Logic and accurate proofs

0268 - Discrete Mathematics

March 8, 2018

1 Instructions

Solve as many problems as you can by March 15. You can work on the problems as a pair, in this case write both names on the paper. Make sure to verify your solutions and solutions of your partner.

2 Problems

Problem 1 What is going on here?!

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{(-1)}\sqrt{(-1)} = \left(\sqrt{(-1)}\right)^2 = -1.$$

A Precisely identify and explain the mistake(s) in this bogus (错误的) proof.

B Every positive real number r has two square roots, one positive and the other negative. The standard convention is that the expression $\sqrt{r}$ refers to the positive square root of r. Assuming familiar properties of multiplication of real numbers, prove that for positive real numbers r and s,

$$\sqrt{r \cdot s} = \sqrt{r} \cdot \sqrt{s}.$$

Problem 2 It is a fact that the Arithmetic Mean（算术平均）is at least as large as the Geometric Mean（几何平均）, namely,

$$\frac{a+b}{2} \geq \sqrt{a \cdot b}$$

for all non negative real numbers a and b. But there's something objectionable（难以接受的）about the following proof of this fact. What's the objection, and how would you fix it?

Bogus proof.

$$\begin{aligned} \frac{a+b}{2} &\overset{(?)}{\geq} \sqrt{ab}, && \text{so} \\ a+b &\overset{(?)}{\geq} 2\sqrt{ab}, && \text{so} \\ a^2+2ab+b^2 &\overset{(?)}{\geq} 4ab, && \text{so} \\ a^2-2ab+b^2 &\overset{(?)}{\geq} 0, && \text{so} \\ (a-b)^2 &\overset{(?)}{\geq} 0, && \text{which we know is true.} \end{aligned}$$

The last statement is true because $a-b$ is a real number, and the square of a real number is never negative. This proves the claim. ■

Problem 3 Hint: Contradiction（反证法）.

A Let $n \in \mathbb{N}$. Explain why if n^2 is even—that is, a multiple of 2—then n is even.

B Explain why if n^2 is a multiple of 3, then n must be a multiple of 3.

C Prove that if $a \cdot b = n$, then either $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$, where a, b, and n are nonnegative real numbers.

Problem 4 Write down with quantifiers and formal logical relations the following sentences. Any logically equivalent statements are fine. Simpler version of each sentence are generally preferred.

A In order for all students to pass this course every student must not skip any homework assignments and in each joint homework none of the students should completely rely on the other person to solve all their problems.

B Both students receive 0 score for a solution to a homework problem if there is the same solution in someone else's homework.

Problem* 5 Lemma 1 Let the coefficients（系数）of the polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_{m-1}x^{m-1} + x^m$ be integers（整数）. Then any real root（实数根）of the polynomial is either integral or irrational（无理数）.

A Explain why the Lemma immediately implies that $\sqrt[m]{k}$ is irrational whenever k is not an m-th power of some integer.

B Carefully prove the Lemma.

You may find it helpful to appeal to:

Fact. If a prime p is a factor of some power of an integer, then it is a factor of that integer. (如果质数 p 是某整数的幂的因数，那么它一定是该整数的因数。)

You may assume this Fact without writing down its proof, but see if you can explain why it is true.