Universal Stratifications and a Bertini-type Theorem

Dima Grigoriev
CNRS, Mathématiques, Université de Lille,
Villeneuve d'Ascq, 59655, France
dmitry.grigoryev@math.univ-lille1.fr
http://logic.pdmi.ras.ru/~grigorev

Pierre Milman
Departement of Mathematics,
University of Toronto,
40 St. George Street, Toronto,
Ontario M5S 2E4, Canada
milman@math.toronto.edu

Abstract

A stratification of the set of critical points of a map is universal in the class of stratifications satisfying the classical Thom and Whitney-a conditions if it is the coarsest among all such stratifications. We show that a universal stratification exists if and only if the 'canonical subbundle' of the cotangent bundle of the source of the map (constructed via operations introduced by Glaeser) is Lagrangian. The proof relies on a new Bertini-type theorem for singular varieties proved via an intriguing use of resolution of singularities. Many examples are provided, including those of maps without universal stratifications.

AMS classification: primary 14B05, secondary 14E15.

Contents

1 Introduction.

The notions and results used here are from several diverse areas, particularly 'Stratification Theory', e.g. as in [18], 'Resolution of Singularities', related to Bertini Theorem in algebraic geometry and 'Complexity' (of constructions in 'Algebraic Geometry'), but the focus of these areas proper is different. Nevertheless, the blend of these areas in the questions raised and the results obtained here appears to be intriguing (and, perhaps, is important). Crucial to our work is the constructiveness of the Glaeser bundle, which is a new invariant that we associate with the classes of stratifications that possess classical Thom and Whitney-a (shortly TWa) properties, see [8], [20], [23], [10] or Section 2. Roughly speaking universal is the coarsest stratification in the latter class of TWa stratifications of the set of critical points of a map.

We show that a universal stratification exists if and only if the 'canonical subbundle' of the cotangent bundle of the source of the map (constructed via operations introduced by Glaeser and referred to in this article as Glaeser bundle) is Lagrangian. ('Thom' and 'Whitney-a' properties, are very basic for any analysis within the subject of "singularities of mappings" and of several other subjects. Thus our criterion of universality applies to all 'reasonable' classes of stratifications.) The proof relies on an Extension Theorem 5.9. Crucial for the proof of the latter is our *Bertini-type Theorem* 7.2. Its proof makes an unusual use of desingularization, see Section 1.3. Examples of universal TWa stratifications and of non Lagrangian Glaeser bundles illustrate our results. Below, $K=\mathbb{R}$ or $\mathbb{C}$ and map $F:K^n\to K^l$ is polynomial (or analytic) and dominating, i. e. $\overline{F(K^n)}=K^l$.

We refer to an open in its closure constructible subset S of the critical points $\mathrm{Sing}(F)$ as **universal** for the class of TWa stratifications if it is open and dense in a component of a stratum for any TWa stratification of $\mathrm{Sing}(F)$. We also refer to the minimal by inclusion set among closed subsets of T^*K^n with fibers over K^n being subspaces of the respective fibers of T^*K^n and containing the differentials of the components of F as the **Glaeser bundle** G_F of map F . We say that a constructible set S is **Gauss regular** when there is a unique continuous extension to all of S of the *Gauss map* of S , i. e. of the map which sends nonsingular $x\in S$ to the tangent space $T_x(S)$ to S at x . (Algebraic curves with analytically irreducible singularities are *Gauss regular*, but are not even C^1-smooth, e. g. curve $\{z^3=w^2\}\subset\mathbb{C}^2$.) Finally, we refer to the subsets of a bundle with the fibers over the base being subspaces of the respective fibers of the bundle as subbundles.

At the first glance it seemed that *Glaeser bundle* G_F could serve the purpose of identifying TWa Gauss regular stratifications with all strata being universal, namely: by means of partitioning of the critical locus by dimension of its fibers (private discussions with M. Gromov, M. Kontsevich, T. Mostowski, A. Parusinski, N. Vorobjov, Y. Yomdin and others). But it does not always work, see Section 10.3 for example of *not Lagrangian* G_F .

Note that a universal stratification of $\mathrm{Sing}(F)$ if exists is essentially unique, as is precisely spelled out in Corollary 4.2 (ii) of Proposition 4.1 . Finally, subbundles $B(\mathcal{S})$ of the cotangent bundle of the source of the mapping F that we associate with any stratification $\mathcal{S}$ (as in the paragraph above Remark 3.3) are closed sets and contain the 'Glaeser bundle' G_F of F if and only if stratification $\mathcal{S}$ is a Gauss regular TWa stratification (Proposition 3.4). Of course our original hope and motivation (as expressed in the previous paragraph) to study TWa stratifications vis-a-vis the notion of an 'universal' stratum is rooted in this observation (for an inexplicable reason previously not mentioned in the literature on stratifications).

1.1 Main construction, results and hopes - briefly.

Construction. We construct a closed *bundle* $G_F \subset T^*K^n$ over the critical points $\mathrm{Sing}(F)$ and partition $\mathrm{Sing}(F)$ into *'quasistrata'* of points with the fibers of G_F of constant dimension. It turns out (see Theorem 5.1) that TWa stratifications of $\mathrm{Sing}(F)$ exist iff G_F is a Lagrangian subbundle of T^*K^n , i. e. the fibers of bundle G_F are orthogonal to the tangent spaces at the smooth points of the quasistrata (e. g. is true when $l = 1$, see [18]). Fibers of G_F are the orthogonal complements (to the tangent spaces at the smooth points of the quasistrata) over an irreducible component S of a quasistratum only if S is **universal** for the class of TWa stratifications of $\mathrm{Sing}(F)$, i. e. for any $\{S'_j\}_j$ in the class, there is a stratum S'_j with $S \cap S'_j$ being open and dense in both S and S'_j . We relax condition of smoothness of strata to a weaker assumption of Gauss regularity. Construction of bundle G_F involves *Glaeser iterations* of replacing the fibers of the successive closures by the respective linear spans (see [9]), stabilizes after $\rho(F) \leq 2n$ iterations (see [5]) and $\dim(G_F) = n$ for $K \neq \mathbb{R}$ (see Claim 3.8 and Remark 3.9).

Main results:

1. *Criterion:* In Theorem 5.1 we prove that TWa stratifications of $\mathrm{Sing}(F)$ with all strata universal exist iff all fibers of Glaeser bundle G_F are the orthogonal complements to the respective tangent spaces to the quasistrata, and then the partition of $\mathrm{Sing}(F)$ by the dimension of fibers of G_F yields the coarsest *universal TWa stratification.*

2. *Extension.* Proof of Theorem 5.1 relies on Theorem 6.1, in which under the assumptions of a version of Whitney-a condition on the initial data we construct an extension (within a Zariski open subset of the variety in question) of a component of the regular loci of singularities to a Gauss regular subvariety with the prescribed values of the continuous extension of its *Gauss map* over that component. Our construction of this extension is by means of

3. *Bertini-type Theorem 7.2 .* The proof of the latter unexpectedly (and essentially) depends on a novel construction of a metric on desingularization that enables us to make use of an ancient trick of logarithmic differentiation, see Section 1.3.

4. Our examples without universal TWa stratifications and of $F_n \colon K^{4n+1} \to K$ with $\rho(F_n) = n$ are in Sections 10.3 and 10.2. Every hypersurface occurs as a quasistrata of some G_F (Remark 10.3), but we wonder whether the quasistrata of all Lagrangian G_F are smooth ? To proceed with our investigation it is essential to clarify the validity of the following

Conjecture. Assume $l = 1$ and $K = \mathbb{C}$. Then all irreducible components of Glaeser bundle G_F are n-dimensional and G_F is the intersection of the subbundles of $T^*K^n|_{\mathrm{Sing}(F)}$ of the orthogonal complements to the tangent spaces to the strata of TWa stratifications.

1.2 Underlying motivations.

We consider stratifications of the set of the critical points $\mathrm{Sing}(F)$ which satisfy Thom and Whitney-a conditions. Our main goal is to identify the **'universal strata'**. To that end we consider a larger class of TWa stratifications with the condition of smoothness of strata relaxed to a weaker assumption of Gauss regularity and require the strata to be open in their respective closures, pairwise disjoint and, of course, to satisfy the classical TWa conditions.

Contribution towards the double-exponential lower bound conjecture. While the notion of the universal stratum is basic and important in its own right our constructive identification in Theorem 5.1 of the **universal stratifications** is a crucial step towards a solution of a long-standing problem regarding the validity of a double-exponential lower bound on the computational complexity of stratifications, see for example [6] or [22] . Due to Theorem 5.1 it suffices to identify an example of a polynomial map F that admits our 'universal' stratification and has the 'quasistrata' associated with the Glaeser bundle G_F being of a 'very high', i. e. double-exponential in the dimension of the source, degree.

It turns out that the irreducible subsets (we call them *Glaeser components*) over which dimension of the fibers of Glaeser bundle G_F equals the codimension of the respective Glaeser component are universal even with respect to the class of TWa Gauss regular stratifications (for the brevity sake we call the latter TWG-stratifications), see Corollary 3.5. We provide various examples of mappings that admit universal TWG-stratifications, but the question of recognition of an individual universal stratum remains open. We expect that the universal strata in general are precisely the Glaeser components over which Glaeser bundle is of dimension n . The latter components we refer to as *Lagrangian* since off their singular locus the restriction of the Glaeser bundle over these components is a Lagrangian submanifold of T^*K^n in the natural symplectic structure of the latter.

Constructive criterion of Theorem 5.1 'opens doors' to the intriguing questions listed at the very end of Section 1.1 and the question of identifying the individual universal strata. Proof of Theorem 5.1 relies on Theorem 6.1, which in its own turn relies on Theorem 7.2. On the other hand Bertini-type Theorem 7.2 and a surprizing use of desingularization in its proof are perhaps some of the most exciting features of this article (see Section 1.3). Their beautiful and important applications include the Extension Theorem 6.1 and our criterion of universality in Theorem 5.1. Numerous examples of Section 10 are devoted to an illustration of constructions and claims (rather than proofs) of our main results. Finally, the estimation of the computational complexity of our constructions in Appendix 11 and Section 9 results in a double-exponential complexity upper bound on the stratifications by the dimension of the fibers of the Glaeser bundles and, as a consequence, on the universal TWa stratifications.

1.3 Key instrument: a Bertini-type Theorem for singular varieties.

Let $\mathcal{G} \hookrightarrow \mathcal{U}$ and $S \hookrightarrow \mathcal{U} \setminus \mathcal{G}$ be nonsingular algebraic (or analytic) subvarieties, $\mathcal{U} \subset \mathbb{C}^n$ open and dense, $\overline{S}$ a subvariety of $\mathcal{U}$ and $\mathcal{G} = \overline{S} \setminus S$. Assume $\{L_j\}_{1 \le j \le k}$ is a collection of $k > 1$ polynomials on $\mathcal{U}$ vanishing on $\mathcal{G}$ with linearly independent $dL_j(x)$ at the points $x \in \mathcal{U}$ and that the pair of S and bundle $\mathcal{B}$ of vector spaces $\mathcal{B}(x) := \cap_j \operatorname{Ker} dL_j(x) \subset \mathbb{C}^n$ for $x \in \mathcal{G}$ satisfy a version of Whitney-a conditions on the pair, i. e. the limits of tangent spaces $T_x(S)$ for points $x \in S$ converging to a point $b \in \mathcal{G}$ contain fiber $\mathcal{B}(b)$ of $\mathcal{B}$ at b . Denote $L(x, c) := \sum_{1 \le j \le k} c_j L_j(x)$, where $(x, c) \in \mathcal{U} \times \mathbb{C}^k$, and $L_c(x) := L(x, c)$.

For Bertini-type Theorem 7.2 the crucial content is that for a 'generic' $c \in \mathbb{C}^k$ not only

$d(L_c|_S)$ does not vanish on $V(L_c) := \{x \in S : L_c(x) = 0\}$, but that for points $x \in V(L_c)$ 'nearby' any fixed point $x^0 \in \mathcal{G}$ there is also a lower bound (by a positive constant) on the sizes of $d(L_c|_S)(x)$ (or, equivalently, that the angles between the tangent spaces to S and to the hypersurface $\{L_c = 0\} \hookrightarrow \mathcal{U}$ are separated by a positive constant from 0).

We sketch the key idea of our proof in the last paragraph of this subsection. To that end we start with a reduction to a 'nonsingular setting' by means of an embedded desingularization $\sigma : \mathcal{N} \to \mathcal{U}$ of $\overline{S}$ with an additional (standard) desingularization property of all $L_j \circ \sigma$ being (locally) monomials that divide each other for appropriate (local) orderings, say $j(a)$ is the index of the 'smallest' amongst monomials $L_j \circ \sigma$ near point a . Thus the *strict transform*

$$N := \overline{\sigma^{-1}(\overline{S} \setminus \sigma(\mathrm{Sing}(\sigma)))} \hookrightarrow \mathcal{N} \ \text{ of } \ \overline{\mathcal{S}} \ \text{ under map } \ \sigma$$

is nonsingular, while the 'additional' property implies nonsingularity of both the strict transform Λ' of $\{L = 0\} \subset \mathcal{U} \times \mathbb{C}^k$ under map $\tilde{\sigma} := \sigma \times id \colon \mathcal{N} \times \mathbb{C}^k \to \mathcal{U} \times \mathbb{C}^k$ and of $\Lambda := (N \times \mathbb{C}^k) \cap \Lambda'$. Note that with initial hypersurfaces $\{x \in \mathcal{U} : L_j(x) = 0\}$, $1 \le j \le k$, declared 'exceptional' the embedded desingularization property of map σ includes the property of 'normal crossing' for the resulting *exceptional set* $\mathcal{E} := (\cup_{1 \le j \le k} \Lambda_j) \cup \mathrm{Sing}(\sigma)$, where Λ_j's are the strict transforms of $\{L_j = 0\}$'s under map σ . The latter property means that set $\mathcal{E}$ is a union of nonsingular *(exceptional)* hypersurfaces which are coordinate hyperplanes for an appropriate choice of local analytic coordinates. (For a complete exposition see Section 8.)

By applying the standard Sard Theorem in this 'nonsingular setting', i. e. to the restriction of the natural projection $N \times \mathbb{C}^k \to \mathbb{C}^k$ to hypersurface Λ , we conclude that $\Lambda_c := \Lambda \cap (N \times \{c\})$ is nonsingular for 'generic' $c \in \mathbb{C}^k$. Consequently for 'generic' $c \in \mathbb{C}^k$ hypersurfaces $\{L = 0\} \cap (\overline{S} \times \{c\})$ are nonsingular off $\sigma(\mathrm{Sing}(\sigma))$. To establish Theorem 7.2 it suffices then to apply the claimed above estimate at noncritical for map σ values $x = \sigma(a)$ 'nearby' critical (also for map σ) values $x^0 = \sigma(b)$ with $b \in \Lambda_c$ (and a 'nearby' b).

We derive the required lower bound on the sizes of $d(L_c|_S)(x)$ (for points $x \in V(L_c)$ 'nearby' $x^0 \in \mathcal{G}$) by means of 'an estimate via a logarithmic differentiation', namely: for any $b \in \Lambda_c \cap \sigma^{-1}(x^0)$ and for a choice of local coordinates x_i , such that the 'exceptional' hyperplanes containing b are $\{x_i = 0\}$ for $1 \le i \le q$ and one of the remaining coordinates is a local equation of Λ_c , we introduce (with a help of Remark 8.2) a metric on $N \setminus \mathrm{Sing}(\sigma)$ 'nearby' point b , namely: we 'declare' collection $\{dx_i/x_i\}_{1 \le i \le q} \cup \{dx_j\}_{q<j}$ to be orthonormal. Also, for noncritical for map σ points $a \in \Lambda_c \cap \sigma^{-1}(x)$ 'nearby' point b we introduce on the spans $\mathcal{L}_a^* \subset T^*_{\sigma(a)}\mathbb{C}^n$ of $\{dL_j(\sigma(a))\}_{1 \le j \le k}$ a new norm equivalent to the original by 'declaring' these collections to be orthonormal. Note that the composite $\sigma_a^*|_{\mathcal{L}_a^*} : \mathcal{L}_a^* \to T_a^*N$ of the pull back by σ with the restriction to T_a^*N vanishes on $(T_{\sigma(a)}(S))^{\perp} \cap \mathcal{L}_a^*$ and coincides with the composite of embeddings $i_a \colon \mathcal{L}_a^*|_S := \mathcal{L}_a^*/(T_{\sigma(a)}(S))^{\perp} \hookrightarrow T^*_{\sigma(a)}(S)$ followed by the pull backs $\sigma_a^* \colon T^*_{\sigma(a)}(S) \to T_a^*(N)$. Also, since the embeddings i_a are isometries it follows that the norms of $\sigma_a^*|_{\mathcal{L}_a^*}$ and of $\mathcal{A}_a := \sigma_a^*|_{\mathcal{L}_a^*|_S} : \mathcal{L}_a^*|_S \to T_a^*(N)$ coincide.

It is an easy consequence of the 'logarithmic differentiation' that in the introduced metric the norms of $\sigma_a^*|_{\mathcal{L}_a^*}$ coincide with the sizes of $\sigma_a^*(dL_{j(a)}(\sigma(a)))$ and that the latter coincide with $|L_{j(a)} \circ \sigma(a)|$ (up to the l_2 norm of the exponents of the monomial $L_{j(a)} \circ \sigma$). Moreover, the sizes of $\mathcal{A}_a(\eta_a) = \sigma_a^*(\eta_a)$ for $a \in \Lambda_c$ coincide with $|L_{j(a)} \circ \sigma(a)|$, where $\eta_a := d(L_c|_S)(\sigma(a))$ The required lower bound on the sizes of η_a (at the points $a \in \Lambda_c$ and 'nearby' point b) then follows from the upper bound by the sizes of $\mathcal{A}_a(\eta_a)$ on the norms of $\mathcal{A}_a$, see Section 8 for complete details.

1.4 Basic terminology, notations and a guide to the article.

Below, for a Gauss regular algebraic (or analytic) set S we denote by $T_a(S)$ the unique limiting position at a of the tangent spaces $T_x(S)$ to S at the nonsingular points $x \in \mathrm{Reg}(S)$ and drop the mention of map F in expression "TWa stratification of $\mathrm{Sing}(F)$ for F", whenever the reference to F is clear. By an irreducible component of a constructible set we mean here its intersection with an irreducible component of its closure. In Section 3 we describe a construction of Glaeser bundle G_F involving iterations (starting with $\mathcal{T} := \{(x,\ \mathrm{Span}\{df_j(x)\}_{1\le j\le l})\}_{x\in K^n}$, where Span denotes the K-linear hull of a family of vectors in $(T_xK^n)^*$) of replacing the fibers of the successive closures by their linear spans. Thus, G_F is the minimal closed bundle of vector spaces over $\mathrm{Sing}(F)$ which contains $\overline{\mathcal{T}}|_{\mathrm{Sing}(F)}$. The correspondence '$F \to G_F$' is *functorial* with respect to isomorphisms preserving fibers of F 'near' its critical value 0 (including with respect to C^1 diffeomorphisms when K is $\mathbb{C}$ or $\mathbb{R}$), see Section 3.

Let quasistrata $\mathcal{G}_k \subset K^n$ consist of the points of $\mathrm{Sing}(F)$ whose fibers of G_F are vector spaces of dimension k. Assuming Thom stratification 'near' $\mathrm{Sing}(F)$ exists, cf. [18] (e. g. when $l = 1$), it follows that $k \ge l$, $\dim(G_F) \le n$ and, consequently, the dimensions of the quasistrata $\mathcal{G}_k$ are smaller or equal $n-k$ (Lemma 3.7 and the remark following). We refer to the bundle G_F as *Lagrangian* whenever all submanifolds $\mathrm{Reg}(G_F|_{\mathcal{G}_k})$ of $K^n \times (K^n)^*$ are Lagrangian in the natural symplectic structure of the latter.

Following Introduction we review in Section 2 the classical notions of Thom and Whitney-a stratifications and the *canonicity* property of the latter extending the classical notions to our TWG-stratifications (for the sake of a stronger version of the canonicity property introduced in Section 1.1 under the name of *universality*). We derive consequences of our constructions related to the notion of Glaeser bundles in Section 3. All of the latter are simple (with the exception of Claim 3.8 which is perhaps the least obvious).

The principal aim of this article is a constructive criterion for the existence of a universal TWG-stratification $\{S_i\}_i$. Our Theorem 5.1 states that $\mathrm{Sing}(F)$ admits a universal TWG-stratification for F iff Glaeser bundle G_F is Lagrangian. Consequently for any universal TWG-stratification $\mathcal{S} = \{S_i\}_i$ of $\mathrm{Sing}(F)$ sets $\mathcal{S}_{(m)} := \bigcup_{\{i:\dim(S_i)=m\}} S_i$ coincide with the quasistrata $\mathcal{G}_{n-m}$ for every m. Partitions $\{S_{k,i}\}_i$ of quasistrata $\mathcal{G}_k$ into pairwise disjoint constructible irreducible sets open in their respective closures induce partitions $\mathcal{S} := \{S_{k,i}\}_{k,i}$ of $\mathrm{Sing}(F)$. For a Lagrangian G_F we consider the (nonempty) class of the latter partitions with an additional property of $\dim(S_{k,i}) = n-k$ for all sets $S_{k,i}$ (see the paragraph preceding Proposition 4.7). We establish in Section 4 the simpler implication of our constructive criterion, namely: if the bundle G_F is Lagrangian then the latter partitions $\mathcal{S}$ form the universal TWG-stratifications of $\mathrm{Sing}(F)$.

A more difficult converse implication is proved in Sections 5, 6, 7 and 8. It relies on Proposition 5.9, which is of interest in its own right. A straightforward generalization of the latter is Extension Theorem 6.1. It provides an extension of a (smooth) stratum $\mathcal{G}$ of a singular locus of a variety S (algebraic or analytic, open in its closure and with $\mathcal{G}$ being essentially its boundary) to a Gauss regular subvariety $\mathcal{G}^+$ of $\overline{S}$ with a prescribed tangent bundle $T_{\mathcal{G}}$ over $\mathcal{G}$ (under the necessary assumptions of our version of Whitney-a condition for the pair of $T_{\mathcal{G}}$ over $\mathcal{G}$ and S). The key ingredient to both is our version of Bertini-type Theorem 7.2 for singular varieties (briefly described in the previous Section 1.3), whose proof in Section 8 makes an essential (and surprizing) use of the resolution of singularities.

In Section 10.2 we construct a family of $F_n\colon K^{4n+1} \to K$ with the *index of stabilization*

$\rho(F_n) = n$. In Section 10.3 we prove that $F := AX^2 + 2B^2XY + CY^2$ does not admit a universal TWG-stratification. Moreover, we show that for an appropriate variation of the former example an arbitrary hypersurface appears as $\mathcal{G}_r$ for some r (see Remark 10.3). We also consider in Sections 10.1 , 10.4 (discriminant-type) examples for which $\{\mathcal{G}_r\}_r$ are universal TWG-stratifications (and exhibit these stratifications explicitly). Finally, in the Appendix Section 11 we provide for the sake of completeness a calculation of the computational complexity of the main construction in our Extension Theorem 6.1.

In abuse of notation in the remainder of the article we identify (occasionaly) the dual $(K^n)^*$ with K^n , the cotangent bundle $T^*(K^n)$ with K^{2n} , denote $dF(x) := \mathrm{Span}\{\{df_i(x)\}_{1\le i\le l}\}$, the variety of zeroes of a polynomial f by $\{f = 0\}$, refer to "Gauss regular" as "G-regular" and to a nonsingular variety as a manifold.

This paper replaces and supersedes our earlier preprint [13].

2 Canonical TWa stratifications.

Recall that the traditional notion of *stratification* $\{S_i\}_i$, say of the set of critical points $\mathrm{Sing}(F)$ of F (meaning the points x such that $\dim(dF(x)) < l$) includes $\mathrm{Sing}(F) = \cup_i S_i$ with pairwise disjoint S_i's; the irreducibility, nonsingularity and openness in its closure of each stratum S_i (in the classical euclidean topology for $K = \mathbb{C}$ or $\mathbb{R}$ connectedness replaces irreducibility); and also the frontier condition, i. e. that for each pair (S_i , S_j) if $\overline{S}_i \cap S_j \neq \emptyset$ then $S_j \subset \overline{S}_i$, as is e. g. in [8], [10]. Also, a pair of constructible nonsingular subsets (Y , X) of K^n satisfies Whitney-a condition provided that $\lim_{m\to\infty}(y_m, T_{y_m}(Y)) = (x, T)$ for a sequence $\{y_m \in Y\}_{m<\infty}$, a point $x \in X$ and a subspace $T \subset K^n$ implies that $T \supset T_x(X)$, see e. g. [7, 8, 10, 19, 20, 25, 26]. Finally, a constructible nonsingular subset $X \subset K^n$ satisfies Thom condition for a dominating map $F : K^n \to K^l$ provided that $\lim_{m\to\infty}(z_m, dF(z_m)) = (x, V)$ for a sequence $\{z_m \in K^n\}_{m<\infty}$ of noncritical points of F , a point $x \in X$ and a (suitable l-dimensional) subspace $V \subset (K^n)^{dual}$ implies that V is orthogonal to $T_x(X)$. Of course stratification $\{S_i\}_i$ of $\mathrm{Sing}(F)$ satisfies Whitney-a or Thom (for a dominating map $F : K^n \to K^l$) condition whenever every pair (S_i , S_j) satisfies Whitney-a or, respectively, every S_i satisfies Thom condition.

In the present article for the sake of a concept of universality introduced in Section 1.1, i. e. of a stronger version of the traditional notion of canonicity (see Remark 2.3 below), we relax condition of smoothness and allow S_i to be G-regular. We consider *Gauss regular stratifications* $\{S_i\}_i$ of $\mathrm{Sing}(F)$, i. e. all S_i are G-regular, irreducible, open in their respective closures and pairwise disjoint (but do not necessarily fulfil the frontier condition, which may occur for TWG-stratifications naturally induced by Glaeser bundles G_F , see example in Remark 10.6). Extension of the notions of Thom (for a map F) and of Whitney-a conditions on stratifications to Gauss regular stratifications is straightforward.

Lemma 2.1 *i) Thom stratifications exist iff the following condition holds:*

(1) any irreducible constructible set $S \subset \mathrm{Sing}(F)$ *contains an open dense subset* $S^o \subset \mathrm{Reg}(S)$ *such that if a sequence* $\{(x_m , dF(x_m)) \subset K^{2n}\}_m$ *has a limit* $\lim_{m\to\infty}(x_m , dF(x_m)) = (x_0 , V)$ *, where* $x_0 \in S^o$ *,* $x_m \in K^n \setminus \mathrm{Sing}(F)$ *and* V *is an* l*-dimensional linear subspace of* $(K^n)^*$ *, then it follows* $V \perp T_{x_0}(S^o)$ *;*

ii) TWa stratifications exist iff (1) and the following condition hold:

(2) for any smooth irreducible constructible set $M \subset \mathrm{Sing}(F)$ *and any irreducible constructible set* $S \subset \mathrm{Sing}(F)$ *there is an open dense subset* $S^o \subset \mathrm{Reg}(S)$ *such that if a*

sequence $\{(x_m\ ,\ V_m) \subset K^n \times (K^n)^*\}_m$ *has a limit* $\lim_{m\to\infty}(x_m\ ,\ V_m) = (x_0\ ,\ V)$ *, where* $x_0 \in S^o$, $x_m \in M$ *and subspaces* V_m *in* $(K^n)^*$ *are orthogonal to* $T_{x_m}(M) \subset K^n$ *, then it follows that subspace* $V \subset (K^n)^*$ *is orthogonal to* $T_{x_0}(S^o) \subset K^n$ *.*

Proof. Since the proofs of i) and ii) are similar, we provide only a proof of ii). First assume that $\{S_i\}_i$ is a TWa stratification. Once again the proofs of properties (1) and (2) are similar and we provide only a proof of (2). Take a unique S_i (respectively, S_j) such that $M \cap S_i$ (respectively, $S \cap S_j$) is open and dense in M (respectively, in S). If $S \setminus \overline{S_i}$ is open and dense in S then the choice of $S^o := (S_j \cap \mathrm{Reg}(S)) \setminus \overline{S_i}$ is as required in (2). On the other hand the remaining assumptions of (2) can not hold which makes (2) valid, but vacuous. (Property (1) holds due to the Thom property of $\{S_i\}_i$.) Otherwise $S \subset \overline{S_i}$ and the choice of $S^o := S_j \cap \mathrm{Reg}(S)$ is as required in (1) and in (2) due to the Thom and Whitney-a properties of $\{S_i\}_i$ respectively. Indeed, it suffices to replace the sequence of (2) by its subsequence for which exists $\lim_{m\to\infty} T_{x_m}(M) =: W$, and then to choose another sequence $\{x'_m\}_m$ of points in $M \cap S_i$ with the 'distance' between respective $(x_m\ ,\ T_{x_m}(M))$ and $(x'_m\ ,\ T_{x'_m}(M))$ converging to zero. Then $W = \lim_{m\to\infty} T_{x'_m}(M)$ and is orthogonal to V . On the other hand due to the Whitney-a property of the pair S_i , S_j it follows that $W \supset T_{x_0}(S_j) \supset T_{x_0}(S)$ and therefore also $T_{x_0}(S)$ is orthogonal to V as required.

Now we assume that (1) and (2) are valid. We construct strata $S_1, S_2, \ldots$ by induction on their codimensions, i. e. $\mathrm{codim}(S_1) \le \mathrm{codim}(S_2) \le \cdots$. So assume that $S_1, \ldots, S_k$ are already produced with $\mathrm{codim}(S_k) = r$, set $\mathrm{Sing}(F) \setminus (S_1 \cup \cdots \cup S_k) =: Z$ being of $\mathrm{codim}(Z) := r_1 > r$ and that Thom and Whitney-a properties are satisfied for stratification $\{S_i\}_{1\le i\le k}$ of $\mathrm{Sing}(F) \setminus Z$. Subsequently for every irreducible component S of Z of $\mathrm{codim}(S) = r_1$ (and by making use of the noetherian property of the Zariski topology of S) we choose a maximal open subset of $\mathrm{Reg}(S)$ which satisfies both property (1) and the property (2) with respect to the choices of sets S_i , for $1 \le i \le k$, as the set M of (2). By additionally choosing each subsequent S_j in $\mathrm{Sing}(F) \setminus (S_1 \cup \cdots \cup S_{j-1})$ for $k < j \le k_1$ we produce strata $S_{k+1}, \ldots, S_{k_1}$ of codimensions r_1 with $\mathrm{codim}((\mathrm{Sing}(F) \setminus (S_1 \cup \cdots \cup S_{k_1})) > r_1$. Such choice ensures Thom and Whitney-a properties of stratification $\{S_i\}_{1\le i\le k_1}$ of set $\cup_{1\le i\le k_1} S_i$, as required in the inductive step, which completes the proof of ii). ∎

Remark 2.2 *Say* $l > 1$ *and* $F : K^n \to K^l$ *is a dominating polynomial mapping. It is not true that then necessarily exists a stratification that satisfies Thom condition with respect to* F *, e. g. consider the 'local' blowing up of the origin:*

$$F : (z_1, \ldots, z_n) \mapsto (z_1\ ,\ z_1 \cdot z_2\ ,\ \ldots\ ,\ z_1 \cdot z_n)$$

(here origin is an isolated critical value). For the validity of properties (2) and, when $l = 1$ *, of (1) of Lemma 2.1, see [26], [24], [17], [25], [10] and, respectively, [18]. For conditions on* F *implying the validity of property (1) of Lemma 2.1 when* $l > 1$ *see e. g. [18], [8], [19].*

Remark 2.3 *Fix a class of stratifications. A stratification* $\{S_i\}_i$ *of* $\mathrm{Sing}(F) = \cup_i S_i$ *is called canonical (or minimal), e. g. in [8] and [22], if for any other stratification* $\{S'_i\}_i$ *of* $\mathrm{Sing}(F) = \cup_i S'_i$ *in this class with* $\mathrm{codim}(S_1) \le \mathrm{codim}(S_2) \le \cdots$ *and* $\mathrm{codim}(S'_1) \le \mathrm{codim}(S'_2) \le \cdots$ *it follows (after possibly reindexing* $\{S'_i\}$ *) that* $S'_1 = S_1, \ldots, S'_k = S_k$ *and* $S'_{k+1} \subsetneq S_{k+1}$ *. Constructed in the proof of Lemma 2.1 Thom and TWa stratifications are canonical in the corresponding classes. These respective canonical stratifications are clearly unique. We extend to Gauss regular stratifications the concepts and constructions introduced above for stratifications.*

3 Dual bundles of vector spaces for TWG-stratifications.

We will repeatedly apply the following construction. Let M , N be constructible sets open in their Zariski closures (by default we consider Zariski topology, sometimes in the case of K being $\mathbb{C}$ or $\mathbb{R}$ we also use euclidean topology). In the analytic case we assume alternatively that M , N are analytic manifolds. Let V , W be vector spaces. For a subset $\mathcal{T} \subset M \times V$ we denote by $\mathcal{T}^{(0)} := \mathcal{T}$ and by $\mathcal{T}^{(1)} \subset M \times V$ a bundle of vector spaces whose fiber $\mathcal{T}^{(1)}_x$ at a point $x \in M$ is the linear hull of the fiber $(\overline{\mathcal{T}})_x$ of the closure $\overline{\mathcal{T}} \subset M \times V$, cf. [9] . Defining in a similar way $\mathcal{T}^{(p+1)}$ starting with $\mathcal{T} := \mathcal{T}^{(p)}$, for $p \geq 0$, results in an increasing chain of (not necessarily closed) bundles of vector spaces and terminates at $\mathcal{T}^{(\rho)}$ such that $\mathcal{T}^{(\rho)} = \mathcal{T}^{(\rho+1)}$ with $\rho \leq 2\dim(V)$ (see [5]). We denote $Gl(\mathcal{T}) = \mathcal{T}^{(\rho)}$ and refer to the smallest $\rho = \rho(\mathcal{T})$ as the index of stabilization. The so called 'Glaeserization' $Gl(\mathcal{T})$ of $\mathcal{T}$ is the minimal closed bundle of vector spaces which contains $\mathcal{T}$. We apply this construction to $\mathcal{T} = \{(x, dF(x))\}$ where x ranges over all noncritical points of F . The result we denote by $G^{(p)} := G^{(p)}_F := \overline{\mathcal{T}^{(p)}}|_{\mathrm{Sing}(F)}$, for $p \geq 0$, and $G := G_F := Gl(\mathcal{T})|_{\mathrm{Sing}(F)}$ (and still refer to the smallest $\rho = \rho(F)$ as the index of stabilization). We mention that according to [15] Thom stratifications with respect to F exist iff $\dim(G^{(0)}) \leq n$, cf. Remark 3.10 and [18]. (We do not make use of the latter criterion in this article.)

Denote $G_x := \pi^{-1}(x) \cap G$, where $\pi : T^*(K^n)|_{\mathrm{Sing}(F)} \to \mathrm{Sing}(F)$ is the natural projection. The proofs of the following Proposition and its corollary are straightforward.

Proposition 3.1 *Let $\mathcal{T}_M \subset M \times V,\ \mathcal{T}_N \subset N \times W$ and $h^{-1} : N \to M, H : N \times W \to M \times V$ be homeomorphisms which commute with the natural projections $N \times W \to N,\ M \times V \to M$. Assume in addition that H is linear on each fiber of these projections and that $H(\mathcal{T}_N) = \mathcal{T}_M$. Then $H(Gl(\mathcal{T}_N)) = Gl(T_M)$, moreover $H(T^{(i)}_N) = T^{(i)}_M$ for every i .*

Corollary 3.2 *Let M , N be nonsingular, $\mathcal{T}_M \subset T^*M,\ \mathcal{T}_N \subset T^*N$. If $h : M \to N$ is an isomorphism such that for the pullback D^*h by h we have $(D^*h)(\mathcal{T}_N) = \mathcal{T}_M$ then $(D^*h)(Gl(\mathcal{T}_N)) = Gl(\mathcal{T}_M)$. Moreover, $(D^*h)(\mathcal{T}^{(i)}_N) = \mathcal{T}^{(i)}_M$ for every i .*

For $K = \mathbb{C}$ or $\mathbb{R}$ the correspondence '$F \to G_F$' and the partition $\{\mathcal{G}_r\}_{l \leq r \leq n}$ of $\mathrm{Sing}(F)$ introduced in Section 1.4 are functorial with respect to the C^1 diffeomorphisms h preserving fibers of F 'near' its critical values.
(For an arbitrary K replace "C^1 diffeomorphisms" above by "isomorphisms".)

With any Gauss regular stratification $\mathcal{S} = \{S_i\}_i$, where $\mathrm{Sing}(F) \supset \cup_i S_i$, we associate a subbundle $B = B(\mathcal{S})$ of $T^*(K^n)|_{\mathrm{Sing}(F)}$ of vector subspaces of $(K^n)^*$ such that for every i and a smooth point $a \in S_i$ the fiber $B_a := (T_a(S_i))^{\perp} \subset (K^n)^*$ and for a singular point a of S_i the fiber B_a is defined by continuity, by making use of S_i being G-regular. Note that the dimension of fibers $\dim(B_a) = \mathrm{codim}(S_i)$ for $a \in S_i$.

Remark 3.3 *Note that for any Gauss regular stratification $\mathcal{S} = \{S_i\}_i$ of $\mathrm{Sing}(F)$ bundle $B(\mathcal{S}) = \cup_i B(\mathcal{S})|_{S_i}$ and for any stratum S_i bundle $B(\mathcal{S})|_{S_i}$ is an irreducible n-dimensional Gauss regular set open in its closure. Thus, $B(\mathcal{S})|_{S_i}$ are irreducible components of $B(\mathcal{S})$.*

Proposition 3.4 *A Gauss regular stratification $\mathcal{S}$ of $\mathrm{Sing}(F)$ is a TWG-stratification iff $G_F \subset B(\mathcal{S})$ and $B(\mathcal{S})$ is closed.*

Proof. It follows by a straightforward application of definitions that Thom and Whitney-a properties for any Gauss regular stratification $\mathcal{S} = \{S_i\}_i$ of $\mathrm{Sing}(F)$ are equivalent to $G^{(1)} \subset B(\mathcal{S})$ and, respectively, that set $B(\mathcal{S})$ is closed. Due to the definition of bundle G proposition follows. ▪

Corollary 3.5 *It follows due to the preceding Remark and Proposition that all n-dimensional irreducible components of G_F appear as irreducible components of $B(\mathcal{S})$ for any TWG-stratification $\mathcal{S} = \{S_i\}_i$ of $\mathrm{Sing}(F)$ (provided that the TWa stratifications of $\mathrm{Sing}(F)$ exist). Therefore every irreducible component $\mathcal{G}$ of $\mathcal{G}_r$ with $(G_F)|_{\mathcal{G}}$ being n-dimensional is a universal stratum.*

Remark 3.6 *Let $\{S_i\}_i$ be a TWG-stratification of $\mathrm{Sing}(F)$. Then for every $0 \le m \le n$ the union $\bigcup_{\dim(S_i)=m} S_i$ coincides with $(\bigcup_{\dim(S_i)\ge m} S_i) \setminus (\bigcup_{\dim(S_i)>m} S_i)$ and therefore is open in its closure. Also due to Proposition 3.4 it is G-regular. Moreover, if we replace any subfamily of $\{S_i\}_i$ of the same dimension m by its union S, we would again obtain a TWG-stratification if only S is open in its closure.*

Below (and throughout the article) $(G_F)_x$ and G_x denote the fibers of bundle G_F at the respective points $x \in \mathrm{Sing}(F)$. Also, $r := \mathrm{codim}(\mathrm{Sing}(F)) := n - \dim(\mathrm{Sing}(F))$.

Lemma 3.7 *The following three statements are equivalent:*

- *TWa stratifications exist;*
- *TWG-stratification exist;*
- *condition (2) of Lemma 2.1 and the following property hold:*

(1') any irreducible constructible set $S \subset \mathrm{Sing}(F)$ contains an open dense subset $S_0 \subset \mathrm{Reg}(S)$ such that for any $x_0 \in S_0$ we have $T_{x_0}(S) \perp (G_F)_{x_0}$.

Lemma 2.1 implies (assuming TWa stratifications of $\mathrm{Sing}(F)$ exist) that $\mathrm{codim}(\mathrm{Sing}(F)) \ge \min_{a\in\mathrm{Sing}(F)}\{\dim(G_a)\} \ge l$ and, due to Lemma 3.7, $\dim(G_F) \le n$.

Proof. For the proof of (1') above note that property (1') with G_{x_0} being replaced by $G^{(1)}_{x_0}$ is a straightforward consequence of the Thom property of stratification $\mathcal{S}$ with respect to F and condition (1) of Lemma 2.1 , which Thom property implies. By making use then of condition (2) of Lemma 2.1 consecutively property (1') with G_{x_0} being replaced by $G^{(p)}_{x_0}$, for $p \ge 1$, follows and implies property (1') as stated, since $G = G^{(p)}$ for $p = \rho(F)$. Otherwise the proof is similar to that of Lemma 2.1 with the exception that we replace $\mathrm{Reg}(S)$ with the maximal (by inclusion) open subset U of $\overline{S}$ to which by continuity the Gauss map of S uniquely extends from $\mathrm{Reg}(S)$. ▪

Claim 3.8 *Assume that Thom stratification of $\mathrm{Sing}(F)$ exists (e. g. if $l = 1$, see [18]), and that $K \ne \mathbb{R}$, then $\mathrm{Sing}(F) = \cup_{j\ge r}\mathcal{G}_j$. Also, then quasistrata $\mathcal{G}_j$ are open and dense in irreducible components of $\mathrm{Sing}(F)$ of dimension $n - j$ (if such exist). In particular, quasistratum $\mathcal{G}_r \ne \emptyset$ and $\dim(G_F) = n$.*

Remark 3.9 *In the example of $F : \mathbb{R}^2 \to \mathbb{R}$ defined by $F := x^3 + x \cdot y^4$ the critical points $\mathrm{Sing}(F) = \{0\}$, the fiber at 0 of the Glaeser bundle G_F is spanned by dx, i. e. is 1-dimensional, and therefore $\dim(G_F) = 1 < 2 =: n$.*

Proof of Claim. It suffices to verify that a generic point of an irreducible component of $\mathrm{Sing}(F)$ of dimension $n-j$ belongs to $\mathcal{G}_j$, since the openness is due to the upper semicontinuity of the function $g\colon x \to \dim(G_x)$.

We first reduce to the case of $l=1$. Indeed, let U be an open set such that $U \cap \mathrm{Sing}(F)$ is smooth, irreducible and of dimension $n-j$. We may assume w.l.o.g. that $0 \in U \cap \mathrm{Sing}(F)$ and that for the 1-st component $f := f_1$ of $F : K^n \to K^l$ the differential $df(0) = 0$ (which anyway holds after a linear coordinate change in the target K^l of map F). By making use of the reduction assumption for f (the case of $l=1$) it follows that $(G_f)_a$ are the orthogonal complements of the tangent spaces $T_a(\mathrm{Sing}(f)) \subset T_a(\mathrm{Sing}(F))$ for a in an open dense subset $\mathcal{V}$ of $U \cap \mathrm{Reg}(\mathrm{Sing}(f))$. We may also assume by shrinking U and replacing 0 , if needed, that $0 \in \mathcal{V}$, that $\dim(G_F)_a$ is constant for $a \in U \cap \mathrm{Sing}(F)$ and that $U \cap \mathrm{Sing}(f) = \mathcal{V}$ is smooth, open and dense in an irreducible component of $\mathrm{Sing}(f)$. Inclusions $\mathrm{Sing}(f) \subset \mathrm{Sing}(F)$ and $(G_f)_a \subset (G_F)_a$, for $a \in \mathrm{Sing}(f)$, are straightforward consequences of the definitions. We continue the proof following

Remark 3.10 *Note that replacing the assumption of the existence of Thom stratification of* $\mathrm{Sing}(F)$ *by the assumption that* $\dim(G^{(0)}) \le n$ *and following the proof above would then imply that* $(G_F)_a = (G_f)_a$ *, for* $a \in \mathcal{V}$ *, and, moreover, that* $\dim(U \cap \mathrm{Sing}(F)) = \dim(U \cap \mathrm{Sing}(f))$ *. In particular, it would follow that* $(G_F)_a$ *are the orthogonal complements of the tangent spaces* $T_a(\mathrm{Sing}(F)) = T_a(\mathrm{Sing}(f))$ *for* $a \in U \cap \mathrm{Sing}(F)$ *, cf. with i) of Lemma 2.1 and a criterion* $\dim(G^{(0)}) \le n$ *for the existence of Thom stratifications of* $\mathrm{Sing}(F)$ *for mapping* F *from [15].*

By making use of the existence of Thom stratifications of $\mathrm{Sing}(F)$ for mapping F and consequently of (1') of Lemma 3.7 applied to F it follows $(G_F)_a$ are orthogonal to $T_a(\mathrm{Sing}(F))$ for $a \in U \cap \mathrm{Sing}(f)$. Therefore, by making use of the inclusions above, it follows that $(G_F)_0 = (G_f)_0$ and $T_0(\mathrm{Sing}(f)) = T_0(\mathrm{Sing}(F))$, in particular implying that $\dim(U \cap \mathrm{Sing}(f)) = \dim(U \cap \mathrm{Sing}(F))$. Hence also $(U \cap \mathrm{Sing}(f)) = (U \cap \mathrm{Sing}(F))$, which suffices by making use of the established above inclusions.

In the case of $l=1$ and by once again making use of (1') of Lemma 3.7 it suffices w.l.o.g. to consider the case of the restriction of F to a plane of dimension j intersecting transversally an irreducible component Z of $\mathrm{Sing}(F)$ of dimension $n-j$ (if such exists) at a , thus reducing the proof to the case of $l=1$ and of a being an isolated critical point. In the latter case it suffices to show that $(G_F)_a = K^n$.

If K is algebraically closed our claim follows since for any $c_2, \ldots, c_n \in K$ due to $F_i(a) := \frac{\partial F}{\partial x_i}(a) = 0$, $1 \le i \le n$, the germ at a of $\Gamma := \{F_i - c_i \cdot F_1 = 0 , 2 \le i \le n\}$ is at least 1-dimensional, thus producing $dx_1 + c_2 \cdot dx_2 + \cdots + c_n \cdot dx_n$ in $(\overline{G_F^{(0)}})_a \subset (G_F)_a$ by means of limits of $dF(a)/||dF(a)||$ along Γ , as required. ∎

4 Universality and Lagrangian bundles.

We start by introducing a partial order on the class of TWG-stratifications of $\mathrm{Sing}(F)$ (note that it differs from the order defined in Ch.1 [8], see Remark 2.3). For any pair $\mathcal{S} = \{S_i\}_i$ and $\mathcal{S}' = \{S'_j\}_j$ of TWG-stratifications of $\mathrm{Sing}(F)$ and for every S_i there exists a unique $j = j(i)$ such that $S_i \cap S'_j$ is open and dense in S_i , and reciprocally for every S'_j there exists a unique $i = i(j)$ such that $S_i \cap S'_j$ is open and dense in S'_j . We say that $\mathcal{S}$ is *larger* than $\mathcal{S}'$ (or, in other words $\mathcal{S}$ is 'almost everywhere finer' than $\mathcal{S}'$) if for every S_i and $j := j(i)$ holds $i = i(j)$. In particular, the universality of a TWG-stratification means that it is the largest with respect to this partial order.

Proposition 4.1 *For a pair of TWG-stratifications* $\mathcal{S}$, $\mathcal{S}'$ *of* $\mathrm{Sing}(F)$ *TWG-stratification* $\mathcal{S}$ *is larger than* $\mathcal{S}'$ *iff bundle* $B := B(\mathcal{S}) \subset B' := B(\mathcal{S}')$ *.*

Proof. Let $\mathcal{S}$ be larger than $\mathcal{S}'$. For each i we have that $S_i \cap S'_j$ (where $j := j(i)$) is open and dense in both S_i , S'_j , while $\dim(S_i \cap S'_j) = \dim(S_i) = \dim(S'_j)$. Therefore, for any point $a \in S_i \cap S'_j$ we have $T_a(S_i) = T_a(S'_j)$, i. e. $B(S_i)_a = B(S'_j)_a$. It follows for any point $b \in S_i$ that $B_b = B(S_i)_b \subset B'_b$ since the Gauss map of $\overline{S_i}$ is continuous on S_i and B' is closed due to Proposition 3.4.

Conversely, let $B \subset B'$. For every S_i let $j := j(i)$, then $S_i \cap S'_j$ is open and dense in S_i . It follows that for any point $a \in S_i \cap S'_j$ inclusion $T_a(S_i) \subset T_a(S'_j)$ holds and therefore $B_a \supset B'_a$ implying that $B_a = B'_a$ and $\dim(S_i) = \dim(S'_j)$, hence $S_i \cap S'_j$ is open and dense in S'_j , i. e. $i(j) = i$. ∎

Proposition 4.1 and Remark 3.6 imply the following corollary.

Corollary 4.2 *i) If* $B(\mathcal{S}) = B(\mathcal{S}')$ *for a pair of TWG-stratifications* $\mathcal{S} = \{S_i\}_i$ *and* $\mathcal{S}' = \{S'_j\}_j$ *of* $\mathrm{Sing}(F)$ *then for all* k *constructible sets* $\mathcal{S}_{(k)} := \bigcup_{\dim(S_i)=k} S_i$ *and* $\mathcal{S}'_{(k)} := \bigcup_{\dim(S'_j)=k} S'_j$ *coincide and are G-regular;*

ii) For universal TWG-stratification $\mathcal{S}$ *of* $\mathrm{Sing}(F)$ *the unions* $\mathcal{S}_{(k)}$, $0 \le k \le n$ *, of equidimensional strata are independent on the choices of* $\mathcal{S}$ *.*

Remark 4.3 *TWG-stratifications exist iff for any point* $x \in \mathrm{Reg}(\mathcal{G}_k)$ *fiber* G_x *of* G_F *at* x *is orthogonal to* $T_x(\mathcal{G}_k)$ *. Indeed, existence of TWG-stratifications of* $\mathrm{Sing}(F)$ *(for* F *) implies claimed orthogonality due to (1') of Lemma 3.7. Conversely, the existence of TWG-stratifications follows from Lemma 3.7 by making use of the existence of Whitney-a stratifications [26], [20], [17], [25].*

For a (constructible) closed subbundle $B \subset T^*(K^n)|_Z$ (of the cotangent bundle $T^*(K^n)$ of K^n restricted over a subset $Z \subset K^n$ and where subbundle, as is common throughout this article, means only that the fibers B_x , $x \in Z$, of B are vector subspaces of the fibers $(K^n)^{dual}$ at $x \in Z$ of bundle $T^*(K^n)|_Z$) we consider its *'quasistrata'*

$$B_{(k)} := \{x \in Z : \dim_K(B_x) = k\}, 0 \le k \le n \ .$$

(This construction applied to bundle $B = G_F$ of course results in quasistrata $B_{(k)} = \mathcal{G}_k$.)

Definition 4.4 *We refer to an irreducible component* $\mathcal{B}$ *of the quasistrata* $B_{(k)}$, $0 \le k \le n$*, as* **Lagrangian** *if for points* $x \in \mathrm{Reg}(\mathcal{B})$ *the tangent spaces* $T_x(\mathcal{B})$ *are the orthogonal complements of* B_x *. We refer to a bundle* B *as Lagrangian whenever all irreducible components of* $B_{(k)}$, $0 \le k \le n$ *, are Lagrangian.*

Remark 4.5 *For any closed bundle* B *Lagrangian components of its quasistrata* $B_{(k)}$ *are automatically G-regular (cf. Remark 3.6) and of dimension* $n-k$ *.*

Remark 4.6 *Claim 3.8 implies that for every* $(n-k)$*-dimensional irreducible component* C *of* $\mathrm{Sing}(F)$ *there exists a Lagrangian component of* $\mathcal{G}_k$ *dense in* C *. Consequently the Lagrangian components of* $\{\mathcal{G}_k\}_{0 \le k \le n}$ *are dense in* $\mathrm{Sing}(F)$ *.*

Due to the upper-semicontinuity of the function $\mathrm{Sing}(F) \ni x \to \dim_K(B_x)$ sets $B_{(k)}$ are constructible and open in their respective closures. We consider *partitions* $\{S_{k,i}\}_i$ of $B_{(k)}$ (and consequently partitions $\mathcal{S} := \{S_{k,i}\}_{k,i}$ of $\mathrm{Sing}(F)$) into pairwise disjoint constructible irreducible sets $S_{k,i}$ open in their respective closures. For a Lagrangian bundle B the class of such partitions with an additional property that $\dim(S_{k,i}) = n-k$ for all $S_{k,i}$ is not empty. (We may construct such partitions for example by means of splitting sets $B_{(k)}$ into a sequence of its irreducible components and then defining $S_{k,i}$ for $i \geq 1$ to be the i-th irreducible component of $B_{(k)}$ without the union of the preceding ones.) According to Proposition 3.4 partitions $\mathcal{S}$ form TWG-stratifications of $\mathrm{Sing}(F)$ whenever bundle $B \supset G_F$ and $B(\mathcal{S})$ is a closed set.

Proposition 4.7 *If closed bundle B is Lagrangian then there is a bijective correspondence between the irreducible components of its quasistrata $B_{(k)}$, $0 \leq k \leq n$, and the irreducible components of B . Also, the irreducible components $\tilde{B}$ of B are of dimension n and* $\mathrm{Reg}(\tilde{B})$ *are Lagrangian submanifolds of $T^*(K^n)$ in the natural symplectic structure of the latter.*

Proof. As a straightforward consequence of Definition 4.4 bundle B is a union of n-dimensional (constructible) sets $B|_{\mathcal{B}}$ with $\mathcal{B}$ being the irreducible components of the quasistrata $B_{(k)}$, $0 \leq k \leq n$, and $\mathrm{Reg}(B|_{\mathcal{B}})$ are Lagrangian submanifolds of $T^*(K^n)$. Therefore the closures of $B|_{\mathcal{B}}$ are the irreducible components $\tilde{B}$ of B implying the remainder of the claims of Proposition 4.7 as well. ∎

Theorem 4.8 *The first two of the following statements are equivalent and imply the third:*
(i) bundle G_F is Lagrangian;
(ii) TWG-stratifications of $\mathrm{Sing}(F)$ *exist and each irreducible component of $\mathcal{G}_k$, $r \leq k \leq n$, is of dimension $n-k$;*
(iii) each irreducible component of G_F is of dimension n .

Remark 4.9 *In the example of Remark 10.2 there are only* 2 *irreducible components of G_F , both are of dimension $n=5$ and G_F is not Lagrangian.*

Proof of Theorem 4.8. First (i) implies (ii) since quasistrata $\{\mathcal{G}_k\}_{r\leq k\leq n}$ form a TWG-stratification due to Proposition 3.4 and Remark 4.5. Now assume (ii). Then (1') of Lemma 3.7 implies that for any irreducible component $\tilde{\mathcal{G}}$ of $\mathcal{G}_k$ there is an open dense subset $\tilde{\mathcal{G}}^{(0)} \subset \tilde{\mathcal{G}}$ such that $T_x(\tilde{\mathcal{G}}) \perp G_x$ holds for any point $x \in \tilde{\mathcal{G}}^{(0)}$. Since $\dim(\tilde{\mathcal{G}}) = n-k$ it follows that G_x is the orthogonal complement to $T_x(\tilde{\mathcal{G}})$ for any point $x \in \tilde{\mathcal{G}}^{(0)}$, which implies (i). Finally, (i) implies (iii) is proved in Proposition 4.7. ∎

Subbundles $B := B(\mathcal{S})$ of $T^*(K^n)|_{\mathrm{Sing}(F)}$ constructed for any TWG-stratification $\mathcal{S}$ of $\mathrm{Sing}(F)$ in the paragraph preceding Remark 3.3 contain bundle G_F and are Lagrangian. Conversely, if a Lagrangian subbundle $B \hookrightarrow T^*(K^n)|_{\mathrm{Sing}(F)}$ contains G_F and partition $\mathcal{S} = \{S_{k,i}\}_{k,i}$ of $\mathrm{Sing}(F)$ is constructed as is described (for a Lagrangian B) above Proposition 4.7 then $\mathcal{S}$ provides a TWG-stratification of $\mathrm{Sing}(F)$ due to Proposition 3.4 and Remark 4.5 and, consequently, $B(\mathcal{S}) = B$. We summarize these observations in

Theorem 4.10 *There is a bijective correspondence between the classes of TWG-stratifications $\mathcal{S}$ of* $\mathrm{Sing}(F)$ *with all quasistrata $B(\mathcal{S})_{(k)}$, $1 \leq k \leq n$, fixed and between closed Lagrangian subbundles of $T^*(K^n)|_{\mathrm{Sing}(F)}$ that contain G_F .*

Moreover Propositions 4.1, 3.4, Theorem 4.8 and Corollary 4.2 imply

Corollary 4.11 *In the bijective correspondence of Theorem 4.10 Lagrangian bundles* G_F *correspond to the universal TWG-stratifications of* $\mathrm{Sing}(F)$.

In the next section we establish the converse statement.

5 A constructive criterion of universality.

Results of this and of the following section essentially depend on the validity of the conclusions of Claim 3.8 (which, in general, are not valid for $K = \mathbb{R}$, cf Remark 3.9). We therefore assume for the remainder of this article that for our dominating polynomial (or analytic) map $F : K^n \to K^l$ bundle G_F is n-dimensional over appropriate open dense subsets of every irreducible component of $\mathrm{Sing}(F)$. (For $K \neq \mathbb{R}$ the latter assumption holds due to Claim 3.8.)

The following Theorem partly justifies the title of the article.

Theorem 5.1 *Universal TWG-stratifications of* $\mathrm{Sing}(F)$ *(with respect to* F *) exist iff bundle* G_F *is Lagrangian.*

Proof of Theorem 5.1. The 'if' implication is the main content of Corollary 4.11 . Below we prove the remaining implication. Let $r := n - \dim(\mathrm{Sing}(F))$.

Assume that the 'only if' implication does not hold and let $\mathcal{G}$ be a not Lagrangian irreducible component of some $\mathcal{G}_k$, $r \leq k \leq n$ with a maximal in the lexicographic ordering pair $(n-k \ , \ m := \dim(\mathcal{G}))$. We recall (see Claim 3.8 or in the case $K = \mathbb{R}$ by an assumption above) that the minimal r for which $\mathcal{G}_r \neq \emptyset$ equals $r = n - \dim(\mathrm{Sing}(F))$. Therefore all irreducible components of $\mathcal{G}_r$ are Lagrangian since $\mathcal{G}_r$ is open in $\mathrm{Sing}(F)$, in particular $k > r$. We have $m = \dim(\mathcal{G}) < n-k$ (see Theorem 4.8) because condition (1') of Lemma 3.7 implies that $\dim(\mathcal{G}_t) \leq n-t$, $r \leq t \leq n$. Denote by $\mathcal{S} = \{S_i\}_i$ a universal TWG-stratification of $\mathrm{Sing}(F) = \cup_i S_i$ whose existence is the assumption of Theorem 5.1 .

Let $R \subset \mathrm{Sing}(F)$. Throughout the remainder of the article we denote by $G^{\perp}|_R \subset T(K^n)|_R$ the bundle of vector spaces whose fibers are the orthogonal complements to the fibers of subbundle $G|_R \subset T^*(K^n)|_R$.

Denote by W the union of all Lagrangian irreducible components of $\{\mathcal{G}_t\}_{r\leq t\leq k}$. Due to the choice of $\mathcal{G}$ it follows that $\cup_{r\leq t<k}\mathcal{G}_t \subset W$. On the other hand, W is the union of all Lagrangian irreducible components of $\{\mathcal{G}_t\}_{r\leq t\leq n}$ with dimensions greater or equal to $n-k$. Hence $\dim(\mathrm{Sing}(F)\setminus W) < n-k$.

Remark 5.2 *Following construction that appears in the proof of Lemma 2.1 (cf. Remark 2.3) one can produce a TWG-stratification* $\mathcal{S}' = \{S'_j\}_j$ *of* $\mathrm{Sing}(F) = \cup_j S'_j$ *extending the family of all irreducible components contained in* W *. Then* $B(\{S_i\}_i)|_W = G|_W$ *due to Propositions 3.4 and 4.1. Similarly,* $B(\{S_i\}_i)|_L = G|_L$ *for* L *being the union (dense in* $\mathrm{Sing}(F)$ *) of all open in* $\mathrm{Sing}(F)$ *Lagrangian components of the appropriate quasistrata* $\mathcal{G}_j$ *(cf. Claim 3.8).*

Plan of proof of the 'only if' implication of Theorem 5.1 is to derive a contradiction with our assumption 'to the contrary' by means of Proposition 5.9 which we prove in Sections 6 and 7, see Remark 6.3. To that end we first show (in Claim 5.3) that W is a union of some strata of $\mathcal{S} = \{S_i\}_i$. Next, in Claim 5.6, we prove that within an appropriate open set $U_{\mathcal{G}}$ with $\mathcal{G} \cap U_{\mathcal{G}}$ dense in $\mathcal{G}$ the latter is the boundary $\overline{S^{\cup}} \setminus S^{\cup}$ of the union $S^{\cup} \subset W$ of all strata of $\mathcal{S}$ of the smallest possible dimension among strata with the boundaries containing $\mathcal{G}$. Here our arguments must take into account a possibility that the boundary of $S^{\cup}$ may differ from the union of the boundaries of the strata contained in $S^{\cup}$, cf. the paragraph preceeding Lemma 2.1. (As a consequence of the latter one is not necessarily able to take as $S^{\cup}$ simply a single stratum of $\mathcal{S}$, though of course $\mathcal{G}$ is contained in the boundaries of some strata of $\mathcal{S}$.) Consequently, Proposition 5.9 provides an irreducible G-regular extension $\mathcal{G}^+$ of an open and dense subset of $\mathcal{G}$ into an appropriate stratum contained in $S^{\cup}$ (picked in Corollary 5.7), which enables an extension of a family of G-regular strata $\{\mathcal{Q}\setminus\overline{\mathcal{G}^+}\}_{\mathcal{Q}\subset W}\bigcup\{\mathcal{G}^+\}$, with $W_1 := \bigcup_{\mathcal{Q}\subset W}(\mathcal{Q}\setminus\overline{\mathcal{G}^+})\bigcup\mathcal{G}^+ \subset \mathrm{Sing}(F)$, to a TWG-stratification $\{\tilde{S}_j\}_j$ of $\mathrm{Sing}(F)$. The latter contradicts the universality of $\mathcal{S}$, which would complete the proof.

Claim 5.3 *Let $\mathcal{Q}$ be a stratum of $\mathcal{S}$. Then either $\mathcal{Q}\cap W = \emptyset$ or $\mathcal{Q}$ is an open and dense subset of a Lagrangian component $\mathcal{P} \subset W$. In particular, W coincides with the union of an appropriate subfamily of $\{S_i\}_i$.*

Proof. Indeed, first consider a stratum $\mathcal{Q}$ of $\mathcal{S}$ such that $\mathcal{Q} \cap W$ is dense in $\mathcal{Q}$ and denote $t := n - \dim(\mathcal{Q})$. Since $\mathcal{Q}$ is G-regular, $B(\mathcal{S}) \supset G$ and $B(\mathcal{S})|_{\mathcal{Q}\cap W} = G|_{\mathcal{Q}\cap W}$ it follows that $\mathcal{Q} \subset \cup_{q\le t}\mathcal{G}_q$ and $\mathcal{Q} \cap W \subset \mathcal{G}_t$ (in particular $t \le k$). On the other hand, set $\mathcal{G}^{(t)} := \cup_{q\ge t}\mathcal{G}_q$ is closed (since function $g\colon x \to \dim(G_x)$ is upper semicontinuous) and therefore $\mathcal{Q} \subset \overline{\mathcal{Q} \cap W} \subset \mathcal{G}^{(t)}$. Hence $\mathcal{Q} \subset \mathcal{G}_t$.

Consider an irreducible component $\mathcal{P}$ of $\mathcal{G}_t$ such that $\mathcal{Q} \cap \mathcal{P}$ is dense in our $\mathcal{Q}$. The latter implies that $\dim(\mathcal{P}) \ge n - t$ and since $\mathcal{P} \subset \mathcal{G}_t$ it follows ($n - t \ge \dim(\mathcal{P})$ and therefore) $\dim(\mathcal{P}) = n - t$. Thus $\mathcal{P}$ is Lagrangian and $\mathcal{P} \subset W$ (since $t \le k$). We conclude that $\mathcal{Q} \subset (\overline{\mathcal{Q} \cap \mathcal{P}}) \cap \mathcal{G}_t \subset \overline{\mathcal{P}} \cap \mathcal{G}_t = \mathcal{P} \subset W$ and $\dim(\mathcal{Q}) = n - t = \dim(\mathcal{P})$, as required.

Now, assume that a stratum $\mathcal{Q}$ of $\mathcal{S}$ has a non-empty intersection with a Lagrangian irreducible component $\mathcal{P} \subset W$ of $\mathcal{G}_t$ (and therefore $\dim(\mathcal{P}) = n - t$ for some $t \le k$). Then, using $B(\mathcal{S})|_{\mathcal{P}\cap\mathcal{Q}} = G|_{\mathcal{P}\cap\mathcal{Q}}$ and in view of the definition of $B(\mathcal{S})$, it follows that $\dim(\mathcal{Q}) = n - t$. As we have shown above $\dim(\mathrm{Sing}(F) \setminus W) < n - k \le n - t$. Therefore $\mathcal{Q} \cap W$ is dense in $\mathcal{Q}$. In the latter case we have already proved that $\mathcal{Q} \subset W$, which completes the proof of the claim. ∎

Corollary 5.4 *Let $\mathcal{Q}$ be a stratum of $\mathcal{S}$ with $\dim(\mathcal{Q}) > \dim(\mathcal{G})$ and $\overline{\mathcal{Q}} \supset \mathcal{G}$. Then $\mathcal{Q} \subset \mathcal{G}_{n-q}$, where $q = \dim(\mathcal{Q}) > n - k > \dim(\mathcal{G})$, and $\mathcal{Q} \subset W$.*

Proof. Due to our assumptions either $\mathcal{G} \cap \mathcal{Q}$ or $\mathcal{G} \cap (\overline{\mathcal{Q}} \setminus \mathcal{Q})$ is dense in $\mathcal{G}$. If $\mathcal{Q} \cap W = \emptyset$ then either $\mathcal{Q} \subset \mathcal{G}^{(k-1)}$ or $\mathcal{Q} \cap (\mathcal{G}_k \setminus W)$ is dense in $\mathcal{Q}$. In the latter case $\dim(\mathcal{Q}) \le \dim(\mathcal{G}_k \setminus W) = \dim(\mathcal{G})$, which is contrary to the choice of $\mathcal{Q}$. And in the former case $\mathcal{G} \subset \overline{\mathcal{Q}} \subset \mathcal{G}^{(k-1)}$ contrary to $\mathcal{G}$ being an irreducible component of $\mathcal{G}_k$. Hence $\mathcal{Q} \cap W \ne \emptyset$ and due to the claim above $\mathcal{Q} \subset W$. ∎

Consider the union $S^{\cup}$ of all strata $\mathcal{Q}$ of $\mathcal{S}$ of the smallest possible dimension with $\overline{\mathcal{Q}} \setminus \mathcal{Q}$ containing $\mathcal{G}$, say $s := \dim(S^{\cup})$.

Remark 5.5 *Due to the upper semi-continuity of function* $g : x \to \dim(G_x)$ *and Claim 3.8 (or, when* $K = \mathbb{R}$ *, the replacing it assumption of the paragraph preceeding Theorem 5.1) the inclusions* $\mathcal{G} \subset \overline{\cup_{r \le t < k} \mathcal{G}_t} \subset \overline{W}$ *hold. Therefore Claim 5.3, Corollary 5.4 and Remark 3.6 imply that* $S^{\cup}$ *is not emply,* $S^{\cup} \subset (\mathcal{G}_{n-s} \cap W) = \mathcal{G}_{n-s}$ *and that* $S^{\cup}$ *is G-regular.*

Claim 5.6 *Let* $\mathcal{W}$ *be an irreducible component of* $\overline{S^{\cup}} \setminus S^{\cup}$ *such that* $\mathcal{W}$ *contains* $\mathcal{G}$ *. Then* $\mathcal{G}$ *is dense in* $\mathcal{W}$ *. (Hence such* $\mathcal{W}$ *is unique). In particular,* $\overline{\mathcal{G}}$ *is an irreducible component of* $\overline{\overline{S^{\cup}} \setminus S^{\cup}}$ *and thus within an appropriate open neighbourhood of* $\mathcal{G}$ *holds*

$$\overline{S^{\cup}} \setminus S^{\cup} = \overline{\mathcal{G}} = \mathcal{G} \ .$$

Proof. Assume the contrary. Then $\dim(\mathcal{W}) > \dim(\mathcal{G})$. Denote by $t_{\mathcal{W}}$ the minimal value of $g \colon x \to \dim(G_x)$ on $\mathcal{W}$ (attained on an open dense subset of $\mathcal{W}$ in view of the upper semicontinuity of function g). Then $t_{\mathcal{W}} \ge t := n - s = \dim(G_x)$ for $x \in S^{\cup} \subset W$ because $\mathcal{W} \subset (\overline{S^{\cup}} \setminus S^{\cup})$. Pick a stratum $\mathcal{Q}$ of $\mathcal{S}$ such that $\mathcal{W} \cap \mathcal{Q}$ is dense in $\mathcal{W}$. Then $\overline{\mathcal{Q}} \supset \mathcal{G}$ and since $\dim(\mathcal{Q}) \ge \dim(\mathcal{W}) > \dim(\mathcal{G})$ inclusion $\mathcal{Q} \subset W$ holds due to Corollary 5.4, implying $(W \cap \mathcal{G}) \supset (\mathcal{Q} \cap \mathcal{G})$. Since $\mathcal{G} \subset (\mathcal{G}_k \setminus W)$ it follows $\mathcal{Q} \cap \mathcal{G}$ is empty, i. e. $\mathcal{G} \subset (\overline{\mathcal{Q}} \setminus \mathcal{Q})$. Since also $\mathcal{Q} \subset W$ and due to the choice of s we conclude that $\dim(\mathcal{Q}) \ge s$. On the other hand $n - \dim(\mathcal{Q}) = \dim(G_x) = t_{\mathcal{W}}$ for $x \in (\mathcal{W} \cap \mathcal{Q})$ by making use of Remark 5.2 and Claim 5.3, which implies $s = n - t \ge n - t_{\mathcal{W}} = \dim(\mathcal{Q})$. Therefore $s = \dim(\mathcal{Q})$ and both $\mathcal{Q} \subset S^{\cup}$ and, due to $\overline{\mathcal{Q}} \cap \overline{\mathcal{W}} \ne \emptyset$, inequality $\mathcal{Q} \cap (\overline{S^{\cup}} \setminus S^{\cup}) \ne \emptyset$ holds, leading to a contradiction. ▪

Corollary 5.7 *Let* $\mathcal{Q}$ *be a stratum of* $\mathcal{S}$ *of* $\dim(\mathcal{Q}) = s$ *with* $\overline{\mathcal{Q}} \setminus \mathcal{Q} \supset \mathcal{G}$ *. Let* $S_* := \overline{\mathcal{Q}} \cap S^{\cup} \supset \mathcal{Q}$ *. Then* S_* *is an irreducible subset of* $W \cap \mathcal{G}_{n-s} = \mathcal{G}_{n-s}$ *and* $\overline{S_*} \setminus S_* = \overline{\mathcal{G}} = \mathcal{G}$ *within an open set* $U_{\mathcal{G}}$ *with* $\mathcal{G} \cap U_{\mathcal{G}}$ *dense in* $\mathcal{G}$ *.*

Proof. Inclusion $S_* \subset S^{\cup} \subset W \cap \mathcal{G}_{n-s} = \mathcal{G}_{n-s}$ is the main content of Corollary 5.4. Note that S_* is irreducible since $\overline{S_*} = \overline{\mathcal{Q}} \supset \mathcal{G}$ and that sets $\mathcal{G} \cap S^{\cup}$ and $(\overline{S_*} \setminus S_*) \cap S^{\cup}$ are both empty. Therefore $S_* \cap \mathcal{G} = \emptyset$ and $(\overline{S^{\cup}} \setminus S^{\cup}) \supset (\overline{S_*} \setminus S_*) \supset \mathcal{G}$. Hence due to Claim 5.6 also $\overline{S_*} \setminus S_*$ coincides with $\mathcal{G}$ on an open neighbourhood of an open dense subset of $\mathcal{G}$. ▪

Remark 5.8 *We may choose an open in* K^n *set* $U_{\mathcal{G}}$ *so that* $\mathcal{G} \cap U_{\mathcal{G}} = \overline{\mathcal{G}} \cap U_{\mathcal{G}} \ne \emptyset$ *. Since* $\overline{\mathcal{Q}} \cap U_{\mathcal{G}} \supset \mathcal{G} \cap U_{\mathcal{G}} \ne \emptyset$ *it follows that* $\mathcal{Q} \cap U_{\mathcal{G}} \ne \emptyset$ *. Consider* $S := S_* \cap U_{\mathcal{G}} \supset \mathcal{Q} \cap U_{\mathcal{G}}$ *(as in Corollary 5.7). Then* $\overline{\mathcal{Q}} \supset \overline{S} \supset \overline{\mathcal{Q} \cap U_{\mathcal{G}}} = \overline{\mathcal{Q}} = \overline{S_*}$ *(due to* $\mathcal{Q}$ *being irreducible) and therefore* $\overline{S} = \overline{S_*}$ *and* S *is irreducible. Hence* $\mathcal{G} \cap U_{\mathcal{G}} = (\overline{S_*} \setminus S_*) \cap U_{\mathcal{G}} \supset (\overline{S} \setminus S) \cap U_{\mathcal{G}} \supset \mathcal{G} \cap U_{\mathcal{G}}$ *, which implies*

$$(\overline{S} \setminus S) \cap U_{\mathcal{G}} = \mathcal{G} \cap U_{\mathcal{G}} = \overline{\mathcal{G}} \cap U_{\mathcal{G}} \tag{1}$$

and that S *is open in its closure. Finally,* S *is G-regular (and is a dense subset of a Lagrangian component of* $\mathcal{G}_{n-s}$ *) since* $S \subset W \cap \mathcal{G}_{n-s} = \mathcal{G}_{n-s}$ *.*

In the remainder of this and in the following Section we use notation $\mathcal{U}$ for $U_{\mathcal{G}}$, $\mathcal{G}$ and S for $\mathcal{G} \cap U_{\mathcal{G}}$ and, respectively, for $S \cap U_{\mathcal{G}}$ from Remark 5.8, in particular S is irreducible.

Proposition 5.9 *There is an irreducible G-regular constructible set* $\mathcal{G}^+$ *open in its closure such that* $\mathcal{G}^+ \subset \overline{S}$ *,* $\dim(\mathcal{G}^+) = n - k$ *and* $\mathcal{G}^+$ *contains an open dense subset of* $\mathcal{G}$ *. Finally*

$$G^{\perp}|_{\mathcal{G}^+ \cap \mathcal{G}} = T(\mathcal{G}^+)|_{\mathcal{G}^+ \cap \mathcal{G}} \ .$$

We prove Proposition 5.9 in Sections 6, 7.

Deduction of Theorem 5.1 from Proposition 5.9. The bundle of vector spaces associated (as in the paragraph preceeding Remark 3.3) with a family of G-regular strata

$$\mathcal{F} := \{\mathcal{Q} \setminus \overline{\mathcal{G}^+}\}_{\mathcal{Q}\subset W} \bigcup \{\mathcal{G}^+\} \quad , \quad W_1 := \bigcup_{\mathcal{Q}\subset W} (\mathcal{Q} \setminus \overline{\mathcal{G}^+}) \bigcup \mathcal{G}^+ \subset \mathrm{Sing}(F) \tag{2}$$

(where the union ranges over all strata $\mathcal{Q}$ of $\mathcal{S}$ such that $\mathcal{Q} \subset W$) coincides over $W_1 \setminus \mathcal{G}^+$ with G, is Lagrangian and is closed due to the latter and Proposition 5.9. Since $W \setminus W_1 \subset \overline{\mathcal{G}^+} \setminus \mathcal{G}^+$ and dimensions of $(\overline{\mathcal{G}^+} \setminus \mathcal{G}^+)$ and $(\mathrm{Sing}(F) \setminus W)$ are less than $n-k$ it follows that $\dim(\mathrm{Sing}(F) \setminus W_1) < n-k$. Therefore, as in the Remark 5.2, the family $\mathcal{F}$ extends to a TWG-stratification $\{\tilde{S}_j\}_j$ of $\mathrm{Sing}(F) = \cup_j \tilde{S}_j$.

As we have established above in Claim 5.3 set W and therefore $\mathrm{Sing}(F) \setminus W$ are the unions of several strata of $\mathcal{S}$. Hence there exists a stratum $\mathcal{P}$ of $\mathcal{S}$ such that $(\mathrm{Sing}(F)\setminus W) \supset \mathcal{P}$ and $\mathcal{G} \cap \mathcal{P}$ is open and dense in $\mathcal{G}$. Since being universal TWG-stratification $\{S_i\}_i$ is larger than $\{\tilde{S}_j\}_j$ it follows by Proposition 4.1 that for any point $x \in \mathcal{G} \cap \mathcal{G}^+ \cap \mathcal{P}$ there is an inclusion $B(\mathcal{P})_x \subset B(\mathcal{G}^+)_x = G_x$ for the fibers of G ; hence $\dim(B(\mathcal{P})_x) \leq \dim(G_x) = k$ and $\dim(\mathcal{P}) \geq n-k$. But on the other hand $n-k \leq \dim(\mathcal{P}) \leq \dim((\mathrm{Sing}(F)\setminus W) < n-k$. Thus the assumption (on the first lines of the proof of Theorem 5.1) of the existence of a non Lagrangian component $\mathcal{G}$ in $\{\mathcal{G}_j\}_j$ leads to a contradiction, i. e. G is Lagrangian. ■

6 Extension Theorem for singular varieties.

Proof of the more difficult implication of our main result Theorem 5.1 we complete in this section. To that end we prove here Proposition 5.9 as a special case of an Extension Theorem 6.1 important in its own right. The main ingredient of the proof of the latter is our Bertini-type Theorem for singular varieties introduced in Section 1.3.

Extension Theorem 6.1 essentially provides an extension of a nonsingular part of the singular locus of an algebraic variety to a Gauss regular subvariety with a prescribed tangent bundle over the singularities under the assumptions of Whitney-a type conditions on the data. (To apply the latter notion in the setting of Proposition 5.9 we will allow the adjacent strata to be Gauss regular.)

Assume $X \hookrightarrow \mathcal{U}$ is an irreducible algebraic (or analytic) subvariety of an open $\mathcal{U} \subset K^n$, that a Gauss regular $S \subset X$ is open and dense in X , $\mathrm{Reg}(X) \subset S$ and that $\mathcal{G} := X \setminus S$ is nonsingular. Besides the pair of strata $\{S \ , \ \mathcal{G}\}$ in $\mathcal{U}$ the data for our version of Whitney-a property includes a subbundle $T_{\mathcal{G}}$ of the restriction over $\mathcal{G}$ of the tangent bundle $T(\mathcal{U})|_{\mathcal{G}}$ of $\mathcal{U}$ such that $T_{\mathcal{G}}$ contains the tangent bundle $T(\mathcal{G})$ of $\mathcal{G}$. When $T_{\mathcal{G}} = T(\mathcal{G})$ and S is nonsingular property **W-a** below is the standard Whitney-a condition on strata $\{S \ , \ \mathcal{G}\}$.

W-a property : if exists $\lim_{i\to\infty}(x_i, T_{x_i}(S)) = (x_0 \ , \ T)$, where $x_0 \in \mathcal{G}$, subspace $T \subset T_{x_0}(\mathcal{U})$, $\{(x_i \ , \ T_{x_i}(S))\}_i \subset S \times T(\mathcal{U})|_S$ and the limit $\lim_{i\to\infty} T_{x_i}(S) = T$ is in the Grassmanian of $(\dim S)$-dimensional subspaces of K^n , then $T \supset (T_{\mathcal{G}})_{x_0}$.

Theorem 6.1 *(***Extension Theorem***)*

Assume $\mathcal{U}$, S , $\mathcal{G}$ and $T_{\mathcal{G}} \subset T(\mathcal{U})|_{\mathcal{G}}$ are as in the preceding paragraph and satisfy property W-a. Then there is an open subset $\mathcal{U}'$ of $\mathcal{U}$ and an irreducible Gauss regular closed subvariety $\mathcal{G}^+$ of $\overline{S} \cap \mathcal{U}'$, such that $\mathcal{G}^+$ contains $\mathcal{G} \cap \mathcal{U}'$, the latter set is open and dense in $\mathcal{G}$ and

$$T_{\mathcal{G}}|_{\mathcal{G}^+\cap\mathcal{G}} = T(\mathcal{G}^+)|_{\mathcal{G}^+\cap\mathcal{G}} \ .$$

Let $m := \dim(\mathcal{G})$ and $k := n - (\dim(T_{\mathcal{G}}) - m)$.

Remark 6.2 *When* $\dim(S) = n - k$ *Theorem 6.1 is obviously valid with* $\mathcal{G}^+ := \overline{S} \cap \mathcal{U}$.

Remark 6.3 *Proposition 5.9 is a special case of Theorem 6.1 with constructed in Section 5 open* $\mathcal{U} := U_{\mathcal{G}} \subset K^n$ *, a G-regular irreducible dense subset* $S \subset W \cap \mathcal{U}$ *of a Lagrangian component of* $\{\mathcal{G}_t\}_{r \le t \le k}$ *with* $\mathcal{G} = \mathcal{U} \cap \overline{\mathcal{S}} \setminus \mathcal{S}$ *(see Remark 5.8) and* $T_{\mathcal{G}} := G^{\perp}|_{\mathcal{G}} \subset T(\mathcal{U})|_{\mathcal{G}}$ *, where* $G := G_F$ *. Bundle* $T(\mathcal{G}) \subset T_{\mathcal{G}}$ *due to Remark 4.3. Finally, the validity of property W-a for* $\{\mathcal{U}\ ,\ S\ ,\ \mathcal{G}\ ,\ T_{\mathcal{G}}\}$ *is equivalent to* $\overline{T(S)^{\perp}}|_{\mathcal{G}} \subset G|_{\mathcal{G}}$ *and is a consequence of bundle* G *being closed in* $T(\mathcal{U})|_{\mathrm{Sing}(F)}$ *and* S *of Remark 5.8 being dense in a Lagrangian component of* $\mathcal{G}_{n-s}$ *. (Note that* $\dim(S) > n - k$ *, see Corollary 5.4.)*

Proof of Theorem 6.1. We assume $K = \mathbb{C}$ (or $\mathbb{R}$) and in the algebraic case extend the result to an arbitrary algebraically closed field employing the Tarski-Lefschetz principle. Application of the Tarski-Lefshetz principle requires the estimates of degrees of the output in terms of the degrees of the input, which here is straightforward due to a constructive and explicit nature of the proof below (see also remarks following Theorem 7.1 below).

First we construct a $(k+m) \times n$ matrix $M = (M_{j,i})_{1 \le j \le k+m\ ,\ 1 \le i \le n}$ with the entries being polynomials over $K = \mathbb{C}$ (or $\mathbb{R}$) in n variables such that for a suitable open $V \subset \mathcal{G}$

$$T_{\mathcal{G}}|_V = T(\mathcal{G})|_V \oplus \mathrm{Ker}(M)|_V\ . \tag{3}$$

In particular, the rank of M equals $k+m$ at all points of V .

Consider a Noether normalisation $\pi : \mathcal{G} \to K^m$ being a restriction of a linear projection $\pi : K^n \to K^m$. Then $K^n = K^m \oplus K^{n-m}$, where $K^{n-m} = \mathrm{Ker}(\pi)$ and $K^m = \pi(K^n)$. We may assume w.l.o.g. that the first m coordinates are the coordinates of the first summand and the last $n-m$ coordinates are the coordinates of the second summand. We choose in the tangent space to K^n a basis $\frac{\partial}{\partial X_i}$ corresponding to X-coordinates. In abuse of notation we identify $K^{n-m} = T_x(K^{n-m}) \subset T_x(K^n)$ for points $x \in K^{n-m}$.

Let $\tilde{\mathcal{U}} \subset K^m$ be an open set such that (3) holds for $V := \pi^{-1}(\tilde{\mathcal{U}}) \cap \mathcal{G}$, $\pi(V) = \tilde{\mathcal{U}}$ and such that the dimension of any fiber of bundle

$$T_{\mathcal{G}}|_V \cap (V \times K^{n-m})$$

equals $n-k-m$ (e. g. any open $\tilde{\mathcal{U}}$ such that over V the tangent spaces to $\mathcal{G}$ are mapped onto K^m isomorphically would do). Then there is a matrix of size $(k+m) \times n$, say M , with the entries being polynomials in n variables such that

$$\mathrm{Ker}(M)|_V = T_{\mathcal{G}}|_V \cap (V \times K^{n-m})\ .$$

Of course we may assume w.l.o.g. that $M_{j,i} = \delta_{j,i}$ for $1 \le j \le m$, $1 \le i \le n$ (where $\delta_{j,i}$ denotes the Kronecker's symbol). This provides matrix M and set V satisfying (3).

One can construct an open subset $\mathcal{U}' \subset \tilde{\mathcal{U}}$ and (by means of an interpolation in K^{n-m} parametrized by points in $\mathcal{U}'$, e. g. as in Appendix) functions $L_j(X)$, $1 \le j \le k$, rational in the first m and polynomial in the last $n-m$ coordinates such that all L_j , $1 \le j \le k$, vanish on $V' := \pi^{-1}(\mathcal{U}') \cap \mathcal{G}$ (while their denominators do not) and for every point $x \in V'$

$$\frac{\partial L_j}{\partial X_i}(x) = M_{j+m\ ,\ i}(x) \ \text{ for } 1 \le j \le k\ ,\ m+1 \le i \le n\ .$$

Multiplying by the common denominator and keeping the same notation L_j , $1 \le j \le k$, for the resulting polynomials we conclude that all L_j vanish on $\mathcal{G}$, their differentials $dL_j(x)$, $1 \le j \le k$, are linearly independent for any $x \in V'$ and also, due to (3), that $\bigcap_{1\le j\le k} \mathrm{Ker}(dL_j)|_{V'} = T_{\mathcal{G}}|_{V'}$. Therefore by shrinking neighbourhood $\mathcal{U}$, if need be, and replacing $\mathcal{G}$ by $\mathcal{G}\cap\mathcal{U}$ we may assume that $\mathcal{U} \subset \pi^{-1}(\mathcal{U}')$, that dL_1 , ... , dL_k are linearly independent at every point in $\mathcal{U}$ and that

$$\bigcap_{1\le j\le k} \mathrm{Ker}(dL_j)|_{\mathcal{G}} = T_{\mathcal{G}} . \tag{4}$$

Remark 6.4 *We may w.l.o.g. assume* $\dim(\{L_j|_S\}_{1\le j\le k}) := \dim_K(\mathrm{Span}\{L_j|_S\}_{1\le j\le k}) \ge 2$ *, where* Span *denotes the* K*-linear hull of a family of functions. Indeed,* $\dim(S) > n-k = \dim(T_{\mathcal{G}}) - \dim(\mathcal{G})$ *implies* $d := \dim(\{L_j|_S\}_{1\le j\le k}) > 0$ *. It remains to exclude the case of* $d = 1$ *. In the latter case we may assume w.l.o.g. that* $\dim(\{L_j|_S\}_{2\le j\le k}) \ge 1$ *and then change* L_1 *by adding to it an appropriate generic element of the square of the ideal* $I_{\mathcal{G}}$ *of all polynomials vanishing on* $\mathcal{G}$ *. This would not change the value of* dL_1 *at the points of* $\mathcal{G}$ *, but on the other hand* $\dim(\{L_j|_S\}_{1\le j\le k})$ *for the new choice of* L_1 *will increase due to dimension of* $I_{\mathcal{G}}^2/I_S$ *as a vector space over* K *being infinite, as required.*

7 Bertini-type Theorem and completion of proof of Extension

To complete the proof of Theorem 6.1 we will use Theorem 7.2 stated below and proved following the completion of the ongoing proof of Extension Theorem 6.1. In this Bertini-type Theorem 7.2 we assume that collection $\{\mathcal{U}, S, \mathcal{G}, T_{\mathcal{G}}\}$ satisfies the **W-a** property and that collection $\{L_j\}_{1\le j\le k}$ of polynomials vanishing on $\mathcal{G}$ with linearly independent differentials over $\mathcal{U}$ satisfies property (4), that $\dim(S) > n-k$ and $\dim(\{L_j|_S\}_{1\le j\le k}) \ge 2$. We then construct a codimension one in S irredicible Gauss regular closed subvariety $\hat{S}_{-1} := \hat{S}_{-1}(S) \hookrightarrow S$ with $\mathcal{G}$ being its boundary such that $\{\mathcal{U}, \hat{S}_{-1}, \mathcal{G}\}$ and bundle $T_{\mathcal{G}}$ over $\mathcal{G}$ satisfy the **W-a** property (and then proceed by induction on $\dim(S)$).

We will use the notion of *normal crossing* : a collection of varieties is a (simultaneous) normal crossing at a point, say a , provided that in appropriate local analytic coordinates centered at this point every variety from this collection and passing through a is a coordinate subspace. (Of course this property is open with respect to the choice of points a .) Due to the assumptions on $\{L_j\}_{1\le j\le k}$ (which match the properties of the collection constructed within the proof of Theorem 6.1 in the previous section) the collection of the hypersurfaces $H_j := \{L_j = 0\}\cap\mathcal{U}$, $1 \le j \le k$, is a normal crossing in $\mathcal{U}$, i. e. at every point of $\mathcal{U}$. Moreover, since S is irreducible (as is the S of Theorem 6.1) it follows that set $\mathrm{Reg}_*(\overline{S})$ of all points in $\overline{S}\cap\mathcal{U}$ at which $\{H_j\}_{1\le j\le k}$ and $\overline{S}$ is a normal crossing is an open and dense subset of $\mathrm{Reg}(\overline{S}\cap\mathcal{U})$ (since $\mathrm{Reg}_*(\overline{S}) \supset \mathrm{Reg}(S) \setminus \bigcup_{H_j \not\supset S} H_j \ne \emptyset$). We also denote $\mathrm{Sing}_*(\overline{S}) := \overline{S}\cap\mathcal{U} \setminus \mathrm{Reg}_*(\overline{S})$.

The exposition of our Bertini-type Theorem 7.2 below is for the case of $K = \mathbb{C}$ or $\mathbb{R}$ (e. g. items ii) and v)). The set up is similar to that preceeding Theorem 6.1 and of the 'output' above of the construction in its proof, i. e. $X \hookrightarrow \mathcal{U}$ is an irreducible algebraic (or analytic) subvariety of an open $\mathcal{U} \subset K^n$, a Gauss regular $S \subset X$ is open and dense in X with $\mathrm{Reg}(X) \subset S$ and a nonsingular $\mathcal{G} := X \setminus S$. Also, the fibers of the bundle $T_{\mathcal{G}}$ over $\mathcal{G}$ are determined by a collection $\{L_j\}_{1\le j\le k}$ of polynomials satisfying the properties listed in the first paragraph of the current section. Let $L(x,c) := \sum_{1\le j\le k} c_j L_j(x)$ for

$c = (c_1, \ldots, c_k)$, $(x, c) \in \mathcal{U} \times \mathbb{C}^k$, and for a 'fixed' c let $L_c(x) := L(x, c)$. In the algebraic case a version of Bertini-type theorem in a form that allows to reduce the proof for an arbitrary algebraically closed field K of characteristic zero to the proof in the $K = \mathbb{C}$ case by employing the Tarski-Lefschetz principle is as follows:

Theorem 7.1 *Assume that* $\dim(\{L_j|_S\}_{1 \le j \le k}) \ge 2$ *and the collection* $\{\mathcal{U}\ ,\ S\ ,\ \mathcal{G}\ ,\ T_\mathcal{G}\}$ *of the preceeding paragraph satisfies the* **W-a** *property. Then for* $c \in K^k$ *off a proper algebraic subset* Z *of* K^k *the hypersurface* $S_{-1} := \{L_c = 0\} \cap S$ *of* S *is nonsingular, there is an irreducible component* $\hat{S}_{-1}$ *of* S_{-1} *whose boundary contains* $\mathcal{G}$ *and* $\{\mathcal{U}\ ,\ \hat{S}_{-1}\ ,\ \mathcal{G}\ ,\ T_\mathcal{G}\}$ *satisfies the* **W-a** *property.*

Application of the Tarski-Lefshetz principle requires the estimates of degrees of the output in terms of the degrees of the input. The estimate on the degrees of polynomials L_j , $1 \le j \le k$, in terms of the degrees of polynomials defining collection $\{\mathcal{U}, \mathcal{S}, \mathcal{G}, \mathcal{T}_\mathcal{G}\}$ is straightforward following the proof of Theorem 6.1 and Remark 6.4 . Also, the estimate on the degrees of polynomials defining algebraic set $Z \subset K^k$ in terms of bounds on the algebraic data after the application of desingularization within the proof of Theorem 7.2 below is straightforward (since set Z is the set of critical values of the appropriate projections from the proof of Theorem 7.2). Finally, the bounds on the algebraic data after the application of desingularization in terms of the degrees of polynomials defining collections $\{\mathcal{U}, \mathcal{S}, \mathcal{G}, \mathcal{T}_\mathcal{G}\}$ and $\{L_j\}_{1 \le j \le k}$ is a consequence of the estimate of complexity of desingularization in [4] .

Theorem 7.2 *(***A Bertini-type Theorem for singular varieties***)*

Assume **W-a** *property for* $\{\mathcal{U}\ ,\ S\ ,\ \mathcal{G}\ ,\ T_\mathcal{G}\}$ *of the paragraph preceeding Theorem 7.1 ,* $\dim(S) > n - k$ *and* $\dim(\{L_j|_S\}_{1 \le j \le k}) \ge 2$ *. Then for a generic* $c \in K^k$ *the following properties hold:*

i) $\{L_c = 0\} \cap \mathrm{Reg}_*(\overline{S})$ *is dense in* $S_{-1} := \{L_c = 0\} \cap S$ *manifold of codimension* 1 *in* S *;*

ii) for compacts $\mathcal{K} \subset (\overline{S} \cap \mathcal{U}) \subset K^n$ *the norms of* $d(L_c|_S)(a) = dL_c(a)|_{T_a(S)}$ *, for* $a \in \{L_c = 0\} \cap \mathrm{Reg}_*(\overline{S}) \cap \mathcal{K}$ *, are larger than a positive constant (depending on* $\mathcal{K}$ *);*

iii) the boundary $(\overline{S_{-1}} \setminus S_{-1}) \cap \mathcal{U}$ *of set* S_{-1} *in* $\mathcal{U}$ *coincides with* $\mathcal{G}$ *;*

iv) $\mathrm{Reg}(S_{-1}) \supset (S_{-1} \cap \mathrm{Reg}(S))$ *and* S_{-1} *is G-regular in* $\mathcal{U}$ *;*

v) for any sequence of points in S_{-1} *and their tangent spaces to* S_{-1} *converging to* $a \in \mathcal{G}$ *and, respectively, to a subspace* $Q \subset T_a(K^n)$ *holds* $Q \supset T_\mathcal{G}(a)$ *implying* $\overline{T(S_{-1})^\perp}|_\mathcal{G} \subset T_\mathcal{G}^\perp$ *;*

vi) replacing S_{-1} *by an irreducible component* $\hat{S}_{-1}$ *of* S_{-1} *whose boundary contains* $\mathcal{G}$ *the properties iii)-v) remain valid and, therefore, property v) with* $\hat{S}_{-1}$ *replacing* S_{-1} *means that collection* $\{\mathcal{U}\ ,\ \hat{S}_{-1}\ ,\ \mathcal{G}\ ,\ T_\mathcal{G}\}$ *satisfies the* **W-a** *property.*

Remark 7.3 *For the sake of clarity we include (though do not make use of) the following:*

- *Of course in ii) of Theorem 7.2 we may equivalently replace "the norms of* $d(L_c|_S)(a) = dL_c(a)|_{T_a(S)}$ *are separated from* 0 *" by "the angles between the gradient* $\mathrm{grad}\, L_c(a)$ *of* L_c *at* a *and the tangent spaces* $T_a(S)$ *to* S *at* a *are separated from* $\pi/2$ *".*
- *Due to* S *being irreducible and* $\{L_c = 0\} \cap S \ne S$ *it follows that the irreducible components of* S_{-1} *are equidimensional. An irreducible component* $\hat{S}_{-1}$ *of* S_{-1} *whose boundary contains* $\mathcal{G}$ *exists since the union of boundaries of the irreducible components of* S_{-1} *contains the boundary* $\mathcal{G}$ *of* S_{-1} *(property iii)) and* $\mathcal{G}$ *is irreducible.*

Deduction of Theorem 6.1 from Theorem 7.2. Starting with $\hat{S}_0 := S$ we construct sets $\hat{S}_{-i} := \hat{S}_{-1}(\hat{S}_{-i+1})$, $1 \le i \le e := \dim(S) - n + k$, consecutively applying e times Theorem 7.2 followed by Remark 6.4. Then due to iii) of Theorem 7.2

$$(\overline{\hat{S}_{-e}} \setminus \hat{S}_{-e}) \cap \mathcal{U} \;=\; \mathcal{G} \tag{5}$$

and, moreover,

$$\overline{T(\hat{S}_{-e})^{\perp}}|_{\mathcal{G}} = T_{\mathcal{G}}^{\perp} \tag{6}$$

since the Gauss map of $\hat{S}_{-e}$ extends as continuous (uniquely) to all of $\mathcal{G}$ (due to v) of Theorem 7.2). Indeed, if a sequence of points from $\hat{S}_{-e}$ converges to a point $a \in \mathcal{G}$ with their tangent spaces to $\hat{S}_{-e}$ converging (in the respective Grassmanian) to a subspace $Q \subset K^n$ then $Q \supset T_{\mathcal{G}}(a)$ and then $Q = T_{\mathcal{G}}(a)$ due to $\dim(Q) = \dim(T_{\mathcal{G}}(a)) = n - k$. Therefore using (5) set $\hat{S}_{-e}$ can be enlarged to an irreducible, G-regular and open in $\overline{\hat{S}_{-e}}$ subset $\mathcal{G}^{+} := \hat{S}_{-e} \cup \mathcal{G}$ of dimension $n-k$ satisfying (6), as required in Theorem 6.1. ∎

8 Proof of Bertini-type Theorem for singular varieties.

We prove iii) for an arbitrary choice of $c \in K^k$. Of course $\dim(\overline{(S_{-1})_a}) \ge \dim(S) - 1 \ge n - k > m = \dim(\mathcal{G})$, where $(S_{-1})_a$ denotes the germ of S_{-1} (as an analytic set) at $a \in \mathcal{G}$. Also $(\mathcal{G})_a \subset \overline{((\overline{S} \cap \{L=0\}) \setminus \mathcal{G})_a}$, where "$(\cdot)_a$" is a notation for the germ of "$\cdot$" at a . On the other hand, $((\overline{S} \cap \{L=0\}) \setminus \mathcal{G})_a = ((\overline{S} \setminus \mathcal{G}) \cap \{L=0\})_a = (S_{-1})_a$, since $(S)_a = (\overline{S} \setminus \mathcal{G})_a$ due to $\mathcal{G}$ being the boundary of S in $\mathcal{U}$. Thus $\mathcal{G} \subset (\overline{S_{-1}} \cap \mathcal{U})$ and (since $S \cap \mathcal{G} = \emptyset$) it follows that $(\overline{S_{-1}} \setminus S_{-1}) \supset \mathcal{G}$. Finally, definition of S_{-1} and $\mathcal{G}$ being the boundary of S in $\mathcal{U}$ imply that $\mathcal{G} = (\overline{S} \setminus S) \cap \mathcal{U} \supset (\overline{S_{-1}} \setminus S_{-1}) \cap \mathcal{U} \supset \mathcal{G}$, as required.

Properties i) and ii) of Theorem 7.2 imply both iv) and v). Inclusion $\mathrm{Reg}(S_{-1}) \supset \{L=0\} \cap \mathrm{Reg}(S) = S_{-1} \cap \mathrm{Reg}(S)$ is a straightforward consequence of i) and ii). The remainder is a consequence of the following property: if the limits of two sequences of subspaces of K^n exist, then the limit of the respective intersections of these subspaces also exists and coincides with the intersection of the limits of the sequences, provided that the angles between the respective subspaces in the sequences are separated from 0 by a positive constant.

'**Property vi) follows from iii)-v)**' using that S_{-1} is open in its closure.

Thus it remains to prove i) and ii). We start with the **Proof of i):**

Reduction to a 'nonsingular setting' via an embedded desingularization followed by a combinatorial one. We start with an *embedded desingularization* $\sigma : \mathcal{N} \to \mathcal{U}$ of $\overline{S} \cap \mathcal{U} \subset \mathcal{U}$ by means of successive blowings up along smooth admissible centers (e. g. as in [16], [1] or [3]) with hypersurfaces $H_j := \{L_j = 0\}$, $1 \le j \le k$, treated as exceptional. We may treat H_j's as exceptional since collection $\{H_j\}_{1 \le j \le k}$ is a normal crossing at the points of $\mathcal{U}$. In particular, the following holds:

0. map $\sigma : \mathcal{N} \setminus \sigma^{-1}(\mathrm{Sing}_*(\overline{S})) \to \mathcal{U} \setminus \mathrm{Sing}_*(\overline{S})$ is an isomorphism;
1. the (so-called) *strict transform* $N := \overline{\sigma^{-1}((\overline{S} \cap \mathcal{U}) \setminus \sigma(\mathrm{Sing}(\sigma)))}$ of $\overline{S} \cap \mathcal{U}$ is smooth;
2. $\mathrm{Sing}_*(\overline{S}) = \sigma(\mathrm{Sing}(\sigma))$ and $\mathrm{Sing}(\sigma) = \sigma^{-1}(\sigma(\mathrm{Sing}(\sigma))) = \cup_{i \ge 1} H_{i+k}$, where each H_{i+k} is a smooth (so-called) *exceptional* hypersurface and in addition each H_{i+k} is the strict transform of the set of the critical points of the successive i-th intermediate blowing up;

3. each $H_i \cap N$, $i \geq 1$, is smooth and $\dim(H_i \cap N) = \dim(N) - 1$ for $i \geq k+1$;
4. the family $\{H_i\}_{i \geq 0}$, where we denote $H_0 := N$, forms a normal crossings in $\mathcal{N}$.

For any hypersurface $\{f = 0\} \subset \mathcal{U}$ the *strict transform* of $\{f = 0\}$ under map σ is

$$\Lambda_{(f)} = \overline{\sigma^{-1}(\{f = 0\}) \setminus \mathrm{Sing}(\sigma)} \subset \mathcal{N} \ .$$

Remark 8.1 *Due to property 2. above the local equation of* $\Lambda_{(f)}$ *can be constructed by factoring out from* $f \circ \sigma$ *the maximal monomial in exceptional hypersurfaces. In particular, assume that* f *depends on parameter* $c \in K^k$ *and map* $\tilde{\sigma} := \sigma \times id \colon \mathcal{N} \times K^k \to \mathcal{U} \times K^k$ *. With* $f|_c$ *being the evaluation of* f *at* c *, hypersurfaces* $\Lambda_{(f|_c)} \subset \mathcal{N}$ *and* $\Lambda_{(f)} \subset \mathcal{N} \times K^k$ *being the strict transforms under maps* σ *and* $\tilde{\sigma}$ *respectively, it follows that if for a particular value of* c *hypersurface* $\Lambda_{(f)}|_c := \Lambda_{(f)} \cap (\mathcal{N} \times \{c\}) \subset \mathcal{N}$ *is smooth then*

$$\Lambda_{(f|_c)} = \Lambda_{(f)}|_c \ , \tag{7}$$

where $\mathcal{N} \times \{c\}$ *is identified with* $\mathcal{N}$ *. Of course for a sufficiently generic value of* $c \in K^k$ *equality (7) holds in any case.*

Denote $\Lambda_j := \Lambda_{(L_j)} \subset \mathcal{N}$, $1 \leq j \leq k$, and $\Lambda := \Lambda_{(L)} \subset \mathcal{N} \times K^k$ (hypersurfaces Λ_j and Λ are the strict transforms of hypersurfaces in $\mathcal{U}$ and in $\mathcal{U} \times K^k$ under maps σ and $\tilde{\sigma}$ respectively). Hypersurfaces Λ_j , $1 \leq j \leq k$, are smooth and together with $\mathrm{Sing}(\sigma)$ form normal crossing in $\mathcal{N}$ due to the choice of admissible centers of blowings up (see e. g. [1] or [3]). In addition, for each j , $1 \leq j \leq k$, the difference between the divisors of $L_j \circ \sigma$ and Λ_j is the exceptional divisor E_j supported on $\mathrm{Sing}(\sigma) = \cup_{i \geq k+1} H_i \subset \mathcal{N}$ (each divisor E_j being of the form $E_j = \sum_i n_{j,i}[H_i]$ and all integers $n_{j,i} \geq 0$).

By means of following the embedded desingularization we started with by a composite of *combinatorial blowings up* (i. e. the blowings up with centers of all successive blowings up being the intersections of some of the accumulated and 'declared' exceptional hypersurfaces, where the latter are the strict transforms Λ_j of $\{L_j = 0\}$, $1 \leq j \leq k$) we may assume (Theorem 1.13 in [1]) that besides properties 0.-4. also holds:

5. the principal ideals generated by $L_j \circ \sigma$, $1 \leq j \leq k$, are (locally) linearly ordered with respect to inclusions implying, in particular, that the pull back $\mathcal{J}$ of the ideal generated by all L_j , $1 \leq j \leq k$, under the map $\sigma : \mathcal{N} \to \mathcal{U}$ is principal and locally 'near' any point a is generated by one of the $L_j \circ \sigma$, $1 \leq j \leq k$. (For such $j = j(a)$ it follows that $a \notin \Lambda_j$.)

Application of Sard Theorem on desingularization. As a consequence of property 5. hypersurface Λ is nonsingular. Indeed, for any point $(a, c) \in \Lambda$ there exists $j := j(a)$, $1 \leq j \leq k$, for which ideal $\mathcal{J} = (L_j \circ \sigma)$ in a neighbourhood of point $a \in \mathcal{N}$. Therefore, function

$$\lambda := \frac{\sum_{1 \leq i \leq k} c_i (L_i \circ \sigma)}{L_j \circ \sigma}$$

is regular at (a, c) and $\frac{\partial \lambda}{\partial c_j}(a, c) = 1$, while $\Lambda = \{\lambda = 0\}$.

The standard version of Sard Theorem implies that for a choice of an appropriate generic $c = (c_1, \ldots, c_k)$ the fiber Λ_c of the restriction to Λ of the natural projection $p \colon \Lambda \to K^k$ is nonsingular in $\sigma^{-1}(\mathcal{U})$. Note that Sard Theorem applies because if $x \in \mathcal{N} \setminus \mathrm{Sing}(\sigma)$ and $c \neq 0$ then a straightforward calculation (by making use of the linear independence of differentials dL_j , $1 \leq j \leq k$, in $\mathcal{U}$) shows that the rank of the Jacobian matrix of projection p at $(x, c) \in \Lambda$ equals k .

To complete the proof of i) we apply Sard Theorem to the restriction of projection p to $(N \times K^k) \cap \Lambda$. Note that $(N \times K^k) \cap \Lambda = \{(x,c) \in N \times K^k : \lambda(x,c) = 0\}$ in the local coordinates on $N \times K^k$ chosen as above and is nonsingular (since the partial derivative of λ with respect to c_j at (x,c) equals 1). Due to an assumption of Theorem 7.2

$$\dim(\{L_j \circ \sigma|_N\}_{1 \le j \le k}) = \dim(\{L_j|_S\}_{1 \le j \le k}) \ge 2 \ .$$

Pick $L_{j_1}|_S$, $L_{j_2}|_S$, $1 \le j_1 < j_2 \le k$, being linearly independent over K . It follows that there is a point $x \in N \setminus \mathrm{Sing}(\sigma)$ and c_{j_1} , $c_{j_2} \in K$ such that

$$c_{j_1} L_{j_1}(\sigma(x)) + c_{j_2} L_{j_2}(\sigma(x)) = 0 \ , \qquad c_{j_1}(dL_{j_1})(\sigma(x)) + c_{j_2}(dL_{j_2})(\sigma(x)) \neq 0$$

holds. Such $x \in N \setminus \mathrm{Sing}(\sigma)$ exists since otherwise

$$(L_{j_2}(dL_{j_1}) - L_{j_1}(dL_{j_2}))(\sigma(x)) = 0 \ \text{ for all } \ x \in N \setminus \mathrm{Sing}(\sigma) \ ,$$

which would imply a linear dependence of $L_{j_1}|_S$, $L_{j_2}|_S$ contrary to their choice. Set $c_j = 0$ for all $j \neq j_1$, j_2 . Then again by means of a straightforward calculation the rank of the Jacobian at (x,c) of projection $p \colon (N \times K^k) \cap \Lambda \to K^k$ equals k and therefore Sard Theorem implies that $N \cap \Lambda_c$ is nonsingular for appropriate generic c , where N is identified with $N \times \{c\}$. Since σ is an isomorphism off $\mathrm{Sing}_*(\overline{S})$ (which is the property 0. of σ) it follows that if $\{L = 0\} \cap \mathrm{Reg}_*(\overline{S}) \neq \emptyset$ then it is a smooth hypersurface of $\mathrm{Reg}_*(\overline{S})$ of dimension $\dim(S) - 1$. To complete the proof of i) it suffices to show that $N \cap \Lambda_c \not\subset \mathrm{Sing}(\sigma) = \cup_{i \ge 1} H_{i+k}$ and that, moreover, $N \cap \Lambda_c \setminus \mathrm{Sing}(\sigma)$ is dense in $N \cap \Lambda_c$.

Both properties follow by specifying an appropriate generic choice of c further, e. g. a choice of c such that Λ_c intersects transversally every $H_J \times \{c\}$ would do, where $H_J = \cap_{j \in J} H_j$ for any acceptable index set $J \subset \{i \ge 0\}$. We achieve the latter by once again applying Sard Theorem to the restriction of projection p to $(H_J \times K^k) \cap \Lambda$. Of course, for J such that $p(H_J \times K^k \cap \Lambda)$ is not dense in K^k it follows that $H_J \times \{c\} \cap \Lambda_c = \emptyset$ for a generic choice of $c \in K^k$, and otherwise Sard Theorem applies and implies for an appropriate generic choice of c the desired transversality, which completes the proof of i).

Proof of ii). We summarize consequences of application of Sard Theorem in

Remark 8.2 *For a choice of an appropriate generic* $c \in K^k$ *it follows that the family* $\{H_i\}_{i \ge 0}$ *with* Λ_c *form a normal crossings in* $\mathcal{N} := \mathcal{N} \times \{c\}$.

Adjusting metrics on $\mathcal{U}$ and on S .

Remark 8.3 *By means of replacing the standard Hermitian metric on* K^n *for* $K = \mathbb{C}$ *, respectively Euclidian for* $K = \mathbb{R}$ *, by an equivalent (over any compact subset of* $\mathcal{U}$ *) Hermitian, respectively Riemannian, metric on* $\mathcal{U} \subset K^n$ *we may assume w.l.o.g. that* $dL_1(a)$, $\ldots$, $dL_k(a)$, $a \in \mathcal{U}$ *, is an orthonormal basis in* $\mathcal{L}_a^* := \mathrm{Span}(\{dL_j(a)\}_{1 \le j \le k})$.

Remark 8.4 *For* $a \in S$ *near* $\mathcal{G}$ *inclusions* $\Omega_a := \mathcal{L}_a^* / \mathcal{L}_a^* \cap T_a(S)^\perp \hookrightarrow T_a(S)^*$ *via the restrictions of functionals from* $\mathcal{L}_a^*$ *to* $T_a(S)$ *are isometries.*

Metrics on desingularization convenient for 'logarithmic differentiation': we introduce metrics on $N \setminus \mathrm{Sing}(\sigma)$ 'nearby' any point $\tilde{b} \in N \cap \Lambda_c \cap \mathrm{Sing}(\sigma) \subset N$ as follows. In a neighbourhood of $\tilde{b}$ the smooth variety N admits an analytic coordinate chart $\mathcal{C}$

with the origin at $\tilde{b}$ and every exceptional hypersurface H intersecting $\mathcal{C}$ by a coordinate hyperplane $\{x_H = 0\}$ of $\mathcal{C}$, unless the intersection is empty. (In the algebraic case we may also use the notion of an affine 'etale' coordinate chart of [1], [2].) In a neighbourhood of $\tilde{b}$ the local ideal $\mathcal{J}_{\tilde{b}}$ is generated by a single $L_j \circ \sigma$ for a suitable j (property 5. of map σ), and the function $h := \lambda|_c$ has a non-vanishing differential at $\tilde{b}$, since $N \cap \Lambda_c$ is nonsingular due to the choice of c as shown in the proof of i). We shrink the neighbourhood $\mathcal{C}$ so that dh does not vanish at all points of $\mathcal{C}$. In addition, due to Remarks 8.2 and 8.1, we may assume that h is one of the non-exceptional coordinates on $\mathcal{C}$. We define an auxiliary norm on $T_{\tilde{a}}(N)^*$ for $\tilde{a} \in \mathcal{C} \setminus \mathrm{Sing}(\sigma)$ via an imposition of the following:

$$\left\{ \frac{dx_H}{x_H}, dx_i \right\}_{H,i} \text{ is an orthonormal basis on } T_{\tilde{a}}(N)^* , \tag{8}$$

where $\{x_H, x_i\}_{H,i}$ are the coordinates in $\mathcal{C}$ with the former ones corresponding to the exceptional hypersurfaces and the latter $\{x_i\}_i$ being the remaining coordinate functions (including function h). A straightforward calculation shows that the Hermitian (Riemannian for $K = \mathbb{R}$) metrics on $\mathcal{C} \backslash \mathrm{Sing}(\sigma)$ introduced by means of (8) do not depend on the coordinate choices that preserve exceptional hypersurfaces, i. e. are isomorphic over compacts in $\mathcal{C}$ (here we do not make use of this fact), cf. [11].

The key estimate by means of 'logarithmic differentiation'. We now will complete the proof of Theorem 7.2 relying on the following lemma

Lemma 8.5 *For $\tilde{a} \in (N \cap \Lambda_c \cap \mathcal{C}) \setminus \mathrm{Sing}(\sigma)$ the norm of $d(L_c \circ \sigma)|_{\tilde{a}} \in T_{\tilde{a}}(N)^*$ equals $|L_j \circ \sigma(\tilde{a})|$. Moreover, the latter majorizes the norm of $((\sigma|_S)^*_{\tilde{a}})|_{\Omega_{\sigma(\tilde{a})}} \colon \Omega_{\sigma(\tilde{a})} \to T_{\tilde{a}}(N)^*$ (up to a multiplicative constant depending only on a compact $\mathcal{K} \subset \mathcal{C}$ for $\tilde{a} \in \mathcal{K}$).*

Remark 8.6 *The norms of the composites $\Psi_{\tilde{a}} \colon \mathcal{L}^*_{\sigma(\tilde{a})} \to T_{\tilde{a}}(N)^*$ of the restrictions to $\mathcal{L}^*_{\sigma(\tilde{a})}$ of the pull backs $\sigma^*_{\tilde{a}} \colon T_{\sigma(\tilde{a})}(\mathcal{U})^* \to T_{\tilde{a}}(\mathcal{N})^*$ with the maps dual to the inclusions $T_{\tilde{a}}(N) \hookrightarrow T_{\tilde{a}}(\mathcal{N})$ coincide with the norms of (linear) maps $((\sigma|_S)^*_{\tilde{a}})|_{\Omega_a} \colon \Omega_a \to T_{\tilde{a}}(N)^*$, i. e. the restrictions to Ω_a of the pull backs $(\sigma|_S)^*_{\tilde{a}} \colon T_a(S)^* \to T_{\tilde{a}}(N)^*$ (since maps $\Psi_{\tilde{a}}$ are also the composites of the quotient maps $\mathcal{L}^*_a \to \Omega_a$ with $((\sigma|_S)^*_{\tilde{a}})|_{\Omega_a}$). Therefore it suffices to majorize (up to a multiplicative constant) the norms of the maps $\Psi_{\tilde{a}}$ by $|L_j(a)|$ for $j = j(\tilde{a})$ (the index j though does not depend on $\tilde{a} \in \mathcal{C}$).*

Lemma 8.5 implies a lower bound (depending on the choice of a compact $\mathcal{K} \subset \mathcal{C}$) on the norms of $(dL|_S)(a) \in \Omega_a$ at the points $a \in \{L = 0\} \cap \mathrm{Reg}_*(\overline{S}) \cap \sigma(\mathcal{K}) = \mathrm{Reg}_*(\overline{S}) \cap \sigma(\Lambda_c \cap \mathcal{K})$. Since σ is a proper map the item ii) of Theorem 7.2 follows. ∎

Proof of Lemma 8.5. Recall (see property 5. of map σ) that $L_j \circ \sigma$, $j = j(\tilde{a})$, coincides (up to an invertible function) with $\prod_{\tilde{b} \in H} x_H^{n_H}$ in $\mathcal{C}$ (w.l.o.g. we may assume that they coincide). Due to Remark 8.1 and, since $h(\tilde{a}) = 0$, it follows that

$$d(L_c \circ \sigma)|_{\tilde{a}} = d((L_j \circ \sigma) \cdot h)|_{\tilde{a}} = L_j(a) \cdot dh|_{\tilde{a}} .$$

Due to the choice of the norms on $T_{\tilde{a}}(N)^*$ (see (8)), for $\tilde{a} \in \mathcal{C} \setminus \mathrm{Sing}(\sigma)$, it follows that the norm of $dh|_{\tilde{a}}$ equals 1. Therefore the norm of $d(L_c \circ \sigma)|_{\tilde{a}}$ is $|L_j(a)|$, as required.

Due to Remark 8.6 it remains to bound the norms of the maps $\Psi_{\tilde{a}} \colon \mathcal{L}^*_a \to T_{\tilde{a}}(N)^*$. Note that because $L_j \circ \sigma$ is (in $\mathcal{C}$) a common factor of all $L_i \circ \sigma$, $1 \leq i \leq k$, and since the norms of $d(L_j \circ \sigma)|_{\tilde{a}}$ due to (8) coincide with $\sqrt{\sum_{\tilde{b} \in H} n_H^2} \cdot |L_j(a)|$, it follows that the norms of all $d(L_i \circ \sigma)|_{\tilde{a}}$ are majorized by $|L_j(a)|$ (up to a multiplicative constant depending only on the choice of $\mathcal{K}$ provided that $\tilde{a} \in (N \cap \Lambda_c \cap \mathcal{K}) \setminus \mathrm{Sing}(\sigma)$). The required upper bound on the norms of $\Psi_{\tilde{a}} \colon \mathcal{L}_a^* \to T_{\tilde{a}}(N)^*$ follows since the latter norms are bounded by $k^{1/2}$ times the maximum of the norms of the images of the orthonormal basis $\{dL_i(a)\}_i$ in $\mathcal{L}_a^*$. ■

9 Complexity of universal TWG-stratifications.

To provide a complexity upper bound on constructing the Glaeser bundle of vector spaces $G_F = G = G^{(\rho)} \supset \cdots \supset G^{(1)} \supset G^{(0)} = \mathcal{T}$ (see Section 3) and the quasistrata $\mathcal{G}_k$ assume that the components $f_i \in \mathbb{Z}[x_1, \ldots, x_n]$, $1 \leq i \leq t$ of the polynomial map $F = (f_1, \ldots, f_t)$ have integer coefficients. Let 2^R bound the absolute values of the coefficients, and integer d bound the degrees of the polynomials. We consider two cases: $K = \mathbb{C}$ or $K = \mathbb{R}$, although one could study other algebraically or real closed effectively represented fields K , then R would bound the bit-size of the coefficients [12].

To proceed consecutively from bundle $G^{(p)}$ to $G^{(p+1)}$, $0 \leq p < \rho \leq 2 \cdot n$, one has to carry out two basic subroutines (see Section 3): to produce the closure of a constructible set, and for a given bundle M to construct the bundle M' of vector spaces whose fibers being linear hulls of the respective fibers of M . An algorithm producing the closure is exhibited in [12]. For the second subroutine given a quantifier-free formula over field K describing M , one can describe M' by means of a formula with quantifiers over K in a straightforward way. To the latter formula one can apply a quantifier elimination algorithm of [12] resulting in a quantifier-free formula describing M' . This yields an upper bound $R^{O(1)} \cdot d^{n^{O(\rho)}}$ on the complexity of constructing G and the complexity of $\mathcal{G}_k$. Recall that the quasistrata $\mathcal{G}_k$ provide a universal TWG-stratification, provided that G is Lagrangian (Corollary 4.11). Note that in an example from Section 10.2 the index of stabilization ρ grows linearly with n .

We would like to mention that a similar double-exponential complexity upper bound $R^{O(1)} \cdot d^{n^{O(n)}}$ on stratifications (though without property of universality) was obtained in [22], [6]. On the other hand, there is an obvious exponential complexity lower bound $R^{O(1)} \cdot d^{O(n)}$.

It would be interesting to understand, whether this double-exponential bound is sharp?

Note that the computational complexity bound of [4] for the resolution of singularities (in terms of the primitive-recursive functions) is a considerably larger bound.

10 Examples.

10.1 A family of $\mathbf{F} : \mathbf{K}^{\mathbf{N}} \to \mathbf{K}$ which admit universal TWG-stratifications.

We give an example of a family of polynomial maps $F : K^N \to K$

$$F := F_n = \sum_{1 \leq i \leq j \leq n} A_{i,j} X_i X_j \in K[\{A_{i,j}\}, \{X_i\}] \quad , \; N = n + \binom{n+1}{2} ,$$

that admit universal TWG-stratifications of $\mathrm{Sing}(F)$. The latter turn out to be stratifications in the traditional sense (of the first paragraph of Section 2) with the index of stabilization $\rho(F) = 1$ (i. e. $G^{(1)} = G_F$).

Of course $\mathrm{Sing}(F) = \{X_i = 0\}_{1\le i\le n}$. Let for the sake of brevity $G := G_F$ and bundle $B := G^{(1)}$ (as in the construction of Section 3 for map F).

Any nonsingular $n\times n$ matrix C over K induces an isomorphism of $K^N \to K^N$, which for brevity we also denote C (of course isomorphisms C preserve the ranks of quadratic forms). Therefore, for any particular point $a = (\{a_{i,j}\}, \{0\}) \in \mathrm{Sing}(F)$ with quadratic form $f_a := \sum_{1\le i\le j\le n} a_{i,j}X_iX_j$ being of a rank q the dimension of the fiber B_a of bundle B at a coincides with the dimension of the fiber of bundle B at the point $a(q)$ with the corresponding quadratic form being $f_{a(q)} = \sum_{1\le i\le q} X_i^2$, e. g. due to Corollary 3.2.

We identify the set of all quadratic forms f_a of rank q with the constructible set $\mathcal{B}_{(q)} \subset \mathrm{Sing}(F)$ of the corresponding points $a \in \mathrm{Sing}(F)$. A straightforward calculation shows that $\dim(\mathcal{B}_{(q)}) = qn - q(q-1)/2$. Once again by means of Corollary 3.2 (and of an appropriate isomorphism $C : K^N \to K^N$) it follows that $\mathcal{B}_{(q)}$ is smooth and that fibers G_a are of a constant dimension $k(q)$ at all points $a \in \mathcal{B}_{(q)}$, i. e. $\mathcal{B}_{k(q)} = \mathcal{B}_{(q)}$. (Since $l = 1$ Thom stratification of $\mathrm{Sing}(F)$ exists by [18] and therefore due to (1') of Lemma 3.7 inequality $k(q) \le \mathrm{codim}\,\mathcal{B}_{k(q)}$ holds.) Below we calculate $k(q)$, which would allow us to conclude (by making use of Theorem 4.8) that each $\mathcal{B}_{k(q)}$ is Lagrangian and therefore that $B = G$, $\mathcal{B}_{k(q)} = \mathcal{G}_{k(q)}$ and that stratification $\{\mathcal{B}_{k(q)}\}_{0\le q\le n}$ of $\mathrm{Sing}(F)$ (by rank of f_a for $a \in \mathcal{B}_{k(q)}$) is a universal TWG-stratification.

Consider curves $\gamma : K \ni t \mapsto \gamma(t) \in K^N$ with the origins at $a(q) = \gamma(0)$ and parametrized by $x \in K^n$ as follows:

$$X_i = t^3x_i\ ,\ 1 \le i \le q\ ;\quad X_j = t^2x_j\ ,\ q < j \le n\ ;\quad A_{ii} = 1\ ,\ 1 \le i \le q\ ;$$
$$A_{jj} = t\ ,\ q < j \le n\ ;\quad A_{ij} = 0\ ,\ i \ne j\quad .$$

A straightforward calculation of the limit along this curve of the normalized differential $dF/||dF||$ shows that $\sum_{1\le i\le n} x_idX_i \in B_{a(q)}$. Consider similarly limits along curves with the same origin at $a(q)$ and defined as follows: $A_{ii} := 1$ if $1 \le i \le q$ and $A_{ij} := 0$ for pairs of i , j with $1 \le i < j \le n$ or $q < i = j \le n$, while $X_i = 0$ for $1 \le i \le q$ and $X_j = tx_j$ for $q < j \le n$. A straightforward calculation implies that the ‘coordinate' projection of $B_{a(q)}$ to the subspace spanned by $\{dA_{ij}\}_{1\le i\le j\le n}$ contains the image under the degree two Veronese map of points $(x_1, ..., x_n) \in K^n$ with coordinates $x_i = 0$ for $1 \le i \le q$. It follows that subspace $B_{a(q)}$ of $(K^N)^*$ contains dX_i for $1 \le i \le n$, and $dA_{j,s}$ for $q < j \le s \le n$, i. e. $k(q) \ge (n + (n-q)(n-q+1)/2) = \mathrm{codim}\,\mathcal{B}_{k(q)}$ implying $k(q) = \mathrm{codim}\,\mathcal{B}_{k(q)}$. Consequently each (de facto smooth) quasistratum $\mathcal{B}_{k(q)}$ is Lagrangian, $G = B$ and, due to Theorem 4.10 and its Corollary 4.11, partition $\{\mathcal{B}_{k(q)}\}_{0\le q\le n}$ of $\mathrm{Sing}(F)$ is a universal TWa stratification of $\mathrm{Sing}(F)$. Summarizing

Proposition 10.1 *For*

$$F = F_n = \sum_{1\le i\le j\le n} A_{i,j}X_iX_j \in K[\{A_{i,j}\}, \{X_i\}]$$

the index of stabilization $\rho(F) = 1$ *and strata* $\mathcal{B}_{k(q)} = \{a = (\{a_{ij}\}, \{0\}) : rk(f_a) = q\} \subset \mathrm{Sing}(F)$ *form a universal TWa stratification of* $\mathrm{Sing}(F)$ *with respect to* F *.*

10.2 A family of examples of $\mathbf{F_n : K^{4n+1} \to K}$ with universal TWG-stratifications and the index of stabilization $\rho(\mathbf{F_n}) = \mathbf{n}$.

Let $q(x,y,u,v,w) := u\cdot x^2 + 2w\cdot x\cdot y + v\cdot y^2$ and produce recursively the following polynomials: $q_1 := q(x_1\ ,\ y_1\ ,\ u_1\ ,\ v_1\ ,\ w)$, $q_{k+1} := q(x_{k+1}\ ,\ y_{k+1}\ ,\ u_{k+1}\ ,\ v_{k+1}\ ,\ q_k(\cdot))$, $k \geq 1$. Denote

$$F := F_n\ , \qquad F_n(\vec{x}\ ,\ \vec{y}\ ,\ \vec{u}\ ,\ \vec{v}\ ,\ w) := q_n(\vec{x}\ ,\ \vec{y}\ ,\ \vec{u}\ ,\ \vec{v}\ ,\ w)\ ,$$

where $\vec{x}\ ,\ \vec{y}\ ,\ \vec{u}\ ,\ \vec{v} \in K^n$ and $x_k\ ,\ y_k\ ,\ u_k\ ,\ v_k$ for $1 \leq k \leq n$ denote their respective k-th coordinates, i. e. F depends on $N = 4n+1$ independent variables. Let $h_k := u_k \cdot v_k - q_{k-1}^2(\cdot)$, $1 \leq k \leq n$. Then $F = u_n \cdot x_n^2 + 2q_{n-1} \cdot x_n \cdot y_n + v_n \cdot y_n^2$ and $\mathrm{Sing}(F) = \{x_n = y_n = 0\}$. By making use of Corollary 3.2 and example from Section 10.1 it follows that for points $a \in \mathrm{Sing}(F)$ with $dq_{n-1}(a) \neq 0$ the fibers of bundle $G^{(1)}$ are

1. $G_a^{(1)} = \mathrm{Span}\{dx_n\ ;\ dy_n\}$ if $h_n(a) \neq 0$, i. e. $\mathcal{G}_2 = \mathrm{Sing}(F)\backslash\{h_n = 0\}$ off $\{dq_{n-1} = 0\}$;

2. $G_a^{(1)} = \mathrm{Span}\{dx_n\ ;\ dy_n\ ;\ dh_n\}$ if $h_n(a) = 0$, $dh_n(a) \neq 0$, i. e. off $\{dq_{n-1} = 0\}$ quasistratum $\mathcal{G}_3 = \mathrm{Sing}(F) \cap \{h_n = 0\} \setminus \{dh_n \neq 0\}$;

3. $G_a^{(1)} = \mathrm{Span}\{dx_n\ ;\ dy_n\ ;\ du_n\ ;\ dv_n\ ;\ dq_{n-1}\}$, if $h_n(a) = 0$, $dh_n(a) = 0$, i. e. $\mathcal{G}_5 = \mathrm{Sing}(F) \cap \{h_n = 0\ ,\ dh_n = 0\}$ off $\{dq_{n-1} = 0\}$.

4. In the cases 1. and 2. fibers $G_a^{(1)} = (\overline{G^{(0)}})_a$, but in the case 3. fibers $G_a^{(1)} \neq (\overline{G^{(0)}})_a = \left\{\omega = U_n du_n + V_n dv_n + Q_{n-1} dq_{n-1} + X_n dx_n + Y_n dy_n\ :\ U_n \cdot V_n = (Q_{n-1}/2)^2\right\}$, where ω denotes a 1-form at a .

Denote $D_1 := \mathrm{Span}\{dx_n\ ;\ dy_n\ ;\ du_n\ ;\ dv_n\}$. Note that

$$dF = x_n^2 du_n + y_n^2 dv_n + 2x_n y_n dq_{n-1} + 2(u_n x_n + q_{n-1} y_n) dx_n + 2(q_{n-1} x_n + v_n y_n) dy_n\ .$$

Results above rely on elementary calculations of Section 10.1 summarized below: $h_n = \det\begin{pmatrix} u_n & q_{n-1} \\ q_{n-1} & v_n \end{pmatrix}$ and for any sequence of points from K^N converging to a point $a \in \mathrm{Sing}(F)$ the following holds

i) the size of $\{\frac{\partial F}{\partial x_n}\ ;\ \frac{\partial F}{\partial y_n}\}$ dominates $\{x_n^2\ ,\ y_n^2\ ,\ 2x_n \cdot y_n\}$ at a if $h_n \not\to 0$,

ii) the limits of $dF/||dF||$ are the 1-forms $\omega = U_n du_n + V_n dv_n + Q_{n-1} dq_{n-1} + X_n dx_n + Y_n dy_n$ with $U_n \cdot V_n = Q_{n-1}^2/4$, since the coefficients of dF at du_n , dv_n , dq_{n-1} satisfy $x_n^2 \cdot y_n^2 = (2x_n \cdot y_n)^2/4$.

When $h_n(a) = 0$ the latter also follows from the orthogonality of $\omega \in G_a^{(1)}$ to $T_a(\{h_n = 0\})$ (see (1') of Lemma 3.7) and $dh_n = v_n \cdot du_n + u_n \cdot dv_n + 2q_{n-1} \cdot dq_{n-1}$, implying that ω is proportional to dh_n , while $u_n \cdot v_n = q_{n-1}^2$ for points in $\{h_n = 0\}$.

We now turn to a simple, but crucial observation that the coefficients of dF at du_n , dv_n , dq_{n-1} satisfy inequality $\sqrt{|x_n|^2 + |y_n|^2} \geq (\sqrt{2})^{-1} \cdot |2x_n \cdot y_n|$. Hence the limits of $dF/||dF||$ evaluated at the points that converge to $\mathrm{Sing}(F) \cap \{dq_{n-1} = 0\}$ are the 1-forms with vanishing coefficients at all differentials of the independent variables on which $q_{n-1}(\cdot)$ depends. In particular, combining with the preceding summary of the arguments of Section 10.1 properties 1. and 2. follow without making assumption $dq_{n-1}(a) \neq 0$ and also

5. $G_a^{(1)} = D_1$ for $a \in Z_{n-1} := \mathrm{Sing}(F) \cap \{h_n = 0,\ dh_n = dq_{n-1} = 0\} \subset \{q_{n-1} = 0\}$ holds.

Summarizing $\mathcal{G}_2 = \mathrm{Sing}(F) \setminus \{h_n = 0\}$, $\mathcal{G}_3 = \mathrm{Sing}(F) \cap \{h_n = 0 \ , \ dh_n \neq 0\}$ and with $\mathcal{G}'_5 := \mathrm{Sing}(F) \cap \{h_n = 0 \ , \ dh_n = 0 \ , \ dq_{n-1}(a) \neq 0\}$ bundle $G^{(1)}|_{\mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}'_5} = G|_{\mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}'_5}$. Also $\mathcal{G}'_5 = \{x_n = y_n = u_n = v_n = q_{n-1} = 0 \ , \ dq_{n-1} \neq 0\}$, and $Z_{n-1} = \{x_n = y_n = u_n = v_n = x_{n-1} = y_{n-1} = 0\} = \mathrm{Sing}(F) \setminus (\mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}'_5)$.

Detour. The two Remarks-Examples below are straightforward consequences of the latter observation and the preceding it summary of the arguments of Section 10.1.

Remark 10.2 *With notations* $G = G_{\tilde{F}}$, $G^{(p)} = G^{(p)}_{\tilde{F}}$ *for a function*

$$\tilde{F} := u \cdot x^2 + 2w^2 \cdot x \cdot y + v \cdot y^2$$

depending on 5 *variables the following holds:*
inequality $\dim G^{(1)}_a \leq 4$ *for all* $a \in Sing(\tilde{F})$ *; bundles* G *and* $G^{(1)}$ *coincide; quasistrata* $\mathcal{G}_2 = \{x = y = 0 \ , \ u \cdot v - w^4 \neq 0\}$, $\mathcal{G}_3 = \{x = y = 0 \ , \ u \cdot v - w^4 = 0 \ , \ (u \ , \ v) \neq 0\}$ *and* $\mathcal{G}_4 = \{0\}$ *are smooth and form as strata a TWa stratification, say* $\mathcal{S}$ *, of* $Sing(\tilde{F})$ *; quasistrata* $\mathcal{G}_2$ *and* $\mathcal{G}_3$ *are Lagrangian, but the quasistratum* $\mathcal{G}_4$ *is not Lagrangian (* $\dim \mathcal{G}_4 = 0 < 5 - 4$ *!). Also,* $\overline{G|_{\mathcal{G}_2}}$ *and* $\overline{G|_{\mathcal{G}_3}}$ *are* 5*-dimensional irreducible components of* G *and* $G|_{\mathcal{G}_4}$ *is in the closure of* $G|_{\mathcal{G}_3}$ *.*

Remark 10.3 *Let non-zero polynomial* $g \in K[z_1, \ldots, z_m]$ *and* $F_g := \tilde{F}(x, y, u, v, g(z))$ *, where* $\tilde{F}$ *is from the preceding Remark. Denote* $G := G_{F_g}$, $G^{(p)} := G^{(p)}_{F_g}$ *. Then for polynomial* F_g *depending on* $m + 4$ *variables the following holds:*
$\dim G^{(1)}_a \leq 4$ *for all* $a \in Sing(F_g)$ *; bundles* G *and* $G^{(1)}$ *coincide; the quasistrata are* $\mathcal{G}_2 = \{x = y = 0 \ , \ u \cdot v - g(z)^4 \neq 0\}$, $\mathcal{G}_3 = \{x = y = 0 \ , \ u \cdot v - g(z)^4 = 0 \ , \ (u \ , \ v) \neq 0\}$ *and* $\mathcal{G}_4 = \{x = y = u = v = g(z) = 0\}$ *; only quasistratum* $\mathcal{G}_4$ *is not Lagrangian; the irreducible components* $\overline{G|_{\mathcal{G}_2}}$ *and* $\overline{G|_{\mathcal{G}_3}}$ *of* G *are* $(m+4)$*-dimensional and* $G|_{\mathcal{G}_4}$ *is in the closure of* $G|_{\mathcal{G}_3}$ *.* **Curiously, an arbitrarily chosen hypersurface** $\{g = 0\}$ **appears as a quasistratum**.

We now turn to a calculation of fibers of $G^{(2)}$ for F . Note that $dq_{n-1} - 2x_{n-1}y_{n-1}dq_{n-2} =$

$$x^2_{n-1}du_{n-1} + y^2_{n-1}dv_{n-1} + 2(u_{n-1}x_{n-1} + q_{n-2}y_{n-1})dx_{n-1} + 2(q_{n-2}x_{n-1} + v_{n-1}y_{n-1})dy_{n-1}$$

and bundles $G = G^{(2)} = G^{(1)}$ off $Z_{n-1} \subset \{x_{n-1} = y_{n-1} = 0\}$. It follows by making use of Corollary 3.2 and of the calculations like in the summary of the arguments of Section 10.1 that for points b from $\mathcal{G}'_5$ converging to a point $a \in Z_{n-1} \subset \{q_{n-1} = 0 \ , \ dq_{n-1} = 0\}$ with $dq_{n-2} \neq 0$ the span of the limits of the 1-forms from the fibers G_b of G (it includes the limits of $dq_{n-1}/||dq_{n-1}||$) coincides with the fibers of bundle $G^{(2)}$, namely:

1'. $G^{(2)}_a = \mathrm{Span}\{dx_{n-1} \ ; \ dy_{n-1}\} \oplus D_1$ if $h_{n-1}(a) \neq 0$, i. e. $\mathcal{G}_6 = Z_{n-1} \setminus \{h_{n-1} = 0\}$ off $\{dq_{n-2} = 0\}$;

2'. $G^{(2)}_a = \mathrm{Span}\{dx_{n-1} \ ; \ dy_{n-1} \ ; \ dh_{n-1}\} \oplus D_1$ if $h_{n-1}(a) = 0, \ dh_{n-1}(a) \neq 0$, i. e. off $\{dq_{n-2} = 0\}$ quasistratum $\mathcal{G}_7 = Z_{n-1} \cap \{h_{n-1} = 0\} \setminus \{dh_{n-1} \neq 0\}$;

3'. $G^{(2)}_a = \mathrm{Span}\{dx_{n-1} \ ; \ dy_{n-1} \ ; \ du_{n-1} \ ; \ dv_{n-1} \ ; \ dq_{n-2}\} \oplus D_1$, if $h_{n-1}(a) = 0$, $dh_{n-1}(a) = 0$, i. e. $\mathcal{G}_9 = Z_{n-1} \cap \{h_{n-1} = 0 \ , \ dh_{n-1} = 0\}$ off $\{dq_{n-2} = 0\}$.

4'. In the cases 1'. and 2'. fibers $G^{(2)}_a = (\overline{G^{(1)}})_a$, but in the case 3'. fibers $G^{(2)}_a \not\subset (\overline{G^{(1)}})_a$ and the latter consists of all 1-forms $\omega \in G^{(2)}_a$ with coefficients U_{n-1} , V_{n-1} , Q_{n-2} at

du_{n-1} , dv_{n-1} , dq_{n-2} that satisfy equation $U_{n-1} \cdot V_{n-1} = (Q_{n-2}/2)^2$. Denote $D_2 :=$ Span$\{dx_{n-1} \ ; \ dy_{n-1} \ ; \ du_{n-1} \ ; \ dv_{n-1}\} \oplus D_1$.

Once again, due to the observation that the coefficient of dq_{n-1} at dq_{n-2} is dominated by its coefficients at du_{n-1} , dv_{n-1} , it follows that for points $b \in \mathrm{Sing}(F)$ converging to a point $a \in \{dq_{n-2} = 0\}$ the limits of the 1-forms from fibers $G^{(1)}_b$ (which by definition include the limits of $dq_{n-1}/||dq_{n-1}||$) consist only of the 1-forms with vanishing coefficients at all differentials of the independent variables on which q_{n-2} depends. In particular, properties 1'. and 2'. follow without making assumption $dq_{n-2}(a) \neq 0$ and the fiber of bundle $G^{(2)}$ at a is

5'. $G^{(2)}_a = D_2$ for $a \in Z_{n-2} := Z_{n-1} \cap \{h_{n-1} = 0 \ , \ dh_{n-1} = dq_{n-2} = 0\} \subset \{q_{n-2} = 0\}$.

Summarizing $\mathcal{G}_5 = \mathcal{G}'_5$, $\mathcal{G}_6 = Z_{n-1} \setminus \{h_{n-1} = 0\}$, $\mathcal{G}_7 = Z_{n-1} \cap \{h_{n-1} = 0 \ , \ dh_{n-1} \neq 0\}$ and with $\mathcal{G}'_9 := Z_{n-1} \cap \{h_{n-1} = 0 \ , \ dh_{n-1} = 0 \ , \ dq_{n-2} \neq 0\}$ bundle $G^{(2)}|_{\mathcal{G}_6 \cup \mathcal{G}_7 \cup \mathcal{G}'_9} = G|_{\mathcal{G}_6 \cup \mathcal{G}_7 \cup \mathcal{G}'_9}$. Also $\mathcal{G}'_9 = Z_{n-1} \cap \{u_{n-1} = v_{n-1} = q_{n-2} = 0 \ , \ dq_{n-2} \neq 0\}$, and
$Z_{n-2} = Z_{n-1} \cap \{u_{n-1} = v_{n-1} = x_{n-2} = y_{n-2} = 0\} = Z_{n-1} \setminus (\mathcal{G}_6 \cup \mathcal{G}_7 \cup \mathcal{G}'_9)$.

Thus $G^{(1)} \neq G^{(2)}$ and $G = G^{(2)}$ off Z_{n-2} . Calculation of fibers of $G^{(p)}$, $p > 2$, for points from Z_{n-2} is similar (recursively on p), in particular implying that $\mathcal{G}_9 = \mathcal{G}'_9$. Summarizing

Proposition 10.4 *Quasistrata $\{\mathcal{G}_r\}_r$ for polynomial F (in $4n+1$ independent variables) are smooth, Lagrangian, form a TWa stratification and hence a universal TWG-stratification. The index of stabilization $\rho(F)$ of F equals n .*

10.3 Example of $\mathbf{F} : \mathbf{K^5} \to \mathbf{K}$ with no universal TWG-stratification.

For $F := \tilde{F}$ from Remark 10.2 we have shown that there is a non Lagrangian quasistratum of $G := G_F$ and therefore due to Theorem 5.1 $\mathrm{Sing}(F)$ does not admit a universal TWG-stratification. In this example of Remark 10.2 quasistratum $\mathcal{G}_4$ and curve $\{x = y = 0 \ , \ u = v = t^2 \ , \ w = t\}$ (defined parametrically) are the non-Lagrangian $\mathcal{G}$ and a G-regular extension $\mathcal{G}^+$ of the proof of Theorem 5.1. (Note that most of the proof of Theorem 5.1 covering three sections starting Section 5 is devoted to a construction of $\mathcal{G}^+$.) Consequently, the partition of $\mathrm{Sing}(\tilde{F})$ by sets $\mathcal{B}_2 := \mathcal{G}_2$, $\mathcal{B}_3 := \mathcal{G}_3 \setminus \mathcal{G}^+$, $\mathcal{B}_4 := \mathcal{G}^+$

is a TWa stratification, say $\tilde{\mathcal{S}}$, and the associated bundle $B(\tilde{\mathcal{S}}) \neq B(\mathcal{S})$, where $\mathcal{S}$ is the TWa stratification of $\mathrm{Sing}(F)$ constructed in Remark 10.2. Illustrating the punch line of the proof of Theorem 5.1, we may now show directly that there does not exist a universal TWG-stratification of $\mathrm{Sing}(\tilde{F})$. Assuming the contrary say $\mathcal{S}^{uni}$ is a universal TWG-stratification of $\mathrm{Sing}(F)$. Denote by $B(\mathcal{S}^{uni})$ its bundle of vector spaces. Construction of the Glaeser bundle G and an elementary Proposition 3.4 imply that over $\mathrm{Sing}(F) \setminus \{0\} = \mathcal{G}_2 \cup \mathcal{G}_3$ bundles G , $B(\mathcal{S})$, $B(\tilde{\mathcal{S}})$ and $B(\mathcal{S}^{uni})$ coincide since quasistrata $\mathcal{G}_2$ and $\mathcal{G}_3$ of G from Remark 10.2 are Lagrangian, see Corollary 3.5. Due to Proposition 4.1 also $G \subset B(\mathcal{S}^{uni}) \subset (B(\mathcal{S}) \cap B(\tilde{\mathcal{S}}))$. Combining with $G_0 = B(\tilde{\mathcal{S}})_0$ then $B(\mathcal{S}^{uni})_0 = G_0 =$ the 4-dimensional subspace of $(T_0(K^5))^{dual}$ orthogonal to $\frac{\partial}{\partial w} \in T_0(K^5)$. Finally, $\mathcal{S}^{uni}$ is universal and $\{0\}$ is a stratum of $\mathcal{S}$ implying $\{0\}$ is a stratum of $\mathcal{S}^{uni}$ and consequently contradiction $B(\mathcal{S}^{uni})_0 = (T_0(K^5))^{dual} \neq G_0 = B(\mathcal{S}^{uni})_0$ follows thus proving

Proposition 10.5 *There is no universal TWG-stratification of* $\mathrm{Sing}(\tilde{F})$ *, where polynomial* $\tilde{F} = u \cdot x^2 + 2w^2 \cdot x \cdot y + v \cdot y^2$.

Remark 10.6 *Formulae of (3) provide a family* $\mathcal{F}$ *of strata of a TWG-stratification of subset* W_1 *of* $\mathrm{Sing}(F)$ *. The latter stratification fails frontier condition and set* W_1 *coincides with* $\mathrm{Sing}(F)$ *when* $F = \tilde{F} = u \cdot x^2 + 2w^2 \cdot x \cdot y + v \cdot y^2$ *. Indeed, with curve* $\{x = y = 0\ ,\ u = t^3\ ,\ v = t^2\ ,\ w = t\}$ *as set* $\mathcal{G}^+$ *of formula (3) it follows that family*

$$\mathcal{F} = \{\mathcal{G}_2 \setminus \mathcal{G}^+\ ;\ \mathcal{G}_3 \setminus \mathcal{G}^+\ ;\ \mathcal{G}^+\}$$

is a family of strata of a TWG-stratification of $\mathrm{Sing}(\tilde{F})$ *. Of course this (naturally induced by a not Lagrangian Glaeser bundle* $G_{\tilde{F}}$ *) TWG-stratification of* $\mathrm{Sing}(\tilde{F})$ *fails the frontier condition since* $\emptyset \neq \{(0,0,0,0,0)\ ,\ (0,0,1,1,1)\} = \overline{\mathcal{G}_3 \setminus \mathcal{G}^+} \cap \mathcal{G}^+ \neq \mathcal{G}^+$ *.*

10.4 A universal TWG-stratification and the 'multiplicities of roots'.

Let

$$f := \hat{f}_{q+2} = \sum_{0 \le i \le q} A_i X^i Y^{q-i} \in K[A_0, \ldots, A_q, X, Y]\ ,$$

where $([A_0 : \cdots : A_q], X, Y) \in \mathbb{P}^q(K) \times K^2$. In this example we consider affine charts $\{A_i \neq 0\} \simeq K^q \times K^2$, $0 \le i \le q$, of $\mathbb{P}^q(K) \times K^2$ and setting $A_i = 1$ the corresponding mappings $F := \hat{f}_{q+2} : K^{q+2} \to K$. Then, similarly to the preceding examples, $\mathrm{Sing}(F) = \{X = Y = 0\}$, admits Thom stratification (due to $l = 1$, see Remark 2.2) and, assuming that all irreducible components of the quasistrata $\mathcal{G}_k$, $n - \dim(\mathrm{Sing}(F)) \le k \le n$ associated with the Glaeser bundle $G := G_F$ of F are of dimension $n - k$ (which we will show below) it follows that also (ii) of Theorem 4.8 applies.

Following the original notations of Section 3 let $G^{(p)} := G^{(p)}_{\hat{f}_n}$. We prove here that the index of stabilization $\rho(\hat{f}_n) = 2$, i. e. that $G^{(1)} \neq G^{(2)} = G$, bundle $G = G_{\hat{f}_n}$ is Lagrangian and that $\{\mathcal{G}_{k+2}\}_{0 \le k \le q/2}$ is a universal TWG-stratification with respect to $\hat{f}_{q+2}$.

Let us fix a point $a^{(0)} = ([a_0^{(0)} : \cdots : a_q^{(0)}], 0, 0) \in \mathrm{Sing}(F)$, for the time being, then polynomial

$$f^{(0)} := \sum_{0 \le i \le q} a_i^{(0)} X^i Y^{q-i} = \prod_j (b_j X - c_j Y)^{m_j}. \tag{9}$$

Plan. First we will calculate $G^{(1)}$. It turns out (Lemma 10.7) that the dimension of the fiber $G^{(1)}_{a^{(0)}}$ coincides with two plus the number of the roots of polynomial $f^{(0)}$ counted with multiplicities $m_j \ge 2$, where 'two' is on the account of $\{dX\ ,\ dY\} \subset G^{(1)}_{a^{(0)}}$. (Of course any fiber of bundle $G^{(1)}$ contains $\{dX\ ,\ dY\}$.) Following Lemma 10.7 we then prove (Proposition 9.8) that bundle $G^{(2)}$ is closed (i. e. $G^{(2)} = \overline{G^{(1)}}$) and, therefore, that the Glaeser bundle $G = G_F$ of $F := \hat{f}_n$ of our example coincides with $G^{(2)}$. (In the process we show that dimension of the fiber of bundle $G^{(2)}$ at a point $a^{(0)} \in \mathrm{Sing}(F)$ is $\sum_j [m_j/2] + 2$, where numbers m_j are the multiplicities of the roots of polynomial $f^{(0)}$ corresponding to $a^{(0)}$ and $[m_j/2]$ denotes the integral part of $m_j/2$.) Finally (following Proposition 9.8), we verify that bundle G is Lagrangian. It turns out that generic points of every quasistrata $\mathcal{G}_{k+2}$ of the universal TWG-stratification corresponding to bundle G are the points, say $a^{(0)}$, such that the respective polynomial $f^{(0)}$ has k double roots and $q - 2 \cdot k$ single roots.

Calculation of $G^{(1)}$ **.** We first verify that for each factor $b_jX - c_jY$ of multiplicity $m_j \geq 2$ the fiber of the closure $(\overline{G^{(0)}})_{a^{(0)}}$ contains

$$v_j := v([c_j : b_j]) = \sum_{0 \leq i \leq q} c_j^i b_j^{q-i} dA_i \ .$$

Consider a line defined (parametrically) as follows:

$$A_i(t) = a_i^{(0)} \ , \ 0 \leq i \leq q \ ; \ X(t) = c_j t \ , \ Y(t) = b_j t \ .$$

Then $\lim_{t \to 0} df/||df||$ along this line equals v_j . Conversely, let $v = \sum_{0 \leq i \leq q} h_i dA_i + cdX + bdY$ with a non-vanishing $(h_0, \dots, h_q) \neq 0$ being the $\lim_{t \to 0} df/||df||$ along a curve

$$(\{A_i(t)\}_{0 \leq i \leq q}, X(t), Y(t)) \subset \mathbb{P}^q(K) \times K^2$$

with the origin at $a^{(0)}$. Making a suitable K-linear homogeneous transformation C of the 2-dimensional plane and applying Corollary 3.2 we may assume w.l.o.g. that $ord_t(X(t)) > ord_t(Y(t))$ and it suffices to show that $X^2 | f^{(0)}$. Assume otherwise, then

$$\mathrm{ord}_t \left\{ \frac{\partial f^{(0)}}{\partial X}, \frac{\partial f^{(0)}}{\partial Y} \right\} = (q-1) \, \mathrm{ord}_t(Y(t)) < \mathrm{ord}_t(X^i Y^{q-i}) \ , \ 0 \leq i \leq q \ ,$$

which contradicts to $(h_0 \ , \ \dots \ , \ h_q) \neq 0$.

Vectors $\{v_j\}_j$ are linearly independent (since they form a van-der-Mond matrix) implying

Lemma 10.7 *For any point* $a^{(0)} \in \mathrm{Sing}(F)$ *fiber* $(G^{(1)})_{a^{(0)}}$ *of bundle* $G^{(1)}$ *coincides with the linear hull of vectors* dX , dY *and the* $\{v_j\}_j$ *for the* j*'s with the multiplicity* m_j *of the factor* $b_jX - c_jY$ *in* $f^{(0)}$ *being* ≥ 2 *. Moreover,* $\dim((G^{(1)})_{a^{(0)}}) - 2$ *is the number of such* j *.*

Calculation of $G^{(2)}$ **.** For every $v = v([c : b])$ let $\mathcal{D}^{(l)}(v)$ denote the linear hull of

$$\left\{ \frac{\partial^l v}{\partial c^i \partial b^{l-i}} \right\}_{0 \leq i \leq l} \ .$$

Then $\{v\} = \mathcal{D}^{(0)}(v) \subset \mathcal{D}^{(1)}(v) \subset \cdots$ due to the Euler's formula. W.l.o.g. we may assume that $b = 1$ (for $b = 0$ we would exchange the roles of b and c) and then $\mathcal{D}^{(l)}(v)$ is the linear hull of the derivatives $\{\frac{\partial^i v}{\partial c^i}\}_{0 \leq i \leq l}$, implying that $\dim(\mathcal{D}^{(l)}(v)) = l + 1 \ , 0 \leq l \leq q$.

Below we calculate the limit $\lim_{t \to 0} (G^{(1)})_{a^{(t)}}$. To that end we consider a curve $\{a^{(t)}\}_t \subset \mathrm{Sing}(F)$ with the origin at $a^{(0)}$, and assume w.l.o.g. that $a_q^{(t)} = 1$ for all t . Due to Lemma 10.7 we may assume (also w.l.o.g.) that for any $t \neq 0$ the multiplicity of every factor of polynomial $f^{(t)} = \sum_{0 \leq i \leq q} a_i^{(t)} X^i Y^{q-i}$ does not exceed 2 and these multiplicities are independent on $t \neq 0$. We may factorise

$$f^{(t)} = \prod_j \prod_p (X - (c_j + e_{j,p}(t))Y)^{m_{j,p}} \ ,$$

where $1 \leq m_{j,p} \leq 2$ and $e_{j,p}(t)$ are the appropriate algebraic functions of t with $e_{j,p}(0) = 0$ for all j , p . Then $\sum_p m_{j,p} = m_j$ for each j with m_j from (9). Let

$\overline{m_j} = \sum_p [m_{j,p}/2]$, where $[m_{j,p}/2]$ is the integral part of $m_{j,p}/2$. Due to Lemma 10.7 it follows that $\dim((G^{(1)})_{a^{(t)}}) = \sum_j \overline{m_j} + 2$ for any $t \neq 0$ and that collection

$$\{v([c_j + e_{j,p}(t) : 1])\}_{m_{j,p}=2} \cup \{dX,\ dY\} \tag{10}$$

is a basis of the fiber $(G^{(1)})_{a^{(t)}}$.

We claim that

$$\lim_{t \to 0} (G^{(1)})_{a^{(t)}} = \bigoplus_j \mathcal{D}^{(\overline{m_j}-1)}(v([c_j : 1])) \oplus \mathrm{Span}\{dX,\ dY\} \ . \tag{11}$$

To that end we observe that the right-hand side of (11) is indeed the direct sum of the vector spaces due to the Hermite's interpolation (which interpolates uniquely a polynomial in terms of the values of its several consecutive derivatives at the given points, cf. Appendix). Therefore the dimension of the right-hand side equals $\sum_j \overline{m_j} + 2$ and to complete the proof of (11) it suffices to verify that the left-hand side of (11) contains its right-hand side.

To this end fix j , denote $m := \overline{m_j}$ and let

$$E^{(i)} := (\{e^i_{j,p}(t)\}_{1 \le p \le m})^T \in K^m \ , \ i \ge 0 \ ,$$

where all p satisfy $m_{j,p} = 2$ (see (10)). Let E be the van-der-Mond matrix of size $m \times m$ with columns $E^{(i)}$, $0 \le i \le m-1$. For an arbitrary choice of $w = (w_0 \ , \ \dots \ , \ w_{m-1}) \in K^m$ let $u := (\{u_p\}_{1 \le p \le m}) := w \cdot E^{-1}$. Since $E^{-1}E^{(i)}(0) = 0$ for every $i \ge m$ it follows for $u^{(i)}(t) := u \cdot E^{(i)}(t)$ that $u^{(i)}(0) = 0$. Therefore

$$\sum_{1 \le p \le m} u_p v([c_j + e_{j,p}(t) : 1]) = \sum_{0 \le s \le m-1} \frac{w_s}{s!} \frac{d^s v([c_j : 1])}{dc^s} + \sum_{m \le i \le q} \frac{u^{(i)}}{i!} \frac{d^i v([c_j : 1])}{dc^i} \ .$$

Claim (11) then follows by letting $t = 0$ in the right-hand side of the latter (in view of the 'arbitrary' choice of w in K^m).

We now specify the choice of curve $\{a^{(t)}\}_{t \in K}$ as such that $\overline{m_j} = [m_j/2]$ (with m_j and $\overline{m_j}$ as above holds for every j , in other words $m_{j,p} = 2$ for $\overline{m_j}$ of p's and, moreover, in the case when number m_j is odd that $m_{j,p_0} = 1$ for a single p_0 . Then due to (11) it follows

Proposition 10.8 *For any point* $a^{(0)} \in \mathrm{Sing}(F)$ *fiber*

$$(\overline{G^{(1)}})_{a^{(0)}} = \bigoplus_j \mathcal{D}^{([m_j/2]-1)}(v([c_j : 1])) \oplus \mathrm{Span}\{dX \ , \ dY\}$$

of $\overline{G^{(1)}}$ *at* $a^{(0)}$ *is a vector space of dimension* $\sum_j [m_j/2] + 2$ *(with* m_j *from (9)). In particular, bundle* $G := G_F = \overline{G^{(1)}}$.

Proof that G is Lagrangian. For every k , $0 \le k \le q/2$, let

$$\mathcal{G}^{(0)}_{k+2} := \{a^{(0)} \in \mathrm{Sing}(F) : f^{(0)} = \prod_{1 \le j \le k} (X - c_j Y)^2 \cdot \prod_{k < s \le q-k} (X - c_s Y)\} \ ,$$

i. e. $f^{(0)}$ has k factors of multiplicity 2 and $q - 2k$ factors of multiplicity 1 . Proposition 10.8 implies that $\mathcal{G}^{(0)}_{k+2} \subset \mathcal{G}_{k+2}$ (see Definition 4.4) and, moreover, that $\mathcal{G}^{(0)}_{k+2}$ is dense in $\mathcal{G}_{k+2}$. On the other hand, $\mathcal{G}^{(0)}_{k+2}$ is open and is isomorphic to the set of all orbits of the group $\mathrm{Sym}(k)\times\mathrm{Sym}(q-2k)$ acting on a set

$$\mathcal{Z} := K^{q-k}\setminus(\bigcup_{1\le i<j\le q-k}\{Z_i=Z_j\})\ ,$$

where $\mathrm{Sym}(k)$ permutes the first k coordinates Z_1 , ... , Z_k and $\mathrm{Sym}(q-2k)$ permutes the last $q-2k$ coordinates Z_{k+1} , ... , Z_{q-k} . It follows $\dim(\mathcal{G}^{(0)}_{k+2})=q-k$. Moreover, $\mathcal{G}^{(0)}_{k+2}=H(\mathcal{Z})$, where H maps Z_1 , ... , Z_k to double roots of $f^{(0)}$ and Z_{k+1} , ... , Z_{q-k} to single roots. It follows that $\mathcal{G}^{(0)}_{k+2}$ is irreducible. Finally, since in this example $\mathrm{Sing}(F)$ admits Thom stratification, quasistrata $\mathcal{G}_{k+2}$ are irreducible and of dimension $n-k-2$ item (ii) of Theorem 4.8 and hence Corollary 4.11 apply and imply the following

Theorem 10.9 *Index of stabilization* $\rho(\hat{f}_{q+2})=2$ *, bundle* $G=G_{\hat{f}_{q+2}}$ *is Lagrangian and* $\{\mathcal{G}_{k+2}\}_{0\le k\le q/2}$ *is a universal TWG-stratification of* $\mathrm{Sing}(F)$ *with respect to* $F:=\hat{f}_{q+2}$ *.*

11 Appendix. Complexity of the construction of a Gauss regular extension with a prescribed tangent bundle over the singularities.

Content. Here we estimate the complexity of the algorithm described in Sections 6, 7 and 8 of extending of a (smooth) singular locus of an algebraic variety to a Gauss regular subvariety with a prescribed tangent bundle over the singularities of the variety. Together with Section 9 it would complete the proofs of the effectiveness of all of the constructions of this work and, moreover, results in a double exponensial upper bound on their computational complexities.

We follow the notations of Sections 5, 6, 7, 8 with an exception that we use K rather than $\mathbb{C}$. The input for this algorithm is a family of polynomials g_p , $M_{j+m,i+m}\in K_0[X_1,\dots,X_n]$ with $p\ge 0$, i , j for a subfield $K_0\subset K$. To establish complexity bounds we assume that elements of K_0 can be represented algorithmically, e. g. one may use here the field of rational or algebraic numbers in place of K_0 , cf. [12]. We also assume the following representation of constructive set $S=\{g_0\cdot g_1\ne 0\ ,\ g_p=0\}_{p\ge 2}$ and of its (smooth) singular locus $\mathcal{G}=\{g_0\ne 0,\ g_p=0\}_{p\ge 1}$, which also assume to be its boundary in open in K^n set $\{g_0\ne 0\}$ (as in Remark 5.8). The output of the algorithm is a Gauss regular subvariety $\mathcal{G}^+$ of $\overline{S}\cap\{g_0\ne 0\}$ (as in Proposition 5.9).

Basically the algorithm consists of 3 subroutines. The first one is choosing a Noether normalisation π for $\mathcal{G}$. The second one is an *implicit parametric interpolation* of polynomials L_j from Section 6. (We refer to the latter as implicit because the interpolation data are given over the subsets of points from $\mathcal{G}$ and thus the data appear implicitly.) The third subroutine is a construction of $\mathcal{G}^+$ proper. To this end we may exploit a choice of algebraically independent coefficients $c_1,\dots,c_k$ at each consecutive application of Theorem 7.2 and thereafter to construct an irreducible component containing $\mathcal{G}$ of the resulting intersection with $\overline{S}\cap\{g_0\ne 0\}$ (cf. vi) of Theorem 7.2 and the deduction of Proposition 5.9). Complexity bounds for Noether normalisation and for constructing irreducible components one may find in [21], and in [12] respectively. We observe that the third subroutine depends only on the complexity of finding irreducible components. We therefore focus on an algorithm for a parametric interpolation. In fact, we design an algorithm for interpolation over the parameters varying in K^m , whereas for the purposes of Section 6 it suffices to have the parameters varying in an open subset $\mathcal{U}' \subset K^m$, which would have simplified the algorithm.

To formulate the complexity bounds we assume that $\deg(g_p) < \delta$, $\deg(M_{j+m,i+m}) < \Delta$ for all p , i , j and the total number of bits in representation of the coefficients (in K_0) of polynomials g_p , $M_{j+m,i+m}$ does not exceed R . Our main result here is the following

Proposition 11.1 *One can interpolate polynomials L_j as required in Section 6 and, moreover, under assumptions listed in the preceding paragraph $\deg(L_j) < \Delta\delta^{O(n)}$ is a bound on the degrees of the resulting L_j . Complexity bound for this interpolation algorithm is $(R\Delta^n\delta^{n^2})^{O(1)}$.*

Combining with the complexity bounds for the first and the third subroutines it follows

Corollary 11.2 *The complexity of the algorithm constructing $\mathcal{G}^+$ is bounded by*

$$R^{O(1)}(\Delta\delta)^{n^{O(1)}}.$$

Proof of Proposition 11.1. We first consider a *non-parametrical* interpolation.

Lemma 11.3 *Let $v_1, \dots, v_t \in K^{n-m}$ and $w_q^{(i)} \in K$, $1 \le q \le t$, $0 \le i \le n-m$. There exists a polynomial $A \in K[X_{m+1}, \dots, X_n]$ of $\deg(A) < 2t(n-m)$ such that*

$$A(v_q) = w_q^{(0)} \ , \ \frac{\partial A}{\partial X_{i+m}}(v_q) = w_q^{(i)} \ , \ 1 \le q \le t \ , \ 1 \le i \le n-m \ .$$

Proof. By making an appropriate linear change of the coordinates in K^m we may assume w.l.o.g. that $v_{q_1}^{(i)} \ne v_{q_2}^{(i)}$, $1 \le q_1 < q_2 \le t$, $1 \le i \le n-m$, where $v_q = (v_q^{(1)} \ , \ \dots, v_q^{(n-m)})$, $1 \le q \le t$. Consider a polynomial

$$A_{q_0} = \prod_{q \ne q_0, 1 \le i \le n-m} (X_{i+m} - v_q^{(i)})^2 \cdot \left(\sum_{1 \le i \le n-m} a_i (X_{i+m} - v_{q_0}^{(i)}) + a_0 \right) \ , \ 1 \le q_0 \le t$$

with indeterminate coefficients a_i , $0 \le i \le n-m$. Then $A_{q_0}(v_q) = \frac{\partial A_{q_0}}{\partial X_{i+m}}(v_q) = 0$, $1 \le i \le n-m$, for every $q \ne q_0$. Equation $A_{q_0}(v_{q_0}) = w_{q_0}^{(0)}$ uniquely determines a_0 and, moreover, equation $\frac{\partial A_{q_0}}{\partial X_{i+m}}(v_{q_0}) = w_{q_0}^{(i)}$ uniquely determines a_i , $1 \le i \le n-m$. Finally, we let $A := \sum_{1 \le q \le t} A_q$. ∎

Of course one can in the same vain interpolate the higher derivatives as well.

We now consider a *parametric* interpolation. In view of Bézout inequality $\deg(\overline{\mathcal{G}}) < \delta^n$ we introduce polynomial

$$\mathcal{A} = \sum_{0 \le e_1 + \dots + e_{n-m} \le 2(n-m)\delta^n} A_E X_{m+1}^{e_1} \cdots X_n^{e_{n-m}}$$

with indeterminate coefficients $a := \{A_E\}_E$, $E = (e_1, \dots, e_{n-m})$ and a quantifier-free formula $\Phi(u, v, a)$ of the theory of algebraically closed fields which says that

$$\text{if } v \in \mathcal{G} \ , \ \pi(v) = u \in K^m \text{ then } \mathcal{A}(v) = 0 \ , \ \frac{\partial \mathcal{A}}{\partial X_{i+m}}(v) = M_{j+m,i+m}(v) \ , \ 1 \le i \le n-m$$

for some j , $1 \le j \le k$ (we fix index j for the time being). Then the formula $\forall u\ \exists a\ \forall v\ \Phi$ is valid due to Lemma 11.3.

An algorithm from [14] commonly referred to as a "shape lemma" yields a representation of $\pi^{-1}(u) \cap \mathcal{G}$. Applied to a system $\{g_p = 0, g_0 \ne 0\}_{p>0}$ the output of this algorithm is a partition of $K^m = \cup_\beta U_\beta$ into constructible subsets such that for each β there are a linear combination $\alpha = \sum_{1\le i\le n-m} \alpha_{i,\beta} v^{(i)}$ of coordinates $v^{(i)}$, $1 \le i \le n-m$, with integer coefficients $\alpha_{i,\beta}$ and rational functions ϕ , $\phi_i \in K_0(X_1, \dots, X_m)[Y]$, $1 \le i \le n-m$, for which the following holds:

- for any $u \in U_\beta$ and any $v = (u, v^{(1)}, \dots, v^{(n-m)}) \in \pi^{-1}(u) \cap \mathcal{G}$ equalities $v^{(i_0)} = \phi_{i_0}(u, \alpha)$, $1 \le i_0 \le n-m$, take place, i. e. α is a primitive element of the field $K_0(u, v^{(1)}, \dots, v^{(n-m)})$ over $K_0(u)$;
- the roots of a univariate polynomial $\phi(u, Y)$ are exactly the values of α while ranging over points $v \in \pi^{-1}(u) \cap \mathcal{G}$.

Furthermore, in formula Φ we replace $v^{(i_0)}$, $1 \le i_0 \le n-m$, by $\phi_{i_0}(u, \alpha)$ and divide the resulting polynomials $\mathcal{A}(\alpha)$ and $\left(\frac{\partial \mathcal{A}}{\partial X_{i+m}}(\alpha) - M_{j+m,i+m}(\alpha)\right)$ by polynomial $\phi(u, \alpha)$ (with the remainders as polynomials in α). Then system Φ_1 obtained by equating to zero all coefficients of the remainders at the powers of α is equivalent to formula $\forall v\ \Phi$, for any $u \in U_\beta$.

One may consider Φ_1 as a linear system with respect to variables a and apply to Φ_1 an algorithm of *parametric Gaussian elimination* (see e. g. [14]). It yields a refinement $K^m = \cup_{\beta'} U'_{\beta'}$ of partition $\cup_\beta U_\beta$ into constructible subsets such that for each β' and for every multiindex E there is rational function $a_E \in K_0(X_1, \dots, X_m)$ such that for any $u \in U'_{\beta'}$ the array of coefficients $a(u) = \{a_E(u)\}_E$ fulfils Φ_1 . For a choice of the unique β' for which $U'_{\beta'}$ is dense in K^m the rational function

$$L_j = \sum_{0 \le e_1 + \cdots + e_{n-m} \le 2(n-m)\delta^n} a_E X_{m+1}^{e_1} \cdots X_n^{e_{n-m}}$$

that corresponds to this β' is as required in Section 6.

Finally we address the complexity issue. In the "shape lemma" construction applied above $\deg(\phi)$, $\deg(\phi_i)$ are bounded by $\delta^{O(n)}$ and by the degrees of the polynomials representing $\{U_\beta\}_\beta$, while the number of $\{U_\beta\}$'s, the total sum of sizes of the coefficients of these polynomials and the complexity of the algorithm do not exceed $R^{O(1)}\delta^{O(n^2)}$ [14]. Therefore the degrees of the polynomials occuring in Φ_1 are bounded by $\Delta\delta^{O(n)}$, while the number of the polynomials, the total sum of sizes of their coefficients and the complexity of constructing Φ_1 do not exceed $(R\Delta^n\delta^{n^2})^{O(1)}$. At the stage of applying the parametric Gaussian elimination to Φ_1 the bounds are similar. Proposition is proved. ∎

Acknowledgements. The authors are thankful to the Max-Planck Institut für Mathematik, Bonn for its hospitality during writing the paper and to the referees for comments helping to improve the exposition.

References

[1] E. Bierstone, P. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math., **128** (1997), 207–302.

[2] E. Bierstone, P. Milman, *Standard Basis along a Samuel stratum, and implicit differentiation*, Fields Inst. Commun., **24**, Amer. Math. Soc., Providence, RI, (1999), 81–113.

[3] E. Bierstone, P. Milman, *Desingularization algorithms. I. Role of exceptional divisors*, Mosc. Math. J., **3** (2003), 751–805.

[4] E. Bierstone, D. Grigoriev, P. Milman, Y. Wlodarczyk, *Effective Hironaka resolution and its complexity*, *Asian J. Math.* (Volume dedicated to Hironaka), Vol. **15**, No. **2** (2011), 193 – 228.

[5] E. Bierstone, P. Milman, W. Pawłucki, *Differential functions defined in closed sets. A problem of Whitney*, Invent. Math., **151** (2003), 329–352.

[6] A. Chistov, *Efficient smooth stratification of an algebraic variety of characteristic zero and its applications*, J. Math. Sci., **113** (2003), 689–717.

[7] A.Dimca, Singularities and topology of hypersurfaces, Springer, 1992.

[8] C. G. Gibson, K. Wirthmüller, A. A. du Plessis, E. J. N. Looijenga, *Topological stability of smooth mappings*, Lect. Notes Math., **552**, 1976.

[9] G. Glaeser, *Etude de quelques algèbres tayloriennes*, J. Analyse Math., **6** (1958), 1–124.

[10] M. Goresky, R. MacPherson, *Stratified Morse theory*, Springer, 1988.

[11] C. Grant, P. Milman, *Metrics for singular analytic spaces*, Pacific J. Math. **168**, (1995), 61–156.

[12] D. Grigoriev, *Computational complexity in polynomial algebra*, Proc. of International Congress of Mathematicians, Berkeley, vol. 2, (1986), 1452–1460.

[13] D. Grigoriev, P. Milman, *Construction of universal Thom-Whitney-a stratifications, their functoriality and Sard-type Theorem for singular varieties*, arXiv:0811.1373 (2008).

[14] D. Grigoriev, N. Vorobjov, *Bounds on numbers of vectors of multiplicities for polynomials which are easy to compute*, Proc. ACM Intern. Conf. Symbolic and Algebraic Computations, Scotland, (2000), 137–145.

[15] J. P. Henry, M. Merle, C. Sabbah, *Sur la condition de Thom stricte pour un morphisme analytique complexe*, Ann. sci. de l'Ecole Normale Supér. Sér. 4, **17** (1984), 227–268.

[16] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero I, II*, Ann. Math., **79** (1964), 109–326.

[17] H. Hironaka, *Subanalytic sets*, Number Theory, Algebraic Geometry and Commutative Algebra, Volume in honour of Y. Akizuki, pp. 453–493. Kinokuniya, Tokyo, 1973.

[18] H. Hironaka, *Stratification and flatness. Real and complex singularities*, Proc. Nordic Summer School, Oslo, publ. Sijhoff & Noordhoff, (1977), 199–265.

[19] V. Kaloshin, *Around the Hilbert-Arnold problem*, CRM Monogr. Ser., **24** (2005), 111–162.

[20] T. C. Kuo, *The ratio test for analytic Whitney stratifications*, Lecture Notes in Mathematics, **192**, (1969/70), 141–149.

[21] A. Logar, *A computational proof of the Noether normalization lemma*, Lect. Notes Comput. Sci., **357**, (1988), 259–273.

[22] T. Mostowski, E. Rannou, *Complexity of the computation of the canonical Whitney stratification of an algebraic set in* $\mathbb{C}^n$, Lect. Notes Comput. Sci., **539** (1991), 281–291.

[23] A. Parusinski, *On the bifurcation set of complex polynomial with isolated singularities at infinity*, Compositio Mathematica, **97**, (1995), 369-384.

[24] R. Thom, *Propriétés Différentielles Locales des Ensembles Analytiques*, Sémin. Bourbaki, **281** (1964/5).

[25] C. T. C. Wall, *Regular Stratifications*, Lect. Notes Math., **468** (1974), 332–344.

[26] H. Whitney, *Tangents to an analytic variety*, Ann. Math., **81** (1965), 496–549.