Approximation and complexity II: iterated integration

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Abstract

We introduce two classes of real analytic functions $W \subset U$ on an interval. Starting with rational functions to construct functions in W we allow the application of three types of operations: addition, integration and multiplication by a polynomial with rational coefficients. In a similar way to construct functions in U we allow integration, addition and multiplication of functions already constructed in U and multiplication by rational numbers. Thus, U is a subring of the ring of Pfaffian functions [Kh].

Two lower bounds on L_∞-norm are proved on a function f from W (or from U, respectively) in terms of the complexity of constructing f.

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Introduction

The well-known Liouvillean theorem states that if

$p(a) = 0, a \neq 0, p = \sum_{0 \leq i \leq m} p_i X^i, p_i \in \mathbf{Z}$

then one can bound from below the absolute value $|a|$ of the algebraic number a in terms of the complexity of the determining polynomial p.

The question arises whether this phenomenon could be extended to solutions v of (ordinary) differential equations $Q(v) = 0$? It is known [B] that in general one can't bound v for non-linear second-order (or higher) equations

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$Q(v) = 0$. Thus, one ought consider solutions of either linear or first-order equations.

In the present paper we introduce two classes of real analytic functions $W \subset U$ on a finite interval I. In both cases we start with rational functions with rational coefficients. To construct functions in W we allow the application of three types of operations: addition, integration and multiplication by a polynomial from $\mathbf{Q}[X]$. Thereby, W is a differential $\mathbf{Q}[X]$-module. While for constructing functions in U we allow the application of integration and substitution of already constructed functions from U into a (multivariate) polynomial with rational coefficients. Thereby, U is a differential $\mathbf{Q}[X]$-algebra. Clearly, U is a subring of the ring of Pfaffian functions [Kh].

Thus, each function f from W (or from U, respectively) is constructed by means of a chain of operations (which involve arithmetic and integration) and one can define the "complexity" of f as the complexity of a corresponding chain.

In section 1 below we prove (theorem 1) a lower bound on the "separator" $\min_{x\in I'} |f(x)|$ for $f \in W$ and a suitable subinterval $I' \subset I$ in terms of I and of the complexity of the corresponding chain for f (in particular, this provides a lower bound on L_∞-norm $\max_{x\in I} |f(x)|$). Moreover, we provide an upper bound on the number of roots of f (lemma 1) which is better than the bound from [Kh] established for the wider class of Pfaffian functions. It is worthwhile also to mention that in [Y] one can find a comprehensive survey on the bounds on the number of zeroes of solutions of diverse classes of differential equations.

In section 2 we prove (theorem 2) a lower bound on $\min_{x\in I'} |f(x)|$ for $f \in U$ and a suitable subinterval $I' \subset I$ again in terms of I and of the complexity of the chain for f under the assumption that in the chain each application of integration introduces a function that is transcendental with respect to the previously constructed functions in the chain. This assumption of purely transcendental chains allows one to avoid introducing in a chain arbitrarily small constants (otherwise, no lower bound would be possible).

Thus, in constructing U (and W) we allow integration. It seems that if we allowed the introduction of the solutions of more general types of first-order differential equations (as e.g., in case of Pfaffian functions [Kh]) then results of a similar sort to ones in the present paper would fail, again because solving such equations would allow the introduction of arbitrarily small constants in a chain.

In [G01] similar results were established for solutions of linear differential equations on an interval.

The picture becomes somewhat easier to study if instead of approximating on an interval, asymptotic approximations on the real line (at infinity) are considered since then introducing small constants is not a problem. In this setting a lower bound on approximations in terms of the complexity for a wider class (than the present class) of Pfaffian functions was obtained in [G93]. Besides, a lower bound for a wider class than in [G01] of *compositions* of solutions of linear differential equations was established in [G92].

One could also view the results of the paper as a trade-off between approximations and complexity. It would be interesting to understand more about this trade-off. We mention that in this direction a lower bound was proved in [CG] on the complexity of approximating algebraic computation trees.

It is worthwhile also to mention that in [K] a version of a differential analog of the Liouville's theorem was proposed in terms of bounds on valuations, while we study approximations in the L_∞-norm.

1 Functions of linear-iterated integration

Denote by $I \subset \mathbf{R}$ a finite closed interval of the length $|I|$ such that

$$I \subset [-b, b], b \geq 1 \tag{1}$$

and for a function g on I denote the L_∞-norm $||g||_I = \max_{x \in I} |g(x)|$ and the "separator" $sep_I(g) = \min_{x \in I} |g(x)| = ||g^{-1}||^{-1}$.

Let $q_0, \ldots, q_t$ be real analytic functions on I, moreover $q_0 \in \mathbf{R}(X)$.We say that $q_0, \ldots, q_t$ constitute a (t, d)-chain of *linear-iterated integration* if for each $0 \leq i \leq t-1$ for appropriate polynomials $p_{0,i}, \ldots, p_{i,i} \in \mathbf{R}[X]$ of degrees $\deg(p_{0,i}), \ldots, \deg(p_{i,i}) \leq d$, we have for the derivative

$$q'_{i+1} = p_{0,i}q_0 + \cdots + p_{i,i}q_i \tag{2}$$

In such a case we sometimes simply say that q_t is a (t, d)-chain. Clearly, a (t, d)-chain is a particular case of a Pfaffian chain [Kh] and thereby, q_t is a Pfaffian function.

First we prove an upper bound on the number $\#(g)$ of the roots of g in I (in fact, this would give the same bound on the number of the roots on the whole real line, provided that the functions $\{q_i\}_{0\leq i\leq t}, g$ were analytic on the whole real line as well). We note that this bound is stronger (being polynomial rather than exponential) than the bound [Kh] which is valid for the wider class of Pfaffian functions.

Lemma 1 *Let*

$$g = \sum_{0\leq j\leq t} v_{j,1}q_{j,1} + \cdots + \sum_{0\leq j\leq t} v_{j,N}q_{j,N} \tag{3}$$

where for every $1 \leq l \leq N$ *the functions* $q_{0,l}, \dots, q_{t,l}$ *form a* (t,d)*-chain, here the polynomials* $v_{j,l} \in \mathbf{R}[X], \deg(v_{j,l}) \leq d$. *Then* $\#(g) \leq O(Nt^2d^2)$, *provided that* $g \not\equiv 0$.

Proof. We express the rational function $q_{0,l} = \widehat{q_{0,l}}/\overline{q_{0,l}}$ where the numerator and the denominator $\widehat{q_{0,l}}, \overline{q_{0,l}}$ have the degrees $\deg(\widehat{q_{0,l}}), \deg(\overline{q_{0,l}}) \leq d$.

By induction on $s = 1, \dots, t$ denoting $S = (d+1)+(2d+1)+\cdots+(sd+1)$ one can represent the derivative

$$g^{(S)} = \sum_{1\leq j\leq t-s} v_{j,1,s}q_{j,1} + q_{0,1,s} + \cdots + \sum_{1\leq j\leq t-s} v_{j,N,s}q_{j,N} + q_{0,N,s}$$

for suitable $v_{j,l,s} \in \mathbf{R}[X], q_{0,l,s} = \widetilde{q_{l,s}}/(\overline{q_{0,l}})^S$ where

$$\deg(v_{j,l,s}) \leq \min\{Sd, (t-j+1)d\}, \deg(\widetilde{q_{l,s}}) \leq O((sd)^2),$$

see (2).

Finally, putting $s = t, T = (d+1)+(2d+1)+\cdots+(td+1)$ we get that $g^{(T)} \in \mathbf{R}(X)$ and taking into account the Rolle's theorem $\#(g) \leq \#(g^{(T)})+T$ when $g^{(T)} \not\equiv 0$ or otherwise we have $\#(g) \leq T-1$, we conclude with the lemma. $\Box$

From now on we assume that the functions $q_0 \in \mathbf{Q}(X), p_{0,i}, \dots, p_{i,i} \in \mathbf{Q}[X]$ determining a chain, have rational coefficients. When $q_0 = \widehat{q_0}/\overline{q_0}$ where $\widehat{q_0}, \overline{q_0} \in \mathbf{Z}[X]$ one says that the height $h(q_0) \leq h$ if the absolute values of all the integer coefficients of $\widehat{q_0}, \overline{q_0}$ do not exceed h. Thus, we say that a (t,d)-chain $q_0, \dots, q_t$ is (t,d,h)-chain (we suppose that $h \geq 2$) if

$$h(q_0), h(p_{j,i}) \le h, 0 \le i \le t-1, 0 \le j \le i \tag{4}$$

Denote by W_t the $\mathbf{Q}[X]$-module generated by functions satisfying $(t, \overline{d}, \overline{h})$-chains for all possible $\overline{d}, \overline{h} \ge 0$. Evidently, $W_0 = \mathbf{Q}(X)$.

Using (2), (4) one can by induction on s estimate

$$h(v_{j,l,s}), h(q_{0,l,s}) \le (hsd)^{O((s^2+t)d^2)}$$

Therefore, at the end of induction one gets

$$g^{(T)} \in \mathbf{Q}(X), h(g^{(T)}) \le (htd)^{O(t^2 d^2 N)}$$

.

Let us suppose that $g^{(T)} \not\equiv 0$. Due to lemma 1 there exists a subinterval

$$I_0 \subset I, |I_0| = \frac{|I|}{O(Nt^2d^2)} \tag{5}$$

which contains no roots of the derivatives $g, g^{(1)}, \dots, g^{(T+1)}$ (in case when $g^{(T+1)} \equiv 0$ we require instead that $g, g^{(1)}, \dots, g^{(T)}$ have no roots in I_0).

The following lemma is similar to lemma 2 from [G01] (cf. also lemma 4 in section 2 below) with the difference that instead of L_∞-norm we estimate here the "separator".

Lemma 2 *Assume that the derivatives $g, g^{(1)}, \dots, g^{(T+1)}$ have no roots in I_0 (in case when $g^{(T+1)} \equiv 0$ we require that $g, g^{(1)}, \dots, g^{(T)}$ have no roots in I_0). Then there exists a subinterval $I_T \subset I_0$ of the length $|I_T| = \frac{|I_0|}{T+1}$ such that*

$$sep_{I_0}(g^{(T)}) \le sep_{I_T}(g^{(T-j)}) \left(\frac{T+1}{|I_0|}\right)^j, 0 \le j \le T$$

Proof. Suppose that one has already produced (by recursion on j) closed subintervals $I_0 \supset I_1 \supset \cdots \supset I_j$ with the lengths $|I_l| = |I_0|\frac{T+1-l}{T+1}$ such that

$$sep_{I_0}(g^{(T)}) \le sep_{I_l}(g^{(T-l)}) \left(\frac{T+1}{|I_0|}\right)^l, 0 \le l \le j < T$$

Denote by $a_1 = |g^{(T-j-1)}(x_1)|, a_2 = |g^{(T-j-1)}(x_2)|$ the values of the function $|g^{(T-j-1)}|$ at the endpoints of the interval $I_j = [x_1, x_2]$. If $a_1 < a_2$

then put $x_0 = x_1 + \frac{|I_0|}{T+1}$ and the subinterval $I_{j+1} = [x_0, x_2]$. Otherwise, if $a_1 > a_2$ then put $x_0 = x_2 - \frac{|I_0|}{T+1}$ and the subinterval $I_{j+1} = [x_1, x_0]$. We have $sep_{I_{j+1}}(g^{(T-j-1)}) = |g^{(T-j-1)}(x_0)|$ since $g^{(T-j-1)}$ is monotone and has no roots in the interval $I_{j+1} \subset I_0$ (whence $|g^{(T-j-1)}|$ is monotone in the same interval as well). Observe that $a_1 \neq a_2$, indeed, otherwise $g^{(T-j)}$ would vanish identically on the interval I_j. Hence

$$|g^{(T-j-1)}(x_0)| \geq sep_{I_j}(g^{(T-j)})(|I_j| - |I_{j+1}|)$$

because $g^{(T-j-1)}$ is monotone and has no roots in the interval $I_j - I_{j+1} \subset I_0$. Thus,

$$sep_{I_{j+1}}(g^{(T-j-1)}) \geq sep_{I_j}(g^{(T-j)}) \frac{|I_0|}{T+1}$$

which completes the proof of the recursive hypothesis for $j+1$. Setting $l = j = T$ we get lemma 2. $\Box$

We represent $g^{(T)} = f_1/f_2$ for $f_1, f_2 \in \mathbf{Z}[X]$ such that $h(f_1), h(f_2) \leq h(g^{(T)})$. Denote an integer

$$a_2 = \lceil 2|I_0|^{-1} \rceil \leq \max\{1, \frac{O(Nt^2d^2)}{|I|}\}$$

(cf. (5) and lemma 2). In case when $a_2 \geq 2$ there exists a pair of rational points $a = \frac{a_1}{a_2}, a' = \frac{a_1+1}{a_2} \in I_0, a_1 \in \mathbf{Z}$, consider a subinterval $I_0' = [a, a']$. In case when $a_2 = 1$ we take integers $a < a'$ to be such that the interval $I_0' = [a, a']$ is the maximal subinterval of I_0 with integer endpoints. In both cases we have $|I_0'| \geq |I_0|/4$. Then

$$|f_2(a)|, |f_2(a')| \leq ||f_2||_{I_T} \leq (htdb)^{O(Nt^2d^2)}$$

(see (1), (2), (3), (4)) and

$$|f_1(a)|, |f_1(a')| \geq a_2^{-O(Nt^2d^2)}$$

(see (2), (3)). We apply lemma 2 to the interval $I_0' = [a, a'] \subset I_0$ and conclude that for a certain subinterval

$$I_T' \subset I_0', |I_T'| = \frac{|I_0'|}{T+1} \geq \frac{|I_0|}{4(T+1)}$$

we have

$$sep_{I_T'}(g)\left(\frac{O(Nt^2d^2)}{|I|}\right)^T \geq sep_{I_0'}(g^{(T)}) = \min\{|g^{(T)}(a)|, |g^{(T)}(a')|\} \geq$$
$$(hNtdb)^{-O(Nt^2d^2)} \min\{1, \left(\frac{|I|}{O(Nt^2d^2)}\right)^{O(Nt^2d^2)}\}$$

This implies the following theorem.

Theorem 1 *Assume that g is a (t,d,h)-chain on an interval $I \subset [-b,b]$, the conditions (1), (2), (3), (4) and $g^{(T)} \not\equiv 0$. Then there exists a subinterval $I' \subset I$ with the length $|I'| \geq \frac{|I|}{O(Nt^2d^2)}$ such that*

$$sep_{I'}(g) \geq \left(\frac{|I|}{hNtdb}\right)^{O(Nt^2d^2)}$$

We note that without any condition on the derivatives of g the lower bound in theorem 1 would fail since e.g., when $t=1, g=g_1$ and an equation $g_1'=0$ as a chain, one could take as g an arbitrarily small constant.

Obviously, the same bound as in theorem 1 holds a fortiori, for the L_∞-norm $||g||_I \geq sep_{I'}(g)$ (the similar remark concerns also theorem 2 in section 2).

2 Functions of iterated integration

Now we consider an extension of the class of functions W_t from the previous section. We define a sequence of analytic on I real functions $\{g_i\}_{1\leq i\leq t}, g$ of *iterated integration* by recursion on i. Namely,

$$g_i' = p_i(X, g_1, \ldots, g_{i-1}), g = p_{t+1}(X, g_1, \ldots, g_t) \qquad (6)$$

where the rational functions $p_i \in \mathbf{Q}(X)[Y_1, \ldots, Y_{i-1}], 1 \leq i \leq t+1$. In other words, one is allowed, in particular, to integrate at a current step the product of functions produced at previous steps. Clearly, the produced ring U_t of all functions of the form g contains W_t, and on the other hand being a subring of the ring of Pfaffian function [Kh].

Any rational function $p \in \mathbf{Q}(X)[Y_1, \dots, Y_t]$ we write in a form $\hat{p}/\overline{p}$ where the polynomials $\hat{p} \in \mathbf{Z}[X, Y_1, \dots, Y_t], \overline{p} \in \mathbf{Z}[X]$ are relatively prime. We assume that

$$\deg(p_i) = \max\{\deg(\hat{p}_i), \deg(\overline{p_i})\} \leq d, h(p_i) = \max\{h(\hat{p}_i), h(\overline{p_i})\} \leq h, \quad (7)$$

$h \geq 2, 1 \leq i \leq t+1$.

Also we suppose that

$$||g_i||_I \leq M, 1 \leq i \leq t, M \geq 1 \quad (8)$$

and that each *complex* root of the denominator $\overline{p_i}$ lies at a distance at least 1 from the interval $I, 1 \leq i \leq t+1$, hence

$$|\overline{p_i}| \text{ is greater or equal to 1 everywhere on } I \quad (9)$$

We observe that the latter conditions provide *upper* bounds on the functions involved in computations with the sequence, and one could interpret the theorem in this section as a lower bound on functions of iterated integration by means of their upper bounds.

We make an assumption that the sequence $\{g_i\}_{1 \leq i \leq t}$ (see (6)) is purely transcendental, i.e. g_{i+1} is algebraically independent over the field $F_i = \mathbf{R}(X, g_1, \dots, g_i)$ for all $0 \leq i < t$. We note that F_i is a differential field. The condition of $\{g_i\}_{1 \leq i \leq t}$ being purely transcendental is similar to the condition in the algorithm due to Risch [R] and allows one to avoid introducing in a sequence (6) functions being arbitrarily small constants which would prevent lower bounds on the functions of iterated integration on I.

The following lemma enables us to eliminate a transcendental integral.

Lemma 3 *Let F be a differential field. Assume that u is algebraically independent over F, besides that its derivative $u' \in F$, and that $F(u)$ has the same subfield of constants as F has. Consider $g = p(u) \in F[u], \deg_u(p) = n$ and denote the polynomials $p_i(u) \in F[u]$ to be such that the derivative $g^{(i)} = p_i(u)$ for $i \geq 0$. Then* $\gcd(p, p_1, \dots, p_n) \in F$.

Proof. We argue by induction on n. The base of induction is evident. For the inductive step consider the leading term of $p = wu^n + \cdots$ where $w \in F$. First, $f \equiv w'g - wg' \neq 0$ since otherwise $g = cw$ for a certain constant c and thereby, u is algebraic over F. Denote the derivatives $f^{(i)} = r_i(u) \in F[u], i \geq 0$. Obviously, $\deg_u(r_0) < \deg(p_0) = n$ and $\deg(r_i) \leq \deg(r_0), i \geq 0$. On the other hand,

$$f^{(i)} = \sum_{0 \leq j \leq i} \binom{i}{j} (w^{(i+1)} g^{(i-j)} - w^{(i-j)} g^{(j+1)}), i \geq 0,$$

therefore $\gcd(p_0, \dots, p_n)$ divides $\gcd(r_0, \dots, r_{n-1})$ in the polynomial ring $F[u]$. This implies the inductive step since $\gcd(r_0, \dots, r_{n-1}) \in F$ by the inductive hypothesis. $\square$

Observe that the condition of conserving the subfield of constants will be fulfilled in our situation because we consider the functions on the interval I and the subfield coincides just with $\mathbf{R}$.

In course of the procedure described below a certain family $\Delta \subset U_t$ of functions is constructed. Later on we bound from above the total number N_0 of the roots of the functions from Δ and now we fix a subinterval $J \subset I$ of length $|J| = |I|/(N_0 + 1)$ which does not contain any such root.

At the first step we represent the function $g = q(g_t)$ where the coefficients of the (univariate) polynomial q belong to the (differential) ring $K_{t-1} = \mathbf{Q}(X)[g_1, \dots, g_{t-1}] \subset F_{t-1}$. Then $g^{(i)} = q_i(g_t)$ for suitable polynomials $q_i(g_t) \in K_t = K_{t-1}[g_t], 0 \leq i \leq d$. Consider $G = \gcd(q, q_1, \dots, q_d)$ in the ring $\mathbf{Q}(X, g_1, \dots, g_{t-1})[g_t]$ being defined up to a factor from $Q_{t-1} = \mathbf{Q}(X, g_1, \dots, g_{t-1})$. Then G belongs to $F_{t-1} = \mathbf{R}(Q_{t-1})$ according to lemma 3, in other words, $\deg_{g_t}(G) = 0$, therefore, $G \in Q_{t-1}$ since gcd does not change when the field of the coefficients is extended $Q_{t-1} \subset F_{t-1}$.

The subresultant theorem (see e.g., [L]) states that one can choose G in such a way that

$$G = A_0 q + A_1 q_1 + \ldots + A_d q_d \in K_{t-1} \tag{10}$$

where the coefficients of the (univariate) polynomials $A_0, \dots, A_d \in K_{t-1}[g_t]$ are appropriate subminors of the Sylvester matrix A of the family of the (univariate) polynomials $q, q_1, \dots, q_d$. More precisely, the usual Sylvester matrix is associated to a pair of polynomials, but one can directly extend it to a family of polynomials (see [G90]).

The size of the matrix A is bounded by $O(\deg(q) + \deg(q_1) + \ldots + \deg(q_d)) \leq O(d^2)$ (see [L], [G90]). Each entry of A written as a certain function $r(X, g_1, \ldots, g_{t-1}) \in K_{t-1}$ can be bounded as follows: $\deg(r) \leq O(d^2), h(r) \leq (hd)^{O(d)}$ (see (7)). Therefore,

$$\deg(A_i) \leq O(d^4), h(A_i) \leq d^{O(d^4+t)} h^{O(d^3)}, ||A_i||_I \leq d^{O(d^4+t)} h^{O(d^3)} M^{O(d^4)}$$

(the latter inequality invokes (8), (9)). Hence $\deg(G) \leq O(d^4), h(G) \leq d^{O(d^4+t)} h^{O(d^3)}$.

The following lemma was proved as lemma 2 [G01] (we use the notations introduced in lemma 2 from section 1).

Lemma 4 *Assume that the derivatives $g, g^{(1)}, \ldots, g^{(T+1)}$ have no roots in I_0 (in case when $g^{(T+1)} \equiv 0$ we require that $g, g^{(1)}, \ldots, g^{(T)}$ have no roots in I_0). Then there exists a subinterval $I_T \subset I_0$ of the length $|I_T| = \frac{|I_0|}{T+1}$ such that*

$$||g^{(j)}||_{I_T} \leq ||g||_{I_0} \left(\frac{T+1}{|I_0|} \right)^j, 0 \leq j \leq T$$

Lemma 4 implies that there exists a subinterval $J_1 \subset J, |J_1| = |J|/(d+1)$ such that

$$||g^{(j)}||_{J_1} \leq ||g||_J \left(\frac{d+1}{|J|} \right)^j, 0 \leq j \leq d$$

.
We include the functions $g, g^{(1)}, \ldots, g^{(d)}$ into the family Δ and thereby, impose the condition that $g, g^{(1)}, \ldots, g^{(d)}$ have no roots in J (cf. the discussion on Δ, J above). Due to that and to (10) we get

$$||G||_{J_1} \leq ||g||_J \max\{1, \left(\frac{d+1}{|J|} \right)^d\} d^{O(d^4+t)} h^{O(d^3)} M^{O(d^4)} \qquad (11)$$

Thus, we have carried out one step of the procedure, constructed G and thereby, eliminated g_t.

After t such steps we achieve by recursion a rational function $L \in \mathbf{Q}(X)$ and nested subintervals $J \supset J_1 \supset \ldots \supset J_t$ such that

$$\deg(L) \le d^{4^t}, h(L) \le h^{d^{5^t}}, |J_t| \ge \frac{|J|}{d^{5^t}}, ||L||_{J_t} \le ||g||_J (Mh)^{d^{6^t}} \max\{1, |J|^{-d^{5^t}}\}$$

(because of (11)).

The family Δ consists of $O(d^{4^t})$ functions each having at most $d^{O(4^t)}$ roots (on I) due to [Kh] since one can view (6) as a Pfaffian chain, therefore, the total number N_0 of roots of the functions from Δ does not exceed $d^{O(4^t)}$, hence

$$|J| \ge \frac{|I|}{d^{O(4^t)}},$$

finally

$$||L||_{J_t} \le ||g||_I (Mh)^{O(d^{6^t})} \max\{1, |I|^{-d^{5^t}}\}$$

(see (11)).

Similar to the end of section 1 there exists a rational point

$$a = a_1/a_2 \in J_t, a_1 \in \mathbf{Z}, 0 < a_2 = \lceil |J_t|^{-1} \rceil.$$

In a similar way we represent $L = L_1/L_2, L_1, L_2 \in \mathbf{Z}[X]$ and we get

$$|L_1(a)| \ge a_2^{-\deg(L_1)} \ge \min\{1, \left(\frac{|I|}{d^{5^t}}\right)^{O(d^{4^t})}\},$$

$$|L_2(a)| \le h(L_2) \deg(L_2) b^{\deg(L_2)} \le h^{d^{5^t}} b^{d^{4^t}}$$

(see (1), (7)). As $|L_1(a)/L_2(a)| \le ||L||_{J_t}$ we conclude that

$$||g||_I \ge (Mh)^{-d^{6^t}} b^{-d^{4^t}} \min\{1, |I|^{d^{5^t}}\} = B$$

taking into account (11).

Moreover, one can estimate an interval $J' \subset I$ such that $|g(x)| \ge B/2$ for any $x \in J'$. Indeed (see (6)),

$$g' = \frac{\partial p_{t+1}}{\partial X} + \frac{\partial p_{t+1}}{\partial g_1} p_1(X) + \cdots + \frac{\partial p_{t+1}}{\partial g_t} p_t(X, g_1, \ldots, g_{t-1})$$

Therefore, $||g^{'}||_I \leq h^2 d(Mt)^{O(d)}$ due to (7), (8), (9). Now take a point $x_0 \in I$ at which $|g(x_0)| = B_0 \geq B$. Then for any point $x \in I$ such that

$$|x - x_0| \leq \frac{B_0}{2h^2 d(Mt)^{O(d)}}$$

we have $|g(x)| \geq B_0/2$.

Thus, the following theorem is proved.

Theorem 2 *If g satisfies a purely transcendental iterated integration sequence (6) and the bounds (1), (7), (8), (9), then there exists a subinterval*

$$J^{'} \subset I, |J^{'}| \geq (Mhb)^{-O(d^{6^t})} \min\{1, |I|^{d^{5^t}}\}$$

such that

$$sep_{J'}(g) \geq (Mhb)^{-O(d^{6^t})} \min\{1, |I|^{d^{5^t}}\}$$

In conclusion let us formulate a conjecture that an upper bound on the number of roots of a function from the class U_t should be better than the one from [Kh].

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