 that visits every vertex in the graph at least once.

Problem* 10 On a paper square board 9×9 one picks k squares 1×1 and makes 2 diagonal cuts inside each of these unit squares. What is the largest possible k such that the board stays in one piece (remains connected)?

Problem* 11 Initially there are 100 bacteria in the colony at the 1-st stage of their evolution. Every second one of the bacteria at some stage i of its evolution can split into 2 new bacterias at $i+1$ stage of their evolution. The colony is called n-stable if for any possible sequence of splittings and at any moment in time there are at least n bacterias at the same stage of evolution. What is the largest possible n?