An Approximation Algorithm for the Number of Zeros of Arbitrary Polynomials over $GF[q]$

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Abstract

We design the first polynomial time (for an arbitrary and fixed field $GF[q]$) (ϵ, δ)-approximation algorithm for the number of zeros of arbitrary polynomial $f(x_1, \ldots, x_n)$ over $GF[q]$. It gives the first efficient method for estimating the number of zeros and nonzeros of multivariate polynomials over small finite fields other than $GF[2]$ (like $GF[3]$), the case important for various circuit approximation techniques (cf. [BS 90]).

The algorithm is based on the estimation of the number of zeros of an arbitrary polynomial $f(x_1, \ldots, x_n)$ over $GF[q]$ in the function on the number m of its terms. The bounding ratio number is proved to be $m^{(q-1)\log q}$ which is the main technical contribution of this paper and could be of independent algebraic interest.

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1 Introduction

Recently there has been a progress in design of efficient approximation algorithms for algebraic counting problems. The first polynomial time (ϵ, δ)-approximation algorithm for the number of zeros of a polynomial $f(x_1, \ldots, x_n)$ over the field $GF[2]$ has been designed by Karpinski and Luby ([KL 91a]) and this result was extended to arbitrary multilinear polynomials over $GF[q]$ by Karpinski and Lhotzky ([KL 91b]).

In this paper we construct the first (ϵ, δ)-approximation algorithm for the number of zeros of an arbitrary polynomial $f(x_1, \ldots, x_n)$ with m terms over an arbitrary (but fixed) finite field $GF[q]$ working in polynomial time in the size of the input, the ratio $m^{(q-1)\log q}$, and $\frac{1}{\epsilon}$, $\log(\frac{1}{\delta})$. (The corresponding (ϵ, δ)-approximation algorithm for the number of *nonzeros* of a polynomial can be constructed to work in time polynomial in the size of the input, the ratio $m^{\log q}$, and $\frac{1}{\epsilon}$, $\log(\frac{1}{\delta})$.)

2 Approximation Algorithm

We refer to [KLM 89], [KL 91a], [KL 91b] for the more detailed discussion of the abstract structure of the Monte-Carlo method for estimating cardinalities of finite sets.

Given $f \in GF[q][x_1, \cdots, x_n]$, $f = \sum_{i=1}^{m} t_i$, and $c \in GF[q]$. Denote

$$\#_c f = |\{(x_1, \ldots, x_n) \in GF[q]^n \mid f(x_1, \ldots, x_n) = c\}|\,.$$

Our (ϵ, δ)-approximation algorithm will have the following overall structure:

Monte Carlo Approximation Algorithm

Input $f \in GF[q][x_1, \cdots, x_n]$, $c \in GF[q]$, $\epsilon > 0$, $\delta > 0$, $(f \not\equiv 0)$

Output $\tilde{Y}$ (such that $\Pr[(1-\epsilon)\#_c f \le \tilde{Y} \le (1+\epsilon)\#_c f] \ge 1 - \delta$)

1. Construct a universe set U (the size $|U|$ of U must be efficiently computable.)
2. Choose randomly with the uniform probability distribution N members u_i from U, $u_i \in U$, $i = 1, 2, \ldots, N$.
3. Construct now from a polynomial f an indicator function $\tilde{f} : U \to \{0, 1\}$ such that $|\tilde{f}^{-1}(1)| = \#_c f$.

4. Compute the number $N = \frac{1}{\beta}\frac{4\log(2/\delta)}{\epsilon^2}$ for $\beta \geq |U|/\#_c f$.

5. Compute for all i, $1 \leq i \leq N$, the values $\tilde{f}(u_i)$ and set $Y_i \leftarrow |U|\tilde{f}(u_i)$.

6. Compute $\tilde{Y} \leftarrow \frac{\sum_{i=1}^{N} Y_i}{N}$.

7. OUTPUT: $\tilde{Y}$.

Correctness of the above algorithm is guaranteed by the following Theorem.

Theorem 1 (Zero-One Estimator Theorem [KLM 89])
Let $\mu = \frac{\#_c f}{|U|}$. Let $\epsilon \leq 2$. If $N \geq \frac{1}{\mu}\frac{4\log(2/\delta)}{\epsilon^2}$, then the above Monte Carlo Algorithm is an (ϵ, δ)-approximation algorithm for $\#_c f$.

We shall distinguish two (technically different) cases:

Case 1. Polynomial $f(x_1, \ldots, x_n)$ over $GF[q]$ is constant free and $c = 0$.

Case 2. Polynomial $f(x_1, \ldots, x_n)$ over $GF[q]$ is arbitrary and $c \neq 0$.

Let us denote $\bar{f} = (f - c)^{q-1} - 1 = \sum_i \bar{t}_i$.
The corresponding universes and indicator functions will be $U_1 = GF[q]^n$, $\tilde{f}_1(s) = 1$ if and only if $f(s) = 1$, and $U_2 = \{(s,i) \mid \bar{t}_i(s) \neq 0\}$, $\tilde{f}_2(s,i) = 1$ if and only if $f(s) = c$ and for no $j < i$, $(s,j) \in U_2$.
Let us observe that $\frac{|U_2|}{\#_c f} \leq m^{q-1} \cdot \frac{|\tilde{G}_{(f-c)^{q-1}-1}|}{\#_c f}$ for $\tilde{G}_{(f-c)^{q-1}-1} = \{(s,i) \mid \bar{t}_i(s) \neq 0$, there is <u>no</u> j, $j < i$ such that $\bar{t}_j(s) \neq 0\}$, see figure 1. (Observe that $|\tilde{G}_{(f-c)^{q-1}-1}| = |\{s \mid$ there is a term $\bar{t}_i$ of $(f-c)^{q-1} - 1$ such that $\bar{t}_i(s) \neq 0\}|$.)

The corresponding bounds $\beta_i \geq \frac{|U_i|}{\#_c f}$ will be proven to satisfy

$$\begin{aligned} \beta_1 &\leq (m+1)^{(q-1)\log q} \qquad \text{and} \\ \beta_2 &\leq m^{q-1}(m+1)^{(q-1)\log q} \; . \end{aligned}$$

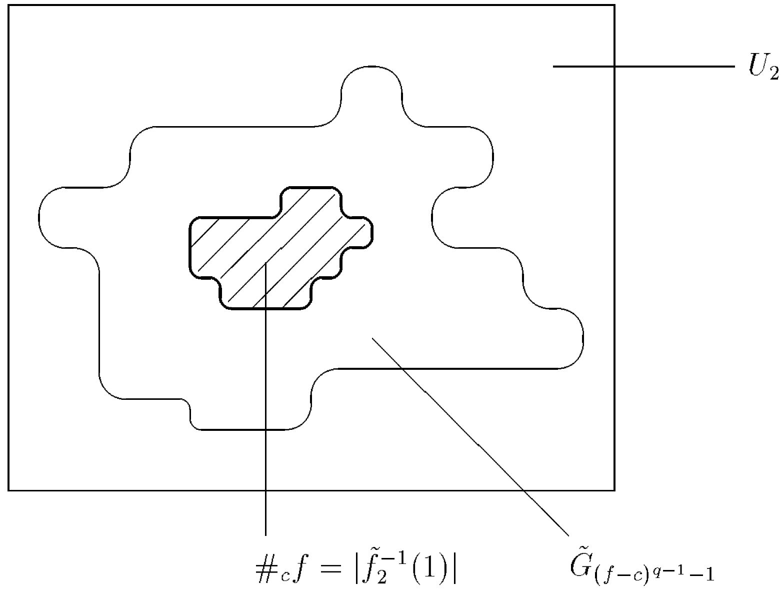

Figure 1

The rest of the paper will be devoted to the proofs of these two bounds.

We shall denote the corresponding algorithms by A_1 and A_2.
Let us analyze the bit complexity of both algorithms (for the corresponding subroutines see [KL 91a], [KL 91b], and [KLM 89]).

Denote by $P(q)$ the bit costs of multiplication and powering over $GF[q]$, $P(q) = O(\log^2 q \log\log q \log\log\log q)$ (cf. [We 87]). The evaluation of the polynomial takes time $O(nmP(q))$ and the overall complexity of the algorithm A_1 is

$$O(nm(m+1)^{(q-1)\log q} P(q) \log(1/\delta)/\epsilon^2)$$

and of the algorithm A_2

$$O(nm(m+1)^{(q-1)(1+\log q)} q \log q P(q) \log(1/\delta)/\epsilon^2)\ .$$

For the fixed finite field $GF[q]$ the running time of both algorithms is bounded by a polynomial of the degree depending on the order of the ground field. The bounds for β_1 and β_2 which are proven polynomial in m only, are the main technical contribution of this paper.

Please note that the condition whether $f = 0$ is *satisfiable* can be checked deterministically for arbitrary polynomial $f \in GF[q][x_1, \ldots, x_n]$ within the bounds stated above because of the following (for a problem of a *black-box* interpolation of f, see [GKS 90]):

Proposition 1. Let $f \in GF[q][x_1, \cdots, x_n]$ and $c \in GF[q]$, the equation $f = c$ is satisfiable if and only if $g = (f-c)^{q-1} - 1$ has at least one nonconstant term.

Proof. $f = c$ is satisfiable iff $(f-c)^{q-1} = 0$ is satisfiable iff the inequality $(f-c)^{q-1} - 1 \neq 0$ is satisfiable. The inequality $(f-c)^{q-1} - 1 \neq 0$ is satisfiable iff there exists in $(f-c)^{q-1} - 1$ at least one nonconstant term. $\square$

3 Main Theorem

Given an arbitrary polynomial $f \in GF[q][X_1, \cdots, X_n]$, $\deg_{X_i} f \leq q-1$, denote $G = G_f = \{(x_1, \cdots, x_n) \mid f(x_1, \cdots, x_n) \neq 0\}$, $\bar{G} = \bar{G}_f = \{(x_1, \cdots, x_n) \mid \exists t_i \in f : t_i(x_1, \cdots, x_n) \neq 0\}$ (For notational reasons from now on in this section, variables will be written in capital (e.g. X_i) and values in small (e.g. x_i)).

Denote by $m = m_f$ the number of terms in f.

By the *support* of a term t we mean the set of indices of variables occurring in t.

Theorem 2 $\frac{|\bar{G}|}{|G|} \leq m^{\log_2 q}$

REMARK. This bound is sharp. Example: for $0 \leq k \leq n$

$$g_k = X_1^{q-1} \cdots X_k^{q-1}(1 - X_{k+1}^{q-1}) \cdots (1 - X_n^{q-1}) .$$

In this case $|\bar{G}| = (q-1)^k q^{n-k}, |G| = (q-1)^k, m = 2^{n-k}$.

Proof. For any subset $J \subset \{1, \cdots, n\}$ define an elementary cylinder $C(J) = \{(x_1, \cdots, x_n) \in GF[q]^n \mid x_j \neq 0 \text{ for } j \in J \text{ and } x_i = 0 \text{ for } i \notin J\}$. Observe that for $J_1 \neq J_2 \quad C(J_1) \cap C(J_2) = \emptyset$. Define the *cone* of J

$$CON(J) = \{(x_1, \cdots, x_n) \in GF[q]^n \mid x_j \neq 0 \text{ for } j \in J\} = \bigcup_{J_1 \supseteq J} C(J_1) .$$

By $f_J \in GF[q][\{X_j\}_{j \in J}]$ we denote the polynomial obtained from f in the following way: mutiply f by the term $X_J = \prod_{j \in J} X_j$, replace each appeared power X_j^q by X_j, make necessary cancellation, denote this intermediate result by $f \cdot X_J$ and finally, substitute zeroes instead of X_i for all $i \notin J$. Remark that each for term of f_J its support coincides with J, moreover $m_{f_J} \leq m_{f \cdot X_J} \leq m_f$.

Lemma 1 *For every* $J \subseteq \{1, \cdots, n\}$
a) $G \cap C(J) = G_{f_J}$ *(here under equality we mean a canonical isomorphism);*
b) $G \cap CON(J) = G_{f \cdot X_J}$.

Proof. Observe that for any point $(x_1, \cdots, x_n) \in C(J)$ (respectively $CON(J)$) $f(x_1, \cdots, x_n) \neq 0$ iff $f_J(\{x_j\}_{j\in J}) \neq 0$ (respectively $fX_J(x_1, \cdots, x_n) \neq 0$), this proves lemma 1.

Lemma 2 *a)* $G \cap C(J) \neq \emptyset$ *iff* $f_J \not\equiv 0$*;*
b) $G \cap CON(J) \neq \emptyset$ *iff* $f \cdot X_J \not\equiv 0$*;*
c) if $f_J \not\equiv 0$ *then* $\bar{G} \supseteq C(J) = \bar{G}_{f_J}$ *and* $\bar{G} \supseteq CON(J) = \bar{G}_{f \cdot X_j}$.

Proof. a) (respectively b)) follows from lemma 1a) (respectively 1b)).
c) follows from the statement that if $f_J \not\equiv 0$ then f contains a term with a support being a subset of J.

We call J *active* if $f_J \not\equiv 0$.

Lemma 3 *Assume* J *is active. Then* $\frac{|\bar{G}_{f_J}|}{|G_{f_J}|} = \frac{|C(J)|}{|G \cap C(J)|} \leq m_{f_J}^{\log_2 q - 1} (\leq m_{f_J}^{\log_2 q})$.

NOTE. This lemma states the theorem for the case of the polynomial f_J.

Proof. We conduct by induction on $|J|$. Remark that $|\bar{G}_{f_J}| = |C(J)| = (q-1)^{|J|}$. Assume that for a certain $j_0 \in J$ the polynomial f_J does not divide by $(X_{j_0} - \alpha)$ for each $\alpha \in GF[q]^*$. Then $f_{J,\alpha} = f_J(X_{j_0} = \alpha) \not\equiv 0$. Then by lemma 2a) we can apply inductive hypothesis to each of these polynomials $f_{J,\alpha}$. Since $|G_{f_J}| = \sum_{\alpha \in GF[q]^*} |G_{f_{J,\alpha}}|$ and $m_{f_{J,\alpha}} \leq m_{f_J}$, we get by induction the statement of the lemma in this case.

Assume now that $\prod_{j\in J} (X_j - \alpha_j) | f_J$ for some $\alpha_j \in GF[q]^*, \ j \in J$. We claim in this case that $m_{f_J} \geq 2^{|J|}$. By lemma 1a) this would prove lemma 3. We prove the claim by induction on $|J|$.
Fix some $j_0 \in J$ and write (uniquely) $f_J = \sum h_{J_1}(X_{j_0}) M_{J_1}$ where M_{J_1} are terms in the variables $\{X_j\}_{j \in J\setminus\{j_0\}}$ and $h_{J_1}(X_{j_0}) \in GF[q][X_{j_0}]$. Then $(X_{j_0} - \alpha_{j_0}) | h_{J_1}(X_{j_0})$ for each M_{J_1}, hence $h_{J_1}(X_{j_0})$ contains at least two terms.
Take a certain $x_{j_0} \in GF[q]^*$ such that $0 \not\equiv f_J(X_{j_0} = x_{j_0}) \in GF[q][\{X_j\}_{j\in J\setminus\{j_0\}}]$ and apply inductive hypothesis of the claim to $f_J(X_{j_0} = x_{j_0})$, taking into account that $m_{f_J} \geq 2 m_{f_J(X_{j_0} = x_{j_0})}$. Lemma 3 is proved.

Lemma 4 *If* $J \subseteq \{1, \cdots, n\}$ *is a minimal (w.r.t. inclusion relation) support of the terms in* f *then* J *is active.*

Proof. Represent (uniquely) $f = f_1 + f_2$ where f_1 is the sum of all terms occurring in f with the support J. Then the polynomial $f_J = X_J f_1 \not\equiv 0$ has the same number of terms as f_1, this proves lemma 4.

Corollary 1 *$\bar{G}$ coincides with the union of the cones $CON(J)$ for all (minimal) active J.*

Now we consider the lattice $\mathcal{L} = 2^{\{1,\cdots,n\}}$ and for $J \in \mathcal{L}$ we denote its cone $con(J) \subseteq \mathcal{L}$, $cone(J) = \{J' | J \subseteq J'\}$. We'll construct a partition $\mathcal{P}$ of the union $\mathcal{G}$ of $con(J)$ for all active J.
Take any linear ordering $\prec$ of the active elements with the only property that if $J_1 \subsetneqq J_2$ for two active elements then $J_1 \succ J_2$ (e.g. as the first element one can take arbitrary maximal one, then a maximal in the rest set etc.).
Associate with any element $J_1 \in \mathcal{G}$ an active element J minimal w.r.t. ordering $\prec$ with the property $J \subseteq J_1$. Then as an element of the partition $\mathcal{P}$ which is attached to an active element J (denote it by $\mathcal{P}(J)$) consists of all such elements of $\mathcal{G}$ which are associated with J.

For any J_1 call a subset $S \subset con(J_1)$ a relative principal ideal with the generator J_1 if for any $J_2 \supseteq J_3 \supseteq J_1$ and $J_2 \in S$ we have $J_3 \in S$.

Lemma 5 *a) $\mathcal{P}$ is a partition of $\mathcal{G}$;*
b) For each active element J, $\mathcal{P}(J)$ is a relative principal ideal with the generator J (with the unique active element J).

Proof. Part a) is clear. To prove part b) consider $J_1 \in \mathcal{P}(J)$ and $J_1 \supseteq J_2 \supseteq J$, then $J_2 \in \mathcal{G}$ (since $\mathcal{G}$ is a union of the cones). We have to prove that J corresponds to J_2.
Assume the contrary and let $J_0 \subseteq J_2$ for some active J_0 such that $J_0 \prec J$, hence $J_0 \subseteq J_1$ and we get a contradiction with $J_1 \in \mathcal{P}(J)$ which proves lemma 5.

Lemma 6 *For any active element J and each $J_1 \in \mathcal{P}(J)$ the sum M_{J_1} of the terms occurring in fX_J with the support J_1 equals to*

$$f_J(\frac{X_{J_1}}{X_J})^{q-1}(-1)^{|J_1 \setminus J|} .$$

Proof. We prove it by induction on $|J_1 \setminus J|$.
The base for $J_1 = J$ is clear. Take any $J_1 \in \mathcal{P}(J)$, then for each $J_1 \supsetneqq J_2 \supseteq J$ we

have $J_2 \in \mathcal{P}(J)$ by lemma 5 and by inductive hypothesis $M_{J_2} = f_J(\frac{X_{J_2}}{X_J})^{q-1}(-1)^{|J_2 \setminus J|}$. Since J_1 is not active we have $f_{J_1} \equiv 0$. Observe that $f_{J_1} = (\sum_{J \subseteq J_2 \subseteq J_1} M_{J_2})\frac{X_{J_1}}{X_J}$. Therefore $f_{J_1} = \frac{X_{J_1}}{X_J}(-f_J(\frac{X_{J_1}}{X_J})^{q-1}(-1)^{|J_1 \setminus J|} + M_{J_1})$ and we obtain

$$M_{J_1} = f_J(\frac{X_{J_1}}{X_J})^{q-1}(-1)^{|J_1 \setminus J|}$$

taking into account that each term in f_J has a support equal to J.
Induction and lemma 6 are proved.

Corollary 2 *For any active element* J

$$m_f \geq m_{f \cdot X_J} \geq m_{f_J} \cdot |\mathcal{P}(J)| \, .$$

Lemma 7 *For any relative principal ideal* $S \subset con(J)$ *with the generator* J *the weight* K *of* S

$$K = \sum_{s \in S} (q-1)^{|s \setminus J|} \leq |S|^{\log_2 q} \, .$$

Proof. We prove by induction on $n - |J|$.
The base for $n = |J|$ (then $|S| = 1$) is obvious. For the inductive step take some index $i_0 \notin J$. Consider a partition of $S = S_0 \cup S_1$ where S_1 (respectively S_0) consists of all elements containing (respectively not containing) i_0. Then S_0 can be considered as a relatively principal ideal with the generator J in the lattice $2^{\{1,\cdots,n\} \setminus \{i_0\}}$. By S_1' denote a subset of $2^{\{1,\cdots,n\} \setminus \{i_0\}}$ obtained from S_1 by deleting i_0 from each element. Then S_1' is also a relative principal ideal (may be empty) with the generator J and $S_1' \subset S_0$, in particular $|S_1| \leq |S_0|$.

According to this partition represent $K = K_0 + (q-1)K_1$ where $K_0 = \sum_{s_0 \in S_0} (q-1)^{|s_0 \setminus J|}$, $K_1 = \sum_{s_1 \in S_1} (q-1)^{|s_1 \setminus J|}$. By inductive hypothesis

$$K \leq |S_0|^{\log_2 q} + (q-1)|S_1|^{\log_2 q} \leq (|S_0| + |S_1|)^{\log_2 q}$$

the latter inequality follows from the convexity of the function $X \to X^{\log_2 q}$ (on the ray $\mathbb{R}_+$ of nonnegative reals), namely rewrite this inequality in the form

$$|S_0|^{\log_2 q} + (2|S_1|)^{\log_2 q} \leq |S_1|^{\log_2 q} + (|S_0| + |S_1|)^{\log_2 q} \, .$$

This completes the proof of the induction and lemma 7.

Corollary 3 *For any active element J*

$$|\bar{G} \cap \bigcup_{J_1 \in \mathcal{P}(J)} C(J_1)| \leq |G \cap C(J)|(m_{fX_J})^{\log_2 q} \leq |G \cap C(J)|(m_f)^{\log_2 q} .$$

Proof. $|\bar{G} \cap \bigcup_{J_1 \in \mathcal{P}(J)} C(J_1)| = (q-1)^{|J|} \cdot \sum_{J_1 \in \mathcal{P}(J)} (q-1)^{|J_1 \setminus J|}$. By lemma 3 $(q-1)^{|J|} \leq |G \cap C(J)|(m_{f_J})^{\log_2 q}$. By lemma 5b) $\mathcal{P}(J)$ is a relative principal ideal, hence $\sum_{J_1 \in \mathcal{P}(J)} (q-1)^{|J_1 \setminus J|} \leq |\mathcal{P}(J)|^{\log_2 q}$ by lemma 7. Therefore we get the corollary 3 applying corollary 2.

Finally, we complete the proof of the theorem summing left and right sides of the inequalities from corollary 3 ranging over all active elements J, taking into account corollary 1, lemma 5a) and lemma 2a).

4 Bounds for β_1 and β_2

We shall apply now Theorem 2 to derive upper bounds for β_1 and β_2.

Theorem 3 *Given any polynomial $f \in GF[q][x_1, \cdots, x_n]$ with m terms and without constant terms. Then*

$$\frac{q^n}{\#_0 f} \leq \beta_1 = (m^{q-1} + 1)^{\log q} \leq (m+1)^{(q-1)\log q} .$$

Proof. Consider the polynomial $g = f^{q-1}$.
For $s \in GF[q]^n$, $f(s) = 0 \Leftrightarrow (f^{q-1} - 1)(s) \neq 0$. Apply Theorem 2 to the polynomial $f^{q-1} - 1 \in GF[q][x_1, \cdots, x_n]$, $|\bar{G}| = q^n$, $|G| = \#_0 f$, and the number of terms of $f^{q-1} - 1$ is $m^{q-1} + 1$. So the exact bound is $(m^{q-1} + 1)^{\log q}$.
$\square$

Theorem 4 *Given any polynomial $f \in GF[q][x_1, \cdots, x_n]$ with m terms and $c \neq 0$. Then*

$$\frac{|\tilde{G}_{(f-c)^{q-1}-1}|}{\#_c f} \leq \beta_2 / m^{q-1} = ((m+1)^{q-1} - 1)^{\log q} \leq (m+1)^{(q-1)\log q} .$$

Proof. For $s \in GF[q]^n$, $f(s) = c \Leftrightarrow (f-c)^{q-1}(s) = 0 \Leftrightarrow (f-c)^{q-1}(s) - 1 \neq 0$. Observe that $(f-c)^{q-1} - 1$ polynomial is constant free. Apply Theorem 2 to the polynomial $(f-c)^{q-1} - 1$ with $|G| = \#_c f$ and $m^{q-1} - 1$ terms which results in $\beta_2 = ((m+1)^{q-1} - 1)^{\log q}$.
$\square$

Observe that in Theorem 4, taking the set $\bar{G}_{(f-c)^{q-1}-1}$ is neccesary as the set $\bar{G}_f$ does not have a polynomial bound for the ratio $\frac{|\bar{G}_f|}{\#_c f}$. Take for example the polynomial

$$(q-2)x_1^{q-1} \cdots x_{n-1}^{q-1} + x_n^{q-1} = -1 \; .$$

$\frac{|\bar{G}_f|}{\#_c f} = \frac{q^{n-1}}{(q-1)^n}$ tends to infinity with growing n and does not satisfy the inequality $\leq q^{q-1}$.

The bounds proven in Theorems 3, and 4 are almost optimal (cf. [GK 90]).

5 Open Problem

Our method yields the first polynomial time (ϵ, δ)-approximation algorithm for the number of zeros of arbitrary polynomials $f \in GF[q][x_1, \ldots, x_n]$ for the fixed field $GF[q]$. Degree of the polynomial bounding the running time of the algorithm depend on the order of the ground field.

Is it possible to remove dependence of the degree on q in the approximation algorithm?

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