Complexity of a Standard Basis of a D-module

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Abstract

We prove a double-exponential upper bound on the degree and on the complexity of constructing a standard basis of a D-module. This generalizes a well known bound on the complexity of a Gröbner basis of a module over the algebra of polynomials. We would like to emphasize that the obtained bound can not be immediately deduced from the commutative case. To get our result we have elaborated a new technique of constructing all the solutions of a linear system over a homogeneous version of a Weyl algebra.

Introduction

Let A be the Weyl algebra $F[X_1,\dots,X_n,\ \frac{\partial}{\partial X_1},\ \dots,\frac{\partial}{\partial X_n}]$ (or the algebra of differential operators $F(X_1,\ \dots,X_n)[\frac{\partial}{\partial X_1},\dots,\frac{\partial}{\partial X_n}]$). Denote for brevity $D_i = \frac{\partial}{\partial X_i}$, $1 \leqslant i \leqslant n$. Any A–module is called D–module. It is well known that an A–module which is a submodule of a free finitely generated A-module has a Janet basis (if A is a Weyl algebra it is called often a standard basis; but in this paper it is natural and convenient to call it a Janet basis also in the case of the Weyl algebra). Historically, it was first introduced in [9]. In more recent times of developing computer algebra Janet bases were studied in [5], [14], [10]. Janet bases generalize Gröbner bases which were widely elaborated in the algebra of polynomials (see e. g.[3]). For Gröbner bases a double-exponential complexity bound was obtained in [12], [6] relying on [1]. Further, more precise results on the same subject (with an independent and self–contained proofs) were obtained in [4].

Surprisingly, no complexity bound on Janet bases was established so far. The reason is unique: the problem is not easy. In the present paper we fill this very essential gap and prove a double-exponential upper bound for complexity. On the other hand, a double-exponential complexity lower bound on Gröbner bases [12], [15] provides by the same token a bound on Janet bases.

Notice also that there has been a folklore opinion that the problem of constructing a Janet basis is easily reduced to the commutative case by considering the associated graded module, and, on the other hand, in the commutative case [6], [12], [4] the double–exponential upper bound is well known. *But it turns out to be a fallacy! From a known system of generators of a D-module one can not obtain immediately any system of generators (even not necessarily a Gröbner basis) of the associated graded module.* The main problem here is to construct such a system of generators of the graded module. It may have the elements of degrees $(dl)^{2^{O(n)}}$, see the notation below. Then, indeed, to the last system of generators of big degrees one can apply the result known in the commutative case and get the bound $((dl)^{2^{O(n)}})^{2^{O(n)}} = (dl)^{2^{O(n)}}$. So new ideas specific to non–commutative case are needed.

We are interested in the estimations for Janet bases of A-submodules of A^l. The Janet basis depends on the choice of the linear order on the monomials (we define them also for $l > 1$). In this paper we consider the most general linear orders on the monomials from A^l. They satisfy conditions (a) and (b) from Section 1 and are called *admissible*. If additionally a linear order satisfies condition (c) from Section 1 then it is called *degree-compatible*. For any admissible linear order *the reduced Janet basis* is chosen canonically and it is uniquely defined, see Section 1. We prove the following result.

THEOREM 1 *For any real number $d \geqslant 2$ and any admissible linear order on the monomials from A^l any A-submodule I of A^l generated by elements of degrees at most d (with respect to the filtration in the corresponding algebra, see Section 1 and Section 9) has a Janet basis with the degrees and the number of its elements less than*

$$(dl)^{2^{O(n)}}. \tag{1}$$

The same upper bound (1) holds for the number of elements of the reduced *Janet basis of the module I with respect to the considered linear order on the monomials.*

If additionally this linear order is degree-compatible or it is arbitrary admissible but $l = 1$ then also the degrees of all the elements the reduced *Janet basis of the module I are bounded from above by (1).*

We prove in detail this theorem for the case of the Weyl algebra A. The proof for the case of the algebra of differential operators is similar. It is sketched in Section 9. From Theorem 1 we get that the Hilbert function $H(I, m)$, see Section 1, of the A-submodule from this theorem is stable for $m \geqslant (dl)^{2^{O(n)}}$ and the absolute values of all coefficients of the Hilbert polynomial of I are bounded from above by $(dl)^{2^{O(n)}}$, cf. e.g., [12]. This fact follows directly from (11), Lemma 12 from Appendix 1, Lemma 2 and Theorem 2. We mention that in [7] the similar bound was shown on the leading coefficient of the Hilbert polynomial.

Now we outline the plan for the proof of Theorem 1. Below the first occurrences of some terms introduced in the paper are italicized. The main tool in the proof is *a homogenized Weyl algebra* ${}^hA$ (or respectively, *a homogenized algebra of differential operators* ${}^hB$). It is introduced in Section 3 (respectively, Section 9). The algebra ${}^hA$ (respectively ${}^hB$) is generated over the ground field F by $X_0, \ldots, X_n$, $D_1, \ldots, D_n$ (respectively over the field $F(X_1, \ldots, X_n)$ by $X_0, D_1, \ldots, D_n$). Here X_0 is a new *homogenizing variable*. In the algebra ${}^hA$

(respectively ${}^hB$) relations (13) Section 3 (respectively (54) Section 9) hold for these generators.

We define *the homogenization* ${}^hI$ of the module I. It is a ${}^hA$–submodule of ${}^hA^l$. The main problem is to estimate the degrees of a system of generators of ${}^hI$. These estimations are central in the paper. They are deduced from Theorem 2 Section 7. This theorem is devoted to the problem of solving systems of linear equations over the ring ${}^hA$; we discuss it below in more detail.

The system of generators of ${}^hI$ gives a system of generators of the graded $\mathrm{gr}(A)$–module $\mathrm{gr}(I)$ corresponding to I. But $\mathrm{gr}(A)$ is a polynomial ring. Hence using Lemma 12 Appendix 1 we get a double–exponential bound $(dl)^{2^{O(n)}}$ on the stabilization of the Hilbert function of $\mathrm{gr}(I)$ and the absolute values of the coefficients of the Hilbert polynomial of $\mathrm{gr}(I)$. Therefore, the similar bound holds for the stabilization of the Hilbert functions of I and the coefficients of the Hilbert polynomial of I, see Section 2.

But the Hilbert functions of the modules I and ${}^hI$ coincide, see Section 3. Hence the last bound holds also for the stabilization of the Hilbert functions of ${}^hI$ and the coefficients of the Hilbert polynomial of ${}^hI$. In Section 5 we introduce the linear order on the monomials from ${}^hA^l$ induced by the initial linear order on the monomials from A^l (the homogenizing variable X_0 is the least possible in this ordering). Further, we define the Janet basis of ${}^hI$ with respect to the induced linear order on the monomials. Such a basis can be obtained by *the homogenization of the elements* of a Janet basis of I with respect to the initial linear order, see Lemma 3 (iii).

For every element $f \in {}^hA$ denote by $\mathrm{Hdt}(f) \in {}^hA$ the greatest monomial of the element f, i.e., each monomial of $f - \mathrm{Hdt}(f)$ is less than $\mathrm{Hdt}(f)$ with respect to the induced linear order on the monomials from ${}^hA$. Let $\mathrm{Hdt}({}^hI) = \{\mathrm{Hdt}(f) : f \in {}^hI\}$ be the set of all the greatest monomials of the elements of the module ${}^hI$, see Section 4. Let ${}^cI \subset {}^cA^l$, see Section 4, be the module over the polynomial ring ${}^cA = F[X_0, \ldots, X_n, D_1, \ldots, D_n]$ generated by all the monomials from $\mathrm{Hdt}({}^hI)$ (they are considered now as elements of ${}^cA^l$). Then the Hilbert functions of the modules ${}^hI$ and ${}^cI$ coincide. Thus, we have the same as above double–exponential estimation for the stabilization of the Hilbert function of ${}^cI$ and the coefficients of the Hilbert polynomial of ${}^cI$. Now using Lemma 13 we get the estimation $(dl)^{2^{O(n)}}$ on the monomial system of generators of ${}^cI$. This gives the bound for the degrees of the elements of the Janet basis of ${}^hI$ and hence by Lemma 11 also the bound from Theorem 1 for the required Janet basis (respectively in the case when the initial order is degree-compatible for the reduced Janet basis) of I. The estimate for degrees of elements the reduced Janet basis in the case $l = 1$ requires special considerations, see Section 8.

REMARK 1 *The problem wether there is a double–exponential upper bound for degrees of elements of the reduced Janet basis with respect to an arbitrary admissible linear order on monomials in the case $l > 1$ remains open. Notice that one can get a description of all the admissible linear orders on monomials from A^l: each linear order corresponds to a rooted tree. But we don't need this description in the present paper.*

The problem of solving systems of linear equations over the homogenized Weyl algebra is central in this paper, see Theorem 2. It is studied in Sections 5–7. A similar problem over the Weyl algebra (without a homogenization) was considered in [7]. The principal idea is to try to extend the well known method from [8] which was elaborated for the algebra of polynomials, to the homogenized Weyl algebra. There are two principal difficulties on this way. The first one is that in the method from [8] the use of determinants is essential which one has to avoid dealing with non-commutative algebras. The second is that one needs a kind of the Noether normalization theorem in the situation under consideration. So it is necessary to choose the leading elements in the analog of the method from [8] with the least possible order ord_{X_0}, where X_0 is a homogenizing variable, see Section 3.

The obtained bound on the degree of a Janet basis implies a similar bound on the complexity of its constructing. Indeed, by Corollary 1 (it is formulated for the case of Weyl algebra but the analogous corollary holds for the case of algebra of differential operators) one can compute the linear space of all the elements $z \in I$ of degrees bounded from above by $(dl)^{2^{O(n)}}$. Hence by Theorem 1 one can compute a Janet basis of I solving linear systems over F of size bounded from above $(dl)^{2^{O(n)}}$ (just by the enumeration of all monomials of degrees at most $(dl)^{2^{O(n)}}$ which are possible elements of $\mathrm{Hdt}(I)$). After that one can compute within the time polynomial in $(dl)^{2^{O(n)}}$ and the size of the input solving linear systems over F also the reduced Janet basis of I provided that the upper bound $(dl)^{2^{O(n)}}$ for the degrees of its elements is known, see Theorem 1.

For the sake of self–containedness in Appendix 1, see Lemma 12, we give a short proof of the double–exponential estimation for stabilization of the Hilbert function of a graded module over a graded polynomial ring. A conversion of Lemma 12 also holds, see Appendix 1 Lemma 13. It is essential for us. The proof of Lemma 13 uses the classic description of the Hilbert function of a homogeneous ideal in $F[X_0, \ldots, X_n]$ via Macaulay constants $b_{n+2}, \ldots, b_1$ and the constant b_0 introduced in [4]. In Appendix 2 we give an independent and instructive proof of Proposition 1 which is similar to Lemma 13. In some sense Proposition 1 is even more strong than Lemma 13 since to apply it one does not need a bound for the stabilization of the Hilbert function. Of course, the reference to Proposition 1 can be used in place of Lemma 13 in our paper.

1 Definition of the Janet basis

Let $A = F[X_1, \ldots, X_n, D_1, \ldots, D_n]$, $n \geqslant 1$, be a Weyl algebra over a field F. So A is defined by the following relations

$$X_v X_w = X_w X_v,\ D_v D_w = D_w D_v,\ D_v X_v - X_v D_v = 1,\ X_v D_w = D_w X_v, \quad v \neq w. \tag{2}$$

By (2) any element $f \in A$ can be uniquely represented in the form

$$f = \sum_{i_1,\ldots,i_n,j_1,\ldots,j_n \geqslant 0} f_{i_1,\ldots,i_n,j_1,\ldots,j_n} X_1^{i_1} \ldots X_n^{i_n} D_1^{j_1} \ldots D_n^{j_n}, \tag{3}$$

where all $f_{i_1,\ldots,i_n,j_1,\ldots,j_n} \in F$ and only a finite number of $f_{i_1,\ldots,i_n,j_1,\ldots,j_n}$ are nonzero. Denote for brevity $\mathbb{Z}_+ = \{z \in \mathbb{Z} : z \geqslant 0\}$ to the set of all nonnegative integers and

$$\begin{array}{l} i = (i_1, \ldots, i_n), \quad j = (j_1, \ldots, j_n), \quad f_{i,j} = f_{i_1,\ldots,i_n,j_1,\ldots,j_n} \\ X^i = X_1^{i_1} \ldots X_n^{i_n}, \quad D^j = D_1^{j_1} \ldots D_n^{j_n}, \quad f = \sum_{i,j} f_{i,j} X^i D^j, \\ |i| = i_1 + \ldots + i_n, \quad i + j = (i_1 + j_1, \ldots, i_n + j_n). \end{array} \tag{4}$$

So $i, j \in \mathbb{Z}_+^n$ are multiindices. By definition the degree of f

$$\deg f = \deg_{X_1,\ldots,X_n,D_1,\ldots,D_n} f = \max\{|i| + |j| \,:\, f_{i,j} \neq 0\}.$$

Let M be a left A-module given by its generators $m_1, \ldots, m_l$, $l \geqslant 0$, and relations

$$\sum_{1 \leqslant w \leqslant l} a_{v,w} m_w, \quad 1 \leqslant v \leqslant k. \tag{5}$$

where $k \geqslant 0$ and all $a_{v,w} \in A$. We assume that $\deg a_{v,w} \leqslant d$ for all v, w. By (5) we have the exact sequence

$$A^k \xrightarrow{\iota} A^l \xrightarrow{\pi} M \to 0 \tag{6}$$

of left A-modules. Denote $I = \iota(A^k) \subset A^l$. If $l = 1$ then I is a left ideal of A and $M = A/I$. In the general case I is generated by the elements

$$(a_{v,1}, \ldots, a_{v,l}) \in A^l, \quad 1 \leqslant v \leqslant k.$$

For an integer $m \geqslant 0$ put

$$A_m = \{a \,:\, \deg a \leqslant m\}, \quad M_m = \pi(A_m^l), \quad I_m = I \cap A_m^l. \tag{7}$$

So now A, M, I are filtered modules with filtrations A_m, M_m, I_m, $m \geqslant 0$, respectively and the sequence of homomorphisms of vector spaces

$$0 \to I_m \to A_m^l \to M_m \to 0$$

induced by (6) is exact for every $m \geqslant 0$. The Hilbert function $H(M, m)$ of the module M is defined by the equality

$$H(M, m) = \dim_F M_m, \quad m \geqslant 0.$$

Each element of A^l can be uniquely represented as an F-linear combination of elements $e_{v,i,j} = (0, \ldots, 0, X^i D^j, 0, \ldots, 0)$, herewith $i, j \in \mathbb{Z}_+^n$ are multiindices, see (4), and the nonzero monomial $X^i D^j$ is at the position v, $1 \leqslant v \leqslant l$. So every element $f \in A^l$ can be represented in the form

$$f = \sum_{v,i,j} f_{v,i,j} e_{v,i,j}, \quad f_{v,i,j} \in F. \tag{8}$$

The elements $e_{v,i,j}$ will be called monomials.

Consider a linear order $<$ on the set of all the monomials $e_{v,i,j}$ or which is the same on the set of triples (v, i, j), $1 \leqslant v \leqslant l$, $i, j \in \mathbb{Z}_+^n$. If $f \neq 0$ put

$$o(f) = \max\{(v, i, j) \,:\, f_{v,i,j} \neq 0\}, \tag{9}$$

see (8). Set

$$o(0) = -\infty < o(f)$$

for every $0 \neq f \in A$. Let us define the leading monomial of the element $0 \neq f \in A^l$ by the formula

$$\mathrm{Hdt}(f) = f_{v,i,j} e_{v,i,j},$$

where $o(f) = (v, i, j)$. Put $\mathrm{Hdt}(0) = 0$. Hence $o(f - \mathrm{Hdt}(f)) < o(f)$ if $f \neq 0$. For $f_1, f_2 \in A^l$ if $o(f_1) < o(f_2)$ we shall write $f_1 < f_2$. We shall require additionally that

(a) for all multiindices i, j, i', j' for all $1 \leqslant v \leqslant l$ if $i_1 \leqslant i'_1, \ldots, i_n \leqslant i'_n$ and $j_1 \leqslant j'_1, \ldots, j_n \leqslant j'_n$ then $(v, i, j) \leqslant (v, i', j')$.

(b) for all multiindices i, j, i', j', i'', j'' for all $1 \leqslant v, v' \leqslant l$ if $(v, i, j) < (v', i', j')$ then $(v, i + i'', j + j'') < (v', i' + i'', j' + j'')$.

Conditions (a) and (b) imply that for all $f_1, f_2 \in A^l$ for every nonzero $a \in A$ if $f_1 < f_2$ then $af_1 < af_2$, i.e., the considered linear order is compatible with the products. Any linear order on monomials $e_{v,i,j}$ satisfying (a) and (b) will be called *admissible*. Consider additionally condition

(c) for all multiindices i, j, i', j' for all $1 \leqslant v, v' \leqslant l$ if $|i| + |j| < |i'| + |j'|$ then $(v, i, j) < (v', i', j')$.

Any linear order on monomials $e_{v,i,j}$ satisfying (a), (b) and (c) will be called *degree-compatible.*

For every subset $E \subset A^l$ put $\mathrm{Hdt}(E) = \{\mathrm{Hdt}(f) \,:\, f \in E\}$. In particular,

$$\mathrm{Hdt}(I) = \{\mathrm{Hdt}(f) \,:\, f \in I\}.$$

So $\mathrm{Hdt}(I)$ is a subset of A^l. By definition the family $f_1, \ldots, f_m$ of elements of I is a Janet basis of the module I if and only if

1) $\mathrm{Hdt}(I) = \mathrm{Hdt}(Af_1) \cup \ldots \cup \mathrm{Hdt}(Af_m)$.

Further, the Janet basis $f_1, \ldots, f_m$ of I is reduced if and only if the following conditions hold.

2) $f_1, \ldots, f_m$ does not contain a smaller Janet basis of I,

3) $\mathrm{Hdt}(f_1) > \ldots > \mathrm{Hdt}(f_m)$.

4) The coefficient from F of every monomial $\mathrm{Hdt}(f_v)$, $1 \leqslant \alpha \leqslant m$, is 1.

5) Let $f_\alpha = \sum_{v,i,j} f_{\alpha,v,i,j} e_{v,i,j}$ be representation (3) for f_α, $1 \leqslant \alpha \leqslant m$. Then for all $1 \leqslant \alpha < \beta \leqslant m$ for all $1 \leqslant v \leqslant l$ and multiindices i, j the monomial $f_{\alpha,v,i,j} e_{v,i,j} \notin \mathrm{Hdt}(Af_\beta \setminus \{0\})$.

Denote by C the ring of polynomials in $X_1, \ldots, X_n$, $D_1, \ldots, D_n$ with coefficients from F (one can take $C = \mathrm{gr}(A)$, see the next section). For every $f \in A^l$ the monomial $\mathrm{Hdt}(f)$ can be considered as an element of C^l. To avoid an ambiguity denote it by $\mathrm{Hdtc}(f) \in C^l$. Now $f_1, \ldots, f_m$ is a Janet basis of the module I if and only if the C-submodule of C^l generated by $\mathrm{Hdtc}(f_\alpha)$, $1 \leqslant \alpha \leqslant m$, contains all the elements $\mathrm{Hdtc}(f)$, $f \in A$. Since the ring C is Noetherian for the considered I there exists a Janet basis. Further the reduced Janet basis of I is uniquely defined.

2 The graded module corresponding to a D–module

Put $A_v = I_v = M_v = 0$ for $v < 0$ and

$$\mathrm{gr}(A) = \oplus_{m \geqslant 0} A_m/A_{m-1},\ \mathrm{gr}(I) = \oplus_{m \geqslant 0} I_m/I_{m-1},\ \mathrm{gr}(M) = \oplus_{m \geqslant 0} M_m/M_{m-1}.$$

The structure of the algebra on A induces the structure of a graded algebra on $\mathrm{gr}(A)$. So $\mathrm{gr}(A) = F[X_1, \dots, X_n, D_1, \dots, D_n]$ is an algebra of polynomials with respect to the variables $X_1, \dots, X_n$, $D_1, \dots, D_n$. Further, $\mathrm{gr}(I)$ and $\mathrm{gr}(M)$ are graded $\mathrm{gr}(A)$-modules. From (7) we get the exact sequences

$$0 \to I_m/I_{m-1} \to (A_m/A_{m-1})^l \to M_m/M_{m-1} \to 0, \quad m \geqslant 0. \tag{10}$$

The Hilbert function of the module $\mathrm{gr}(M)$ is defined as follows

$$H(\mathrm{gr}(M), m) = \dim_F M_m/M_{m-1}, \quad m \geqslant 0.$$

Obviously

$$H(M,m) = \sum_{0 \leqslant v \leqslant m} H(\mathrm{gr}(M), v), \quad H(\mathrm{gr}(M), m) = H(M,m) - H(M, m-1). \tag{11}$$

for every $m \geqslant 0$.

Denote for an arbitrary $a \in M$ by $\mathrm{gr}(a) \in \mathrm{gr}(M)$ the image of a in $\mathrm{gr}(M)$.

LEMMA 1 *Assume that $b_1, \dots, b_s$ is a system of generators of I. Let $\nu_i = \deg b_i$, $1 \leqslant i \leqslant s$. Suppose that for every $m \geqslant 0$*

$$I_m = \Big\{ \sum_{1 \leqslant v \leqslant \mu} c_v b_v \, : \, c_v \in A, \quad \deg c_v \leqslant m - \nu_v, \quad 1 \leqslant i \leqslant s \Big\}. \tag{12}$$

Then $\mathrm{gr}(b_1), \dots, \mathrm{gr}(b_s)$ is a system of generators of the $\mathrm{gr}(A)$-module $\mathrm{gr}(I)$.

PROOF This is straightforward.

3 Homogenization of the Weyl algebra

Let X_0 be a new variable. Consider the algebra ${}^hA = F[X_0, X_1, \dots, X_n, D_1, \dots, D_n]$ given by the relations

$$\begin{array}{l} X_v X_w = X_w X_v, \ D_v D_w = D_w D_v, \quad \text{for all} \quad v, w, \\ D_v X_v - X_v D_v = X_0^2, \ 1 \leqslant v \leqslant n, \quad X_v D_w = D_w X_v \quad \text{for all} \quad v \neq w. \end{array} \tag{13}$$

The algebra ${}^hA$ is Noetherian similarly to the Weyl algebra A. By (13) an element $f \in {}^hA$ can be uniquely represented in the form

$$f = \sum_{i_0, i_1, \dots, i_n, j_1, \dots, j_n \geqslant 0} f_{i_0, \dots, i_n, j_1, \dots, j_n} X_0^{i_0} \dots X_n^{i_n} D_1^{j_1} \dots D_n^{j_n}, \tag{14}$$

where all $f_{i_0, \dots, i_n, j_1, \dots, j_n} \in F$ and only a finite number of $f_{i_0, \dots, i_n, j_1, \dots, j_n}$ are nonzero. Let i, j be multiindices, see (4). Denote for brevity

$$\begin{array}{l} i = (i_1, \dots, i_n), \quad j = (j_1, \dots, j_n), \quad f_{i_0, i, j} = f_{i_0, \dots, i_n, j_1, \dots, j_n} \\ f = \sum_{i_0, i, j} f_{i_0, i, j} X_0^{i_0} X^i D^j. \end{array} \tag{15}$$

By definition the degrees of f

$$\begin{array}{l} \deg f = \deg_{X_0, \dots, X_n, D_1, \dots, D_n} f = \max\{i_0 + |i| + |j| \, : \, f_{i_0, i, j} \neq 0\}, \\ \deg_{D_1, \dots, D_n} f = \max\{|j| \, : \, f_{i_0, i, j} \neq 0\}, \\ \deg_{D_\alpha} f = \max\{j_\alpha \, : \, f_{i_0, i, j} \neq 0\}, \quad 1 \leqslant \alpha \leqslant n, \\ \deg_{X_\alpha} f = \max\{i_\alpha \, : \, f_{i_0, i, j} \neq 0\}, \quad 1 \leqslant \alpha \leqslant n. \end{array}$$

Set $\operatorname{ord} 0 = \operatorname{ord}_{X_0} 0 = +\infty$. If $0 \neq f \in {}^hA$ then put

$$\operatorname{ord} f = \operatorname{ord}_{X_0} f = \mu \Longleftrightarrow f \in X_0^\mu({}^hA) \setminus X_0^{\mu+1}({}^hA), \quad \mu \geqslant 0. \tag{16}$$

For every $z = (z_1, \ldots, z_l) \in {}^hA^l$ put

$$\operatorname{ord} z = \min_{1 \leqslant i \leqslant l} \{\operatorname{ord} z_i\}, \quad \deg z = \max_{1 \leqslant i \leqslant l} \{\deg z_i\}.$$

Similarly one defines $\operatorname{ord} b$ and $\deg b$ for an arbitrary $(k \times l)$-matrix b with coefficients from ${}^hA$. More precisely, one consider here b as a vector with kl entries.

The element $f \in {}^hA$ is homogeneous if and only if $f_{i_0,i,j} \neq 0$ implies $i_0 + |i| + |j| = \deg f$, i.e., if and only if f is a sum of monomials of the same degree $\deg f$. The homogeneous degree of a nonzero homogeneous element f is its degree. The homogeneous degree of 0 is not defined (0 belongs to all the homogeneous components of ${}^hA$, see below).

Further, the m-th homogeneous component of ${}^hA$ is the F-linear space

$$({}^hA)_m = \left\{ z \in {}^hA \,:\, z \text{ is homogeneous} \;\&\; \deg z = m \text{ or } z = 0 \right\}$$

for every integer m. Now ${}^hA$ is a graded ring with respect to the homogeneous degree. By definition the ring ${}^hA$ is a homogenization of the Weyl algebra A.

We shall consider the category of finitely generated graded modules G over the ring ${}^hA$. Such a module $G = \oplus_{m \geqslant m_0} G_m$ is a direct sum of its homogeneous components G_m, where m, m_0 are integers. Every G_m is a finite dimensional F-linear space and $({}^hA)_p G_m \subset G_{p+m}$ for all integers p, m. If G and G' are two finitely generated graded ${}^hA$-modules then $\varphi : G \to G'$ is a morphism (of degree 0) of the graded modules if and only if φ is a morphism of ${}^hA$-modules and $\varphi(G_m) \subset G'_m$ for every integer m.

The element $z \in {}^hA$ (respectively $z \in A$) is called to be the term if and only if $z = \lambda z_1 \cdot \ldots \cdot z_\nu$ for some $0 \neq \lambda \in F$, integer $\nu \geqslant 0$ and $z_w \in \{X_0, \ldots, X_n, D_1, \ldots, D_n\}$ (respectively $z_w \in \{X_1, \ldots, X_n, D_1, \ldots, D_n\}$), $1 \leqslant w \leqslant \nu$.

Let $z = \sum_j z_j \in A$ be an arbitrary element of the Weyl algebra A represented as a sum of terms z_j and $\deg z = \max_j \deg z_j$. One can take here, for example, representation (3) for z. Then we define the homogenization ${}^hz \in {}^hA$ by the formula

$${}^hz = \sum_j z_j X_0^{\deg z - \deg z_j}.$$

By (2), (13) the right part of the last equality does not depend on the chosen representation of z as a sum of terms. Hence ${}^hz$ is defined correctly. If $z \in {}^hA$ then ${}^az \in A$ is obtained by substituting $X_0 = 1$ in z. Hence for every $z \in A$ we have ${}^{ah}z = z$, and for every $z \in {}^hA$ the element $z = {}^{ha}z X_0^\mu$, where $\mu = \operatorname{ord} z$.

For an element $z = (z_1, \ldots, z_l) \in A^l$ put $\deg z = \max_{1 \leqslant i \leqslant l}\{\deg z_i\}$ and

$${}^hz = \left({}^hz_1 X_0^{\deg z - \deg z_1}, \ldots, {}^hz_l X_0^{\deg z - \deg z_l} \right) \in {}^hA^l.$$

Similarly one defines $\deg a$ and the homogenization ${}^ha$ for an arbitrary $(k \times l)$-matrix $a = (a_{v,w})_{1 \leqslant v \leqslant k,\, 1 \leqslant w \leqslant l}$ with coefficients from A. More precisely, one consider here a as a vector with kl entries. Hence if $b = (b_{v,w})_{1 \leqslant v \leqslant k,\, 1 \leqslant w \leqslant l} = {}^ha$ then $b_{v,w} = {}^ha_{v,w} X_0^{\deg a - \deg a_{v,w}}$ for all v, w.

Further, the m-th homogeneous component of ${}^hA^l$ is

$$({}^hA^l)_m = \left\{ {}^hz \,:\, z \in A^l \;\&\; \deg z = m \text{ or } z = 0 \right\}$$

For an F-linear subspace $X \subset A^l$ put ${}^hX$ to be the least linear subspace of ${}^hA^l$ containing the set $\{{}^hz \,:\, z \in X\}$. If X is a A-submodule of A^l then ${}^hX$ is a graded submodule of ${}^hA^l$. The graduation on ${}^hX$ is induced by the one of ${}^hA^l$.

For an element $z = (z_1, \ldots, z_l) \in {}^hA^l$ put ${}^az = ({}^az_1, \ldots, {}^az_l) \in A^l$. For a subset $X \subset {}^hA^l$ put ${}^aX = \{{}^az \,:\, z \in X\} \subset A^l$. If X is a F-linear space then ${}^aX$ is also a F-linear space. If X is a graded submodule of ${}^hA^l$ then ${}^aX$ is a submodule of A^l.

Now ${}^hI$ is a graded submodule of ${}^hA^l$. Further, ${}^{ah}I = I$. Let $({}^hI)_m$ be the m-th homogeneous component of ${}^hI$. Then

$$ {}^h(I_m) = \oplus_{0 \leqslant j \leqslant m} ({}^hI)_j, \quad m \geqslant 0, \tag{17}$$
$$ {}^a(({}^hI)_m) = I_m, \quad m \geqslant 0. \tag{18}$$

and (18) induces the isomorphism $\iota \,:\, ({}^hI)_m \to I_m$ of linear spaces over F. Set ${}^hM = {}^hA^l/{}^hI$. Hence ${}^hM$ is a graded ${}^hA$-module and we have the exact sequence

$$0 \to {}^hI \to {}^hA^l \to {}^hM \to 0. \tag{19}$$

Now the m-th homogeneous component $({}^hM)_m$ of ${}^hM$

$$({}^hM)_m = ({}^hA^l)_m/({}^hI)_m \simeq A^l_m/I_m \tag{20}$$

by the isomorphism ι. We have the exact sequences

$$0 \to ({}^hI)_m \to ({}^hA^l)_m \to ({}^hM)_m \to 0, \quad m \geqslant 0. \tag{21}$$

By definition the Hilbert function of the module ${}^hM$ is

$$H({}^hM, m) = \dim_F ({}^hM)_m, \quad m \geqslant 0.$$

By (20) we have $H(M, m) = H({}^hM, m)$ for every $m \geqslant 0$, i.e., the Hilbert functions of M and ${}^hM$ coincide.

LEMMA 2 *Let $b_1, \ldots, b_s$ be a system of homogeneous generators of the ${}^hA$-module ${}^hI$. Then*

$$\mathrm{gr}({}^ab_1), \ldots, \mathrm{gr}({}^ab_s) \in \mathrm{gr}(A)^l$$

is a system of generators of $\mathrm{gr}(A)$*-module* $\mathrm{gr}(I)$.

PROOF By (18) ${}^a(({}^hI)_m) = I_m$. Now the required assertion follows from Lemma 1. The lemma is proved.

4 The Janet bases of a module and of its homogenization

Each element of ${}^hA^l$ can be uniquely represented as an F-linear combination of elements $e_{v,i_0,i,j} = (0, \ldots, 0, X_0^{i_0} X^i D^j, 0, \ldots, 0)$, herewith $0 \leqslant i_0 \in \mathbb{Z}$, $i, j \in \mathbb{Z}^n_+$

are multiindices, see (4), and the nonzero monomial $X_0^{i_0}X^iD^j$ is at the position v, $1 \leqslant v \leqslant l$. So every element $f \in {}^hA^l$ can be represented in the form

$$f = \sum_{v,i_0,i,j} f_{v,i_0,i,j}e_{v,i_0,i,j}, \quad f_{v,i_0,i,j} \in F. \tag{22}$$

and only a finite number of $f_{v,i_0,i,j}$ are nonzero. The elements $e_{v,i_0,i,j}$ will be called monomials.

Let us replace everywhere in Section 1 after the definition of the Hilbert function the ring A, the monomials $e_{v,i,j}$, the multiindices i, i', i'', triples (v,i,j), (v,i',j'), the module I and so on by the ring ${}^hA$, monomials $e_{v,i_0,i,j}$, the pairs (i_0,i), (i'_0,i'), (i''_0,i'') (they are used without parentheses), quadruples (v,i_0,i,j), (v,i'_0,i',j'), the homogenization ${}^hI$ and so on respectively. Thus, we get the definitions of $o(f)$, $\mathrm{Hdt}(f)$ for $f \in {}^hA^l$, new conditions (a) and (b) which define admissible linear order on the monomials of ${}^hA^l$, new condition (c) and the definition of the degree-compatible linear order, new conditions 1)–5), the definitions of the set $\mathrm{Hdt}({}^hI)$, Janet basis and reduced Janet basis of ${}^hI$. For example, $o(0) = +\infty$, $\mathrm{Hdt}(0) = 0$, and if $0 \neq f \in {}^hA^l$ then

$$\begin{aligned}
&o(f) = \max\{(v,i_0,i,j) \,:\, f_{v,i_0,i,j} \neq 0\},\\
&\mathrm{Hdt}(f) = f_{v,i_0,i,j}e_{v,i_0,i,j}, \quad \text{where} \quad o(f) = (v,i_0,i,j),\\
&\mathrm{Hdt}({}^hI) = \{\mathrm{Hdt}(f) \,:\, f \in {}^hI\}.
\end{aligned}$$

the new conditions (a) and (b) are the following:

(a) for all indices i_0, i'_0, all multiindices i,j,i',j' for all $1 \leqslant v \leqslant l$ if $i_0 \leqslant i'_0$, $i_1 \leqslant i'_1, \ldots, i_n \leqslant i'_n$ and $j_1 \leqslant j'_1, \ldots, j_n \leqslant j'_n$ then $(v,i_0,i,j) \leqslant (v,i'_0,i',j')$.

(b) for all indices i_0, i'_0, i''_0, all multiindices i,j,i',j',i'',j'' for all $1 \leqslant v, v' \leqslant l$ if $(v,i_0,i,j) < (v',i'_0,i',j')$ then $(v, i_0+i''_0, i+i'', j+j'') < (v', i'_0+i''_0, i'+i'', j'+j'')$.

The existence of a Janet basis of ${}^hI$ and the uniqueness of the reduced Janet basis with respect to an admissible linear order are proved similarly to the existence of a Janet basis of I and the uniqueness of the reduced Janet basis I, see Section 1. The Janet basis of ${}^hI$ is homogeneous if and only if it consists of homogeneous elements from ${}^hA^l$. Since the module ${}^hI$ is homogeneous, the family of homogeneous components of any Janet basis of ${}^hI$ is a homogeneous Janet basis of ${}^hI$. Hence the reduced Janet basis of ${}^hI$ is homogeneous (here we leave the details to the reader).

Let $<$ be an admissible linear order on the monomials from A^l, or which is the same, on the triples (v,i,j), see Section 1. So $<$ satisfies conditions (a) and (b). Let us define the linear order on the monomials $e_{v,i_0,i,j}$ or, which is the same, on the quadruples (v,i_0,i,j). This linear order is induced by $<$ on the triples (v,i,j) and will be denoted again by $<$. Namely, for two quadruples (v,i_0,i,j) and (v',i'_0,i',j') put $(v,i_0,i,j) < (v',i'_0,i',j')$ if and only if $(v,i,j) < (v',i',j')$, or $(v,i,j) = (v',i',j')$ but $i_0 < i'_0$. Notice that this induced linear order satisfies conditions (a) and (b) (in the new sense).

REMARK 2 *If $f_1, \ldots, f_m$ is a Janet basis of I (respectively homogeneous Janet basis of ${}^hI$) satisfying 1)–4) then there are the unique $c_{\alpha,\beta} \in A$ (respectively* *homogeneous* $c_{\alpha,\beta} \in {}^hA$*),* $1 \leqslant \alpha < \beta \leqslant m$*, such that*

$$f_\alpha + \sum_{\alpha<\beta\leqslant m} c_{\alpha,\beta} f_\beta, \quad 1 \leqslant \alpha \leqslant m,$$

is a reduced Janet basis of I *(respectively reduced homogeneous Janet basis of* ${}^hI$*), cf. [3].*

Obviously a linear order $<$ on monomials from A^l (respectively ${}^hA^l$) is degree compatible if and only if for any two monomials z_1, z_2 the inequality $\deg z_1 < \deg z_2$ implies $z_1 < z_2$.

LEMMA 3 *The following assertions hold.*

(i) Let $f_1, \ldots, f_m$ *be a (reduced) Janet basis of* I *with respect to the linear order* $<$ *and additionally the order* $<$ *is degree-compartible. Then* ${}^hf_1, \ldots, {}^hf_m$ *is a (reduced) homogeneous Janet basis of the module* ${}^hI$ *with respect to the induced linear order* $<$*.*

(ii) Conversely, suppose that the initial order $<$ *is degree-compartible, and* $g_1, \ldots, g_m$ *be a (reduced) homogeneous Janet basis of the module* ${}^hI$ *with respect to the induced linear order* $<$*. Then* ${}^ag_1, \ldots, {}^ag_m$ *is a (reduced) Janet basis of* I *with respect to the linear order* $<$*.*

(iii) Suppose that the initial order $<$ *is arbitrary admissible. Let* $g_1, \ldots, g_m$ *be a homogeneous Janet basis of the module* ${}^hI$ *with respect to the induced linear order* $<$*. Then* ${}^ag_1, \ldots, {}^ag_m$ *is a Janet basis of* I *with respect to the linear order* $<$*. Moreover,* ${}^{ha}g_w = g_w$ *for all* $1 \leqslant w \leqslant m$*.*

PROOF This follows immediately from the definitions.

Let $f \in {}^hA^l$ and the module ${}^hI$ be as above. Let us show that there is the unique element $g \in {}^hA^l$ such that

$$g = \sum_{v,i_0,i,j} g_{v,i_0,i,j} e_{v,i_0,i,j}, \quad g_{v,i_0,i,j} \in F, \tag{23}$$

$f - g \in {}^hI$ and if $g_{v,i_0,i,j} \neq 0$ then $e_{v,i_0,i,j} \notin \mathrm{Hdt}({}^hI)$. Indeed, if there are two such elements $g \neq g'$ then $0 \neq g - g' \in {}^hI$ but $\mathrm{Hdt}(g - g') \notin \mathrm{Hdt}({}^hI)$ and we get a contradiction. To prove the existence of g we shall suppose without loss of generality that f is homogeneous and show additionally that the sum in the left part if (23) is taken over (v, i_0, i, j) such that $i_0 + |i| + |j| = \deg f$. One can represent

$$f = \sum_{v,i_0,i,j} f_{v,i_0,i,j} e_{v,i_0,i,j}, \quad f_{v,i_0,i,j} \in F,\ i_0 + |i| + |j| = \deg f.$$

We use the induction on the number $\nu(f)$ of (v, i_0, i, j) in the last sum such that $e_{v,i_0,i,j} \in \mathrm{Hdt}({}^hI)$ and $e_{v,i_0,i,j} \leqslant \mathrm{Hdt}(f)$. If $\nu(f) > 0$ then there is a homogeneous $z \in {}^hI$ such that $\mathrm{Hdt}(z) = \mathrm{Hdt}(f)$, $\deg z = \deg f$. Then $\nu(f - z) < \nu(f)$. The required assertion is proved.

The element g from (23) is called the normal form of f with respect to the module ${}^hI$. We shall denote $g = \mathrm{nf}({}^hI, f)$. Obviously $\mathrm{nf}({}^hI, ({}^hA^l)_m) \subset ({}^hA^l)_m$ is a linear subspace and

$$\dim_F \mathrm{nf}({}^hI, ({}^hA^l)_m) = l\binom{m+2n}{2n} - H({}^hI, m) = H({}^hA^l/{}^hI, m).$$

Let ${}^cA = F[X_0, \ldots, X_n, D_1, \ldots, D_n]$ be the polynomial ring in the variables $X_0, \ldots, X_n, D_1, \ldots, D_n$. Each monomial $e_{v,i_0,i,j}$ can be considered also as an element of ${}^cA^l$. Hence $\mathrm{Hdt}(f)$ can be considered as an element of ${}^cA^l$ for every $f \in {}^hA^l$. To avoid an ambiguity we shall denote it by $\mathrm{Hdtc}(f) \in {}^cA^l$. Put $\mathrm{Hdtc}({}^hI) = \{\mathrm{Hdtc}(f) : f \in {}^hI\}$. So the sets $\mathrm{Hdt}({}^hI)$ and $\mathrm{Hdtc}({}^hI)$ are in the one–to–one correspondence.

Denote by ${}^cI \subset {}^cA^l$ the graded submodule of ${}^cA^l$ generated by $\mathrm{Hdtc}({}^hI)$. Then one can easily see that the set of monomials from the module ${}^cI$ coincides with $\mathrm{Hdtc}({}^hI)\setminus\{0\}$. Further, for every $m \geqslant 0$ the F-linear space of homogeneous elements ${}^cI_m$ is generated by the monomials $e_{v,i_0,i,j}$ such that there is $0 \neq f \in {}^hI_m$ with $o(f) = (v, i_0, i, j)$. The Hilbert function

$$H({}^cI, m) = \dim_F\{(z_1, \ldots, z_l) \in {}^cI \,:\, \forall\, i\,(\deg z_i = m \quad \text{or} \quad z_i = 0\,)\},$$
$$H({}^cA^l/{}^cI, m) = l\binom{m+2n}{2n} - H({}^cI, m).$$

Let $f \in {}^cA^l$ and the module ${}^cI$ be as above. Then there is the unique element $g \in A^l$ such that

$$g = \sum_{v,i_0,i,j} g_{v,i_0,i,j} e_{v,i_0,i,j}, \quad g_{v,i_0,i,j} \in F,$$

$f - g \in {}^hI$ and if $g_{v,i_0,i,j} \neq 0$ then $e_{v,i_0,i,j} \notin \mathrm{Hdtc}({}^hI)$ (the proof is similar to the one of the existence and uniqueness of g from (23)). The element g is called the normal form of f with respect to the module ${}^cI$, cf. [4]. We shall denote $g = \mathrm{nf}({}^cI, f)$. Obviously, $\mathrm{nf}({}^cI, ({}^cA^l)_m) \subset ({}^cA^l)_m$ is a linear subspace and

$$\dim_F \mathrm{nf}({}^cI, ({}^cA^l)_m) = l\binom{m+2n}{2n} - H({}^cI, m) = H({}^cA^l/{}^cI, m).$$

Since by the given definitions the F-linear spaces $\mathrm{nf}({}^cI, ({}^cA^l)_m)$ and $\mathrm{nf}({}^hI, ({}^hA^l)_m)$ are generated by the same monomials we have for every $m \geqslant 0$

$$\dim_F \mathrm{nf}({}^cI, ({}^cA^l)_m) = \dim_F \mathrm{nf}({}^hI, ({}^hA^l)_m,$$
$$H({}^hA^l/{}^hI, m) = H({}^cA^l/{}^cI, m), \quad H({}^hI, m) = H({}^cI, m).$$

Therefore, see Section 3,

$$H(I, m) = H({}^cI, m), \quad m \geqslant 0. \tag{24}$$

5 Bound on the kernel of a matrix over the homogenized Weyl algebra

LEMMA 4 *Let $k \geqslant 1$ and $l \geqslant 1$ be integers. Let $b = (b_{i,j})_{1\leqslant i\leqslant k,\, 1\leqslant j\leqslant l}$ be a matrix where $b_{i,j} \in {}^hA$ are homogeneous elements for all i, j. Let $\deg b_{i,j} < d$,*

$d \geqslant 2$, for all i, j. Assume that there are integers $d_j \geqslant 0$, $1 \leqslant i \leqslant k$, and $d'_i \geqslant 0$, $1 \leqslant j \leqslant l$, such that

$$\deg b_{i,j} = d_i - d'_j \tag{25}$$

for all nonzero $b_{i,j}$, and additionally d'_j are chosen to be minimal possible (this means that there do not exist integers $\widetilde{d}_i, \widetilde{d'_j}$ similar to d_i, d'_j such that $\widetilde{d'_j} \leqslant d'_j$ for all $1 \leqslant j \leqslant l$ and at least one of the last inequalities is strict). Then $d_i < \min\{k+1, l\}d$, $d'_j < \min\{k, l-1\}d$ for all i, j

Further, assume that $k = l - 1$. Then there are homogeneous elements $z_1, \ldots, z_l \in {}^hA$ such that $(z_1, \ldots, z_l) \neq (0, \ldots, 0)$,

$$\sum_{1 \leqslant j \leqslant l} b_{i,j} z_j = 0, \quad 1 \leqslant i \leqslant l - 1. \tag{26}$$

There is an integer $\mu \geqslant 0$ such that for all $1 \leqslant j \leqslant l - 1$ if $z_j \neq 0$ then $\deg z_j = \mu + d'_j$ and hence all nonzero $b_{i,j} z_j$ have the same degree depending only on i. Further,

$$\deg z_j \leqslant (2n+2)l \max_{1 \leqslant i \leqslant k} \{d_i\} < (2n+2)l^2 d, \quad 1 \leqslant j \leqslant l. \tag{27}$$

Besides that, if all $b_{i,j}$ do not depend on X_n (i.e., they can be represented as sums of monomials which do not contain X_n) then one can choose also $z_1, \ldots, z_l$ satisfying additionally the same property. Finally, dividing by an appropriate power of X_0 one can assume without loss of generality that $\min\{\operatorname{ord} z_i : 1 \leqslant i \leqslant l\} = 0$.

PROOF Let us prove at first that $d_i < \min\{k+1, l\}d$ and $d'_j < \min\{k, l-1\}d$ for all i, j and arbitrary $k, l \geqslant 1$. We define the relation of equivalence on the set of pairs $P = \{(v, w) : 1 \leqslant v \leqslant k \,\&\, 1 \leqslant w \leqslant l \,\&\, b_{v,w} \neq 0\}$ as follows. Put $(v, w) \sim (v', w')$ if and only if there is a sequence of pairs $(v_1, w_1), \ldots, (v_\nu, w_\nu)$, $\nu \geqslant 1$, from P such that

1) $(v, w) = (v_1, w_1)$, $(v', w') = (v_\nu, w_\nu)$,

2) $v_\alpha = v_{\alpha+1}$ or $w_\alpha = w_{\alpha+1}$ for every $1 \leqslant \alpha \leqslant \nu - 1$.

Let $\pi \subset P$ be the class of equivalence with respect to $\sim$. Then there is a pair $(p, q) \in \pi$ such that $d'_q = 0$ since the numbers d'_j are chosen minimal possible. Besides that for all $(v, w), (v', w') \in \pi$ one can always choose a sequence $(v_1, w_1), \ldots, (v_\nu, w_\nu)$ as above and satisfying additionally the following five properties:

3) $(v_\alpha, w_\alpha) \neq (v_{\alpha+1}, w_{\alpha+1})$ for every $1 \leqslant \alpha \leqslant \nu - 1$,

4) if $v_\alpha = v_{\alpha+1}$ then $w_{\alpha+1} = w_{\alpha+2}$ for every $1 \leqslant \alpha \leqslant \nu - 2$,

5) if $w_\alpha = w_{\alpha+1}$ then $v_{\alpha+1} = v_{\alpha+2}$ for every $1 \leqslant \alpha \leqslant \nu - 2$,

6) $v_\alpha = v_{\alpha+1}$ for some $1 \leqslant \alpha \leqslant \nu - 1$ implies $v_\beta \neq v_\alpha$ for every $\beta \neq \alpha, \alpha + 1$,

7) $w_\alpha = w_{\alpha+1}$ for some $1 \leqslant \alpha \leqslant \nu - 1$ implies $w_\beta \neq w_\alpha$ for every $\beta \neq \alpha, \alpha + 1$,

(we leave the details to the reader). Now conditions 1)–7) imply the number of pairs

$$\#\{(w_\alpha, w_{\alpha+1}) \,:\, w_\alpha \neq w_{\alpha+1} \,\&\, 1 \leqslant \alpha \leqslant \nu-1\} \leqslant \min\{k, l-1\}.$$

Further, if $w_\alpha \neq w_{\alpha+1}$ then $v_{\alpha+1} = v_\alpha$ and $|d'_{w_{\alpha+1}} - d'_{w_\alpha}| = |\deg b_{v_{\alpha+1},w_{\alpha+1}} - \deg b_{v_\alpha,w_\alpha}| < d$. Hence $d'_{w_\nu} < \min\{k, l-1\}d + d'_{w_0}$. For $(v_0, w_0) = (p, q)$ and an arbitrary $(v, w) = (v_\nu, w_\nu) \in \pi$ we get $d'_w < \min\{k, l-1\}d$. Finally, $\deg b_{v,w} = d_v - d'_w < d$ implies $d_v < \min\{k+1, l\}d$. The required inequalities are proved.

Now suppose that $\deg b_{i,j} = \deg b$ for all nonzero $b_{i,j}$ and $k = l-1$. Let us prove the existence of $z_1, \ldots, z_l$ and obtain an estimate for $\deg z_j$ in this case. Consider the linear mapping

$$\begin{aligned} &({}^hA)^l_{m-\deg b} \longrightarrow ({}^hA)^{l-1}_m, \\ &(\,z_1, \ldots, z_l\,) \mapsto \Big(\textstyle\sum_{1\leqslant j\leqslant l} b_{i,j}z_j\Big)_{1\leqslant i\leqslant l-1}. \end{aligned} \tag{28}$$

If

$$l\binom{m-\deg b+2n}{2n} > (l-1)\binom{m+2n}{2n} \tag{29}$$

then the kernel of (28) is nonzero. But (29) holds if

$$\prod_{1\leqslant w\leqslant 2n}\Big(1+\frac{\deg b}{m+w-\deg b}\Big) < \frac{l}{l-1}. \tag{30}$$

Further, (30) is true if $(1+\deg b/(m-\deg b))^{2n} < l/(l-1)$. The last inequality follows from $m \geqslant (2n+1)\deg b/\log(l/(l-1))$. Hence also from $m \geqslant (2n+1)l\deg b$. Thus, the existence of $z_1, \ldots, z_l$ is proved, and even more all nonzero z_j have the same degree $((2n+1)l-1)\deg b$ which does not depend on j. Notice that in the considered case we prove a more strong inequality $\deg z_j \leqslant (2n+1)ld$ for all $1 \leqslant j \leqslant l$.

Finally, let $k = l-1$ and the degrees $\deg b_{i,j}$ are arbitrary satisfying (25). Multiplying the i-th equation of system (26) to $X_0^{\max_w\{d_w\}-d_i}$ we shall suppose without loss of generality that all d_i are equal. Let us substitute $z_jX_0^{d'_j}$ for z_j in (26). Now the degrees of all the nonzero coefficients of the obtained system are equal to $\max_{1\leqslant i\leqslant k}\{d_i\}$ and are less than ld. Thus, replacing in the considered above case of $\deg b_{i,j} = \deg b$ the bound d by $\max_{1\leqslant i\leqslant k}\{d_i\} < ld$ we get the required $(z_1, \ldots, z_l)$ such that $\deg z_j = ((2n+1)l-1)\max_{1\leqslant i\leqslant k}\{d_i\} + d'_j$ or $z_j = 0$ for all $1 \leqslant j \leqslant l$ and the estimation

$$\deg z_j \leqslant (2n+1)l\max_{1\leqslant i\leqslant k}\{d_i\} + d'_j < (2n+2)l\max_{1\leqslant i\leqslant k}\{d_i\} < (2n+2)l^2d$$

for all j.

Suppose that $a_1, \ldots, a_l$ do not depend on X_n. We represent $z_i = \sum_j z_{i,j}X_n^j$, $1 \leqslant i \leqslant l$, where all $z_{i,j}$ do not on X_n. Let $\alpha = \max_i\{\deg_{X_n} z_i\}$. Obviously in this case one can replace $(z_1, \ldots, z_l)$ by $(z_{1,\alpha}, \ldots, z_{l,\alpha})$. The lemma is proved.

REMARK 3 *Lemma 4 remains true if one replaces in its statement condition (26) by*

$$\sum_{1\leqslant j\leqslant l} z_jb_{i,j} = 0, \quad 1 \leqslant i \leqslant l-1, \tag{31}$$

The proof is similar.

REMARK 4 *Let the elements $b_{i,j}$ be from Lemma 4. Notice that there are integers $\delta'_i \geqslant 0$, $1 \leqslant i \leqslant k$, and $\delta_j \geqslant 0$, $1 \leqslant j \leqslant l$, such that*

$$\deg b_{i,j} = \delta_j - \delta'_i$$

for all nonzero $b_{i,j}$, and $\min_{1 \leqslant i \leqslant k}\{\delta'_i\} = 0$. Namely, $\delta'_i = -d_i + \max_{1 \leqslant i \leqslant k}\{d_i\}$, $\delta_j = -d'_j + \max_{1 \leqslant i \leqslant k}\{d_i\}$.

REMARK 5 *Let $b_{i,j} \in {}^hA$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$, be homogeneous elements and there are integers $\widetilde{d}_i$, $1 \leqslant i \leqslant k$, $\widetilde{d'_j}$, $1 \leqslant j \leqslant l$, such that the degrees $\deg b_{i,j} = \widetilde{d}_i - \widetilde{d'_j}$ for all nonzero $b_{i,j}$. Then there are also integers $d_i \geqslant 0$, $1 \leqslant i \leqslant k$, $d'_j \geqslant 0$, $1 \leqslant j \leqslant l$, such that (25) holds for all nonzero $b_{i,j}$.*

6 Transforming a matrix with coefficients from ${}^hA$ to the trapezoidal form

Let b be the matrix from Lemma 4 and integers $k, l \geqslant 1$ are arbitrary. Hence (25) holds. Let $b = (b_1, \ldots, b_l)$ where $b_1, \ldots, b_l \in {}^hA^k$ be the columns of the matrix b (notice that in Lemma 1 and Lemma 2 all b_i are rows of size l; so now we change the notation). By definition $b_1, \ldots, b_l$ are linearly independent over ${}^hA$ from the right (or just linearly independent if it will not lead to an ambiguity; in what follows in this paper if it is not stated otherwise "linearly independent" means "linearly independent from right") if and only if for all $z_1, \ldots, z_l \in {}^hA$ the equality $b_1 z_1 + \ldots + b_l z_l = 0$ implies $z_1 = \ldots = z_l = 0$. By (25) in this definition one can consider only homogeneous $z_1, \ldots, z_l$. For an arbitrary family $b_1, \ldots, b_l$ from Lemma 4 (with arbitrary k, l) one can choose a maximal linearly independent from the right subfamily $b_{i_1}, \ldots, b_{i_r}$ of $b_1, \ldots, b_l$. By Lemma 4 we have $r \leqslant k$. It turns out that r does not depend on the choice of a subfamily. More precisely, we have the following lemma.

LEMMA 5 *Let $c_j = \sum_{1 \leqslant i \leqslant l} b_i z_{i,j}$, $1 \leqslant j \leqslant r_1$, where $z_{i,j} \in {}^hA$ are homogeneous elements. Suppose that there are integers d''_j, $1 \leqslant j \leqslant r_1$, such that for all i, j the degree $\deg z_{i,j} = d'_i - d''_j$ if $z_{i,j} \neq 0$. Assume that c_j, $1 \leqslant j \leqslant r_1$, are linearly independent over ${}^hA$ from the right. Then $r_1 \leqslant r$, and if $r_1 < r$ there are $c_{r_1+1}, \ldots, c_r \in \{b_{i_1}, \ldots, b_{i_r}\}$ such that c_j, $1 \leqslant j \leqslant r$, are linearly independent over ${}^hA$ from the right.*

PROOF The proof is similar to the case of vector spaces over a field and we leave it to the reader.

We denote $r = \mathrm{rankr}\{b_1, \ldots, b_l\}$ and call it the rank from the right of $b_1, \ldots, b_l$. In the similar way one can define rank from the left of $b_1, \ldots, b_l$. Denote it by $\mathrm{rankl}\{b_1, \ldots, b_l\}$. It is not difficult to construct examples when $\mathrm{rankr}\{b_1, \ldots, b_l\}$ $\neq \mathrm{rankl}\{b_1, \ldots, b_l\}$. The aim of this section is to prove the following result.

LEMMA 6 *Let b be the matrix with homogeneous coefficient from ${}^hA$ satisfying (25), see above. Suppose that $d \geqslant 2$ and $\deg b_{i,j} < d$ for all i, j. Let $l_1 = \mathrm{rankr}\{b_1, \ldots, b_l\}$ and $b_1, \ldots, b_{l_1}$ be linearly independent. Hence $0 \leqslant l_1 \leqslant l$ and*

$k \geqslant l_1$. Then there is a matrix $(z_{j,r})_{1\leqslant j,r\leqslant l_1}$ (if $l_1 = 0$ then this matrix is empty) with homogeneous entries $z_{j,r} \in {}^hA$ and a square permutation matrix σ of size k satisfying the following properties.

(i) There are integers d''_r, $1 \leqslant r \leqslant l_1$ such that for all $1 \leqslant j, r \leqslant l_1$ the degree $\deg z_{j,r} = d'_j - d''_r$ or $z_{j,r} = 0$, and hence all the nonzero elements $b_{i,j}z_{j,r}$, $1 \leqslant j \leqslant l_1$ have the same degree $d_i - d''_r$ depending only on i, r. Further,

$$\deg z_{j,r} \leqslant (2n + 2)l_1 \max_{1\leqslant i\leqslant k}\{d_i\} \leqslant (2n + 2)l_1^2 d. \qquad (32)$$

(ii) Set the matrix $e = (e_{i,j})_{1\leqslant i\leqslant k,\, 1\leqslant j\leqslant l_1} = \sigma(b_1, \ldots, b_{l_1})z$ where $(b_1, \ldots, b_{l_1})$ is the matrix consisting from the first l_1 columns of the matrix b. Then

$$e = \begin{pmatrix} e' \\ e'' \end{pmatrix},$$

where $e' = \mathrm{diag}(e'_{1,1}, \ldots, e'_{l_1,l_1})$ is a diagonal matrix with l_1 columns and each $e'_{j,j}$, $1 \leqslant j \leqslant l_1$, is nonzero.

(iii) $\mathrm{ord}\, e_{i,j} \geqslant \mathrm{ord}\, e'_{j,j}$ for all $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l_1$.

Besides that, if all $a_{i,j}$ (and hence all $b_{i,j}$) do not depend on X_n (i.e., they can be represented as sums of monomials which do not contain X_n) then one can choose also $z_{j,r}$ satisfying additionally the same property. Finally, dividing by an appropriate power of X_0 one can assume without loss of generality that $\min\{\mathrm{ord}\, z_{j,r} : 1 \leqslant j \leqslant l_1\} = 0$ for every $1 \leqslant r \leqslant l_1$.

PROOF At first we shall show how to construct z, e and σ such that (ii) and (iii) hold. We shall use a kind of Gauss elimination and Lemma 4. Namely, we transform the matrix e. At the beginning put

$$e = (e_1, \ldots, e_{l_1}) = (b_1, \ldots, b_{l_1}).$$

We shall perform some ${}^hA$-linear transformations of columns and permutations of rows of the matrix e and replace each time e by the obtained matrix. These transformation do not change the rank from the right of the family of columns of e. At the end we get a matrix e satisfying the required properties (ii), (iii).

We have $\mathrm{rankr}(e) = l_1$. If $l_1 = 0$, i.e, e is an empty matrix, then this is the end of the construction: z is an empty matrix. Suppose that $l_1 > 0$. Let us choose indices $1 \leqslant i_0 \leqslant k$, $1 \leqslant j_0 \leqslant l_1$, such that $\mathrm{ord}\, e_{i_0,j_0} = \min_{1\leqslant j\leqslant l_1}\{\mathrm{ord}\, e_j\}$. Permuting rows and columns of e we shall assume without loss of generality that $(i_0, j_0) = (1, 1)$.

By Lemma 4 we get elements $w_{i,1}, w_{i,i} \in {}^hA$ of degrees at most $(2n + 2)4d$ such that $e_{1,1}w_{1,i} = e_{1,i}w_{i,i}$, $1 \leqslant i \leqslant l_1$, and $\mathrm{ord}\, w_{i,i} = 0$ for every $1 \leqslant i \leqslant l_1$. Set $w' = (-w_{1,2}, \ldots, -w_{1,l_1})$, and $w'' = \mathrm{diag}(w_{2,2}, \ldots, w_{l_1,l_1})$ to be the diagonal matrix. Put

$$w = \begin{pmatrix} 1, & w' \\ 0, & w'' \end{pmatrix}$$

to be the square matrix with l_1 rows. We replace e by ew. Now

$$e = \begin{pmatrix} e_{1,1}, & 0 \\ E_{2,1}, & E_{2,2} \end{pmatrix},$$

where $E_{2,2}$ has $l_1 - 1$ columns and

$$\min_{1\leqslant j\leqslant l_1}\{\operatorname{ord} b_j\} = \operatorname{ord} e_{1,1} = \min_{1\leqslant j\leqslant l_1}\{\operatorname{ord} e_j\} \tag{33}$$

(for the new matrix e).

Let us apply recursively the described construction to the matrix $E_{2,2}$ in place of e. So using only linear transformations of columns with indices $2,\dots,l_1$ and permutation of rows with indices $2,\dots,k$ we transform e to the form

$$\sigma e\tau = \begin{pmatrix} e_{1,1}, & 0 \\ E'_{2,1}, & E'_{2,2} \\ E''_{2,1} & E''_{2,2} \end{pmatrix}, \quad \tau = \begin{pmatrix} 1, & 0 \\ 0, & \tau' \end{pmatrix}$$

where σ is a permutation matrix and τ' is a square matrix with $l_1 - 1$ rows (it transforms $E_{2,2}$), the matrix $E'_{2,2} = \operatorname{diag}(e_{2,2},\dots,e_{l_1,l_1})$ is a diagonal matrix with $l_1 - 1 \geqslant 0$ columns, and all the elements $e_{2,2},\dots,e_{l_1,l_1} \in {}^hA$ are nonzero. We shall assume without loss of generality that $\sigma = 1$ is the identity matrix. We replace e by $e\tau$. Condition (iii) holds for the obtained e and, more than that, by (iii) applied recursively to $(E_{2,2}, E'_{2,2}, E''_{2,2})$ (in place of (e, e', e'')), and (33) the same equalities (33) are satisfied for the new obtained matrix e.

Let $E'_{2,1} = (e_{2,1},\dots,e_{l_1,1})^t$ where t denotes transposition. By Lemma 4 there are nonzero elements $v_{1,1},\dots,v_{l_1,1} \in {}^hA$ of degrees at most

$$(2n+2)(\max\{\deg e_{i,j} \,:\, 1 \leqslant i \leqslant l_1 \,\&\, j = 1, i\} + 1)l_1^2 \tag{34}$$

such that $e_{i,1}v_{1,1} = e_{i,i}v_{i,1}$ and $\min\{\operatorname{ord} v_{1,i} \,:\, 1 \leqslant i \leqslant l_1\} = 0$ for all $1 \leqslant i \leqslant l_1 - 1$. Let $v' = (-v_{2,1},\dots,-v_{l_1,1})^t$ and v'' be the identity matrix of size $l_1 - 1$. Put

$$v = \begin{pmatrix} v_{1,1}, & 0 \\ v', & v'' \end{pmatrix}.$$

Let us replace e by ev. Put $z = w\tau v$. Recall that without loss of generality $\sigma = 1$ is the identity permutation. We have $e = (b_1,\dots,b_{l_1})z$. These Gauss elimination transformations of e do not change the rank from the right of the family of columns of e. It can be easily proved using the recursion on l, cf. Lemma 8 below. Now the matrix e satisfies required conditions (ii), (iii) and $\sigma = 1$.

Let us change the notation. Denote the obtained matrix z by z'. Let $z' = (z'_1,\dots,z'_{l_1})$ where z'_j is the j-th column of z'. Our aim now is to prove the existence of the matrix z satisfying (i)–(iii). By Lemma 4 for every $1 \leqslant r \leqslant l_1$ there are homogeneous elements $z_{j,r} \in {}^hA$, $1 \leqslant j \leqslant l_1$, such that $(z_{1,r},\dots,z_{l_1,r}) \neq (0,\dots,0)$, the degrees $\deg z_{j,r} = d'_r + \mu_r$ or $z_{j,r} = 0$ for all $1 \leqslant j \leqslant l_1$ and

$$\sum_{1\leqslant j\leqslant l_1} b_{i,j}z_{j,r} = 0 \quad \text{for every} \quad 1 \leqslant i \leqslant l_1,\ i \neq r, \tag{35}$$

and estimations for degrees (32) hold. Put the matrix $z = (z_{j,r})_{1\leqslant j,r\leqslant l_1}$ and $d''_r = -\mu_r$. Let $z = (z_1,\dots,z_{l_1})$ where z_j is the j-th column of z. Hence $z_j = (z_{1,j},\dots,z_{l_1,j})^t$.

LEMMA 7 *For every $1 \leqslant r \leqslant l_1$ we have*

$$\sum_{1 \leqslant j \leqslant l_1} b_{r,j} z_{j,r} \neq 0. \tag{36}$$

Further, for every $1 \leqslant r \leqslant l_1$ there are nonzero homogeneous elements $g'_r, g_r \in {}^hA$ such that $z'_r g'_r = z_r g_r$.

PROOF Consider the matrix (z', z_r) with l_1 rows and $l_1 + 1$ columns. By Lemma 4 there are homogeneous elements $h_1, \dots, h_{l_1+1} \in {}^hA$ (they depend on r) such that $(h_1, \dots, h_{l_1+1}) \neq (0, \dots, 0)$ and the following property holds. Denote $h = (h_1, \dots, h_{l_1+1})^t$, $h' = (h_1, \dots, h_{l_1})^t$. Then

$$z'h' + z_r h_{l_1+1} = 0 \tag{37}$$

(we don't need at present any estimation on degrees from Lemma 4; only the existence of h). Denote by b'' the submatrix consisting of the first l_1 rows of the matrix $(b_1, \dots, b_{l_1})$. Multiplying (37) to b'' from the left we get

$$b''z'h' + b''z_r h_{l_1+1} = 0. \tag{38}$$

But $b''z'$ is a diagonal matrix with nonzero elements on the diagonal, see (ii) (for z' in place of z). Hence by (35) and (38) $h_j = 0$ for every $j \neq r$.

Now suppose that $h_r = 0$. Then $h' = 0$. Hence by (37) and since $z_r \neq 0$ we have $h_{l_1+1} = 0$. Hence $h = (0, \dots, 0)^t$ and we get a contradiction.

Suppose that $h_{l_1+1} = 0$. Then by (38) we have $h_r = 0$. Hence $h = (0, \dots, 0)^t$ and again we get a contradiction.

Thus, $h_r \neq 0$ and $h_{l_1+1} \neq 0$. Now (38) implies (36). Put $g'_r = h_r$ and $g_r = -h_{l_1+1}$. We have $z'_r g'_r = z_r g_r$ by (37). The lemma is proved.

Let us return to the proof of Lemma 6. Now (i)–(iii) are satisfied by Lemma 7. The last assertions of Lemma 6 are proved similarly to the ones of Lemma 4. Lemma 6 is proved.

7 An algorithm for solving linear systems with coefficients from ${}^hA$.

Let $u = (u_1, \dots, u_l) \in {}^hA^l$. Let all nonzero u_j be homogeneous elements of the degree $-d'_j + \rho$ for an integer ρ. Suppose that $-d'_j + \rho < d'$ for an integer $d' > 1$. Let $b = (b_{i,j})_{1 \leqslant i \leqslant k,\, 1 \leqslant j \leqslant l}$ be the matrix with k rows and l columns from the statement of Lemma 6 (but now k and l are arbitrary). So $\deg b_{i,j} = d_i - d'_j < d$ for all i, j and $d \geqslant 2$. Let $Z = (Z_1, \dots, Z_k)$ be unknowns. Consider the linear system

$$\sum_{1 \leqslant i \leqslant k} Z_i b_{i,j} = u_j, \quad 1 \leqslant j \leqslant l, \tag{39}$$

or, which is the same,

$$Zb = u.$$

Denote

$$\operatorname{ord} u = \min_{1 \leqslant i \leqslant l} \{\operatorname{ord} u_i\}. \tag{40}$$

The similar notations will be used for other vectors and matrices. In this section we shall describe an algorithm for solving linear systems over ${}^hA$ and *prove the following theorem for an infinite field F* (this theorem for an finite field F is easily reduced to the case of an infinite field F; but we shall not use this theorem for a finite field F in this paper).

THEOREM 2 *Suppose that system (39) has a solution over ${}^hA$. Then one can represent the set of all solutions of (39) over ${}^hA$ in the form*

$$J + z^*,$$

where $J \subset {}^hA^l$ is a ${}^hA$-submodule of all the solutions of the homogeneous system corresponding to (39) (i.e., system (39) with all $u_j = 0$) and z^* is a particular solution of (39). Moreover, the following assertions hold.

(A) One can choose z^* such that $\operatorname{ord} z^* \geqslant \operatorname{ord} u - \nu$, where $\nu \geqslant 0$ is an integer bounded from above by $(dl)^{2^{O(n)}}$. The degree $\deg z^*$ is bounded from above by $d' + (dl)^{2^{O(n)}}$.

(B) There exists a system of generators of J of degrees bounded from above by $(dl)^{2^{O(n)}}$. The number of elements of this system of generators is bounded from above by $k(dl)^{2^{O(n)}}$.

The constants from $O(n)$ in assertions (A) and (B) are absolute. Besides that, if all $b_{i,j}$ and u_j do not depend on X_n (i.e., they can be represented as sums of monomials which do not contain X_n) then z^* and all the generators of the module J also satisfy this property.

PROOF Let $l_1 = \operatorname{rankr}(b_1, \ldots, b_l)$. Permuting equations of (39) we shall assume without loss of generality that $(b_1, \ldots, b_{l_1})$ are linearly independent from the right over ${}^hA$. Let σ, z, e, e', e'' be the matrices from Lemma 6. Similarly to the proof of Lemma 6 we shall assume without loss of generality that $\sigma = 1$. Denote by b' the submatrix of b consisting of the first l_1 columns of b, i.e., $b' = (b_1, \ldots, b_{l_1})$. By Lemma 4 there are nonzero homogeneous elements $q_{1,1}, \ldots, q_{l_1,l_1}$ of degrees at most

$$(2n + 2)(\max\{\deg e_{i,i} \,:\, 1 \leqslant i \leqslant l_1\} + 1)l_1^2$$

such that $e_{1,1}q_{1,1} = e_{i,i}q_{i,i}$ and $\min\{\operatorname{ord} q_{i,i} \,:\, 1 \leqslant i \leqslant l_1\} = 0$. Set $q = \operatorname{diag}(q_{1,1}, \ldots, q_{l_1,l_1})$ to be the diagonal matrix. Let $\nu_0 = \operatorname{ord} e_{1,1}q_{1,1}$. Then by Lemma 6 (iii) $\operatorname{ord}(b'zq) \geqslant \nu_0$. Let $X_0^{\nu_0}\delta = b'zq$. Then δ is a matrix with coefficients from ${}^hA$ and

$$\delta = (\delta_{i,j})_{1 \leqslant i \leqslant k,\, 1 \leqslant j \leqslant l_1} = \begin{pmatrix} \delta' \\ \delta'' \end{pmatrix},$$

where $\delta' = \operatorname{diag}(\delta_{1,1}, \ldots, \delta_{l_1,l_1})$ is a diagonal matrix with homogeneous coefficients from ${}^hA$ and all the elements on the diagonal are nonzero and equal, i.e., $\delta_{j,j} = \delta_{1,1}$ for every $1 \leqslant j \leqslant l_1$. Besides that, $\operatorname{ord} \delta_{1,1} = 0$. Further, $\delta'' = (\delta_{i,j})_{l_1+1 \leqslant i \leqslant k,\, 1 \leqslant j \leqslant l_1}$. We have $\operatorname{ord}(uzq) \geqslant \nu_0$, since, otherwise, system (39) does not have a solution. Obviously $\operatorname{ord} u \leqslant \operatorname{ord}(uzq)$. Denote $u' = (u'_1, \ldots, u'_{l_1}) = X_0^{-\nu_0} uzq \in {}^hA^l$. Hence $\operatorname{ord} u' \geqslant \operatorname{ord}(u) - \nu_0$.

By Lemma 6 (i) and since q is the diagonal matrix with nonzero homogeneous entries on the diagonal there are integers d''_j, $1 \leqslant j \leqslant l_1$, such that for all i, j the degree

$$\deg \delta_{i,j} = d_i - d''_j \tag{41}$$

or $\delta_{i,j} = 0$. Besides that, by the same reason there is an integer ρ' such that $\deg u'_j = -d''_j + \rho'$ or $u'_j = 0$ for all $1 \leqslant j \leqslant l_1$ (here we leave the details to the reader).

Consider the linear system

$$Z\delta = u'. \tag{42}$$

LEMMA 8 *Suppose that system (39) has a solution over ${}^hA$. Then linear system (42) is equivalent to (39), i.e., the sets of solutions of systems (42) and (39) over ${}^hA$ coincide.*

PROOF The system $Zb'z = uz$ is equivalent to (39) by Lemma 5. System (42) is equivalent to $Zb'z = uz$ since the ring ${}^hA$ does not have zero–divisors. The lemma is proved.

REMARK 6 *We have* $\mathrm{rankr}(b_1, \ldots, b_l) = l_1$. *Hence by Lemma 6 for every $l_1 + 1 \leqslant j \leqslant l$ there are homogeneous $z_{j,j}, z_{1,j}, \ldots, z_{l_1,j} \in {}^hA$ such that $z_{j,j} \neq 0$ and $b_j z_{j,j} + \sum_{1 \leqslant r \leqslant l_1} b_r z_{r,j} = 0$ and all $\deg z_{j,j}$, $\deg z_{r,j}$ are bounded from above by $(2n+2)(l_1+1)^2 d$. Put $u'_j = u_j z_{j,j} + \sum_{1 \leqslant r \leqslant l_1} u_r z_{r,j}$, $l_1 + 1 \leqslant j \leqslant l$. Then system (39) has a solution if and only if system (42) has a solution and $u'_j = 0$ for all $l_1 + 1 \leqslant j \leqslant l$. This follows from Lemma 8 and Lemma 5. But in what follows for our aims it is sufficient to use only Lemma 8.*

REMARK 7 *Assume that $\deg_{X_n} b_{i,j} \leqslant 0$ for all i, j, i.e., the elements of the matrix b do not depend on X_n. Then by Lemmas 4 and 6 and the described construction all the elements of the matrices $b, z, q, \delta, \delta', \delta''$ also do not depend on X_n.*

By Lemma 4 and Remark 3 for every $l_1 + 1 \leqslant i \leqslant k$ there are homogeneous elements $g_{i,i}, g_{i,j} \in {}^hA$, $1 \leqslant j \leqslant l_1$, such that

$$g_{i,i}\delta_{i,j} = g_{i,j}\delta_{1,1}, \quad 1 \leqslant j \leqslant l_1,$$

all the degrees $\deg g_{i,i}$, $\deg g_{i,j}$, $1 \leqslant j \leqslant l_1$, are bounded from above by

$$(2n+2)(l_1+1)^2(\max\{\deg \delta_{i,j} \,:\, 1 \leqslant j \leqslant k\} + 1)$$

and $\min_{1 \leqslant j \leqslant l_1}\{\mathrm{ord}\, g_{i,i}, \mathrm{ord}\, g_{i,j}\} = 0$. Hence $\mathrm{ord}\, g_{i,i} = 0$ for every $l_1 + 1 \leqslant i \leqslant k$ since $\mathrm{ord}\, \delta_{1,1} = 0$.

We need an analog of the Noether normalization theorem from commutative algebra, cf. also Lemma 3.1 [7].

LEMMA 9 *Let $h \in {}^hA$ be an arbitrary nonzero element and the degree $\deg h = \varepsilon$. There is a linear automorphism of the algebra ${}^hA$*

$$\alpha \,:\, {}^hA \to {}^hA, \quad \alpha(X_i) = \sum_{1 \leqslant j \leqslant n} (\alpha_{1,i,j} X_j + \alpha_{2,i,j} D_j),$$

$$\alpha(D_i) = \sum_{1 \leqslant j \leqslant n} (\alpha_{3,i,j} X_j + \alpha_{4,i,j} D_j), \quad \alpha(X_0) = X_0, \quad 1 \leqslant i \leqslant n,$$

such that all $\alpha_{s,i,j} \in F$, $\deg_{D_n} \alpha(h) = \varepsilon$. *Moreover, one can choose* α *such that additionally for every* $H \in {}^hA$ *if* $\deg_{X_n} H = 0$ *then* $\deg_{X_n} \alpha(H) = 0$.

PROOF At first it is not difficult to construct a linear automorphism β such that $\beta(X_0) = X_0$, $\beta(X_n) = X_n$, $\beta(D_n) = D_n$,

$$\beta(X_i) = \beta_{1,i}X_i + \beta_{2,i}D_i, \quad \beta(D_i) = \beta_{3,i}X_i + \beta_{4,i}D_i, \quad 1 \leqslant i \leqslant n,$$

and $\beta(h)$ contains a monomial $a_{i_1,\ldots,i_n} D_1^{i_1}, \ldots, D_n^{i_n}$ with $a_{i_1,\ldots,i_n} \neq 0$ and $i_1 + \ldots + i_n = \varepsilon$, i.e., $\varepsilon = \deg_{D_1,\ldots,D_n} \beta(h)$. After that one can find an automorphism γ such that $\gamma(X_0) = X_0$,

$$\begin{array}{ll} \gamma(X_i) = X_i, & \gamma(D_i) = D_i + \gamma_i D_n, \quad 1 \leqslant i \leqslant n-1, \\ \gamma(X_n) = X_n - \sum_{1 \leqslant i \leqslant n-1} \gamma_i X_i, & \gamma(D_n) = D_n, \end{array}$$

where $\gamma_i \in F$ for all $1 \leqslant i \leqslant n-1$ and $(\gamma \circ \beta)(h)$ contains a monomial aD_n^{ε} with a coefficient $0 \neq a \in F$. Put $\alpha = \gamma \circ \beta$. Obviously if $H \in {}^hA$ and $\deg_{X_n} H = 0$ then $\deg_{X_n} \alpha(H) = 0$. The lemma is proved.

Put $h = \delta_{1,1} g_{l_1+1,l_1+1} g_{l_1+2,l_1+2} \ldots g_{k,k}$. So $h \in {}^hA$ is a nonzero homogeneous element and $\operatorname{ord} h = 0$. Applying Lemma 9 to h we obtain an automorphism α. In what follows to simplify the notation we shall suppose without loss of generality that $\alpha = 1$. So h contains a monomial aD_n^{ε} with a coefficient $0 \neq a \in F$, where $\varepsilon = \deg h$. It follows from here that

$$\deg_{D_n} \delta_{1,1} = \deg \delta_{1,1}, \quad \deg_{D_n} g_{i,i} = \deg g_{i,i},\ l_1 + 1 \leqslant i \leqslant k. \tag{43}$$

Let $z = (z_1, \ldots, z_k) \in {}^hA^k$ be a solution of (42). Then (43) implies that one can uniquely represent

$$z_i = z'_i g_{i,i} + \sum_{0 \leqslant r < \deg g_{i,i}} z_{i,r} D_n^r, \quad l_1 + 1 \leqslant i \leqslant k, \tag{44}$$

where $z'_i, z_{i,s} \in {}^hA$, the degrees $\deg_{D_n} z_{i,s} \leqslant 0$ for all $l_1 + 1 \leqslant i \leqslant k$, $0 \leqslant s < \deg_{D_1} g_{i,i}$. Again by (43) one can uniquely represent

$$u'_j = u''_j \delta_{1,1} + \sum_{0 \leqslant s < \deg \delta_{1,1}} u'_{j,s} D_n^s, \quad 1 \leqslant j \leqslant l_1,$$

where $u''_j, u'_{j,s} \in {}^hA$, the degrees $\deg_{D_n} u'_{j,s} \leqslant 0$ for all $1 \leqslant j \leqslant l_1$, $0 \leqslant s < \deg_{D_1} g_{i,i}$. Finally, by (43) for all $l_1 + 1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l_1$, $0 \leqslant r < \deg_{D_1} g_{i,i}$, one can uniquely represent

$$D_n^r \delta_{i,j} = \delta_{i,r,j} \delta_{1,1} + \sum_{0 \leqslant s < \deg \delta_{1,1}} \delta_{i,r,j,s} D_n^s,$$

where $\delta_{i,r,j}, \delta_{i,r,j,s} \in {}^hA$, the degrees $\deg_{D_n} \delta_{i,r,j,s} \leqslant 0$ for all considered i, r, j, s. Put

$$\begin{aligned} \mathcal{I} &= \{\, (i,r) \,:\, l_1 + 1 \leqslant i \leqslant k \,\&\, 0 \leqslant r < \deg g_{i,i} \,\}, \\ \mathcal{J} &= \{\, (j,s) \,:\, 1 \leqslant j \leqslant l_1 \,\&\, 1 \leqslant s < \deg \delta_{1,1} \,\}. \end{aligned}$$

Therefore,

$$z_j = -\sum_{l_1+1\leqslant i\leqslant k} z_i' g_{i,j} - \sum_{(i,r)\in\mathcal{I}} z_{i,r}\delta_{i,r,j} + u_j'', \quad 1\leqslant j\leqslant l_1, \tag{45}$$

$$\sum_{(i,r)\in\mathcal{I}} z_{i,r}\delta_{i,r,j,s} = u_{j,s}', \quad (j,s)\in\mathcal{J}. \tag{46}$$

Let us introduce new unknowns $Z_{i,r}$, $(i,r)\in\mathcal{I}$. By (44)–(46) system (39) is reduced to the linear system

$$\sum_{(i,r)\in\mathcal{I}} Z_{i,r}\delta_{i,r,j,s} = u_{j,s}', \quad (j,s)\in\mathcal{J}. \tag{47}$$

More precisely, any solution of system (39) is given by (44), (45) where $z_i'\in {}^hA$ are arbitrary and $z_{i,r}$ is a solution of system (46) over ${}^hA$ (we underline that here this solution $z_{i,r}$ may depend on D_n although one can restrict oneself by solutions $z_{i,r}$ which do not depend on D_n). Note that all $\delta_{i,r,j,s}$ and $u_{j,s}'$ are homogeneous elements of ${}^hA$. Put $d_{i,r} = d_i + r$, $(i,r)\in\mathcal{I}$ and $d_{j,s}' = d_j'' + s$, $(j,s)\in\mathcal{J}$, $\widetilde{\rho} = \rho'$ where d_j, d_i'', ρ' are introduced above, see (41). Then $\deg\delta_{i,r,j,s} = d_{i,r} - d_{j,s}'$ or $\delta_{i,r,j,s} = 0$, and $\deg u_{j,s}' = -d_{j,s}' + \widetilde{\rho}$ or $u_{j,s}' = 0$ for all $(i,r)\in\mathcal{I}$, $(j,s)\in\mathcal{J}$. This follows immediately from the described construction (we leave the details to the reader).

Now all the coefficients of system (47) do not depend on D_n. As we have proved if the coefficients of (39) do not depend on X_n then the coefficients of (47) also do not depend on X_n, and hence in the last case they do not depend on X_n, D_n.

If the coefficients of (47) depend on X_n we perform an automorphism $X_n\mapsto D_n$ $D_n\mapsto -X_n$, $X_i\mapsto X_i$, $D_i\mapsto D_i$, $1\leqslant i\leqslant n-1$. Now the coefficients of system (47) do not depend on X_n (but depend on D_n).

After that we apply our construction recursively to system (47).

The final step of the recursion is $n=0$ (although in the statement of theorem $n\geqslant 1$, see Section 1; we are interested only in Weyl algebras). In this case $\mathcal{I}=\mathcal{J}=\emptyset$. Hence using (45) for $n=0$ we get the required z^* and J for $n=0$.

Thus, by the recursive assumption we get a particular solution $Z_{i,r} = z_{i,r}^*$, $(i,r)\in\mathcal{I}$, of system (47), an integer ν_1 (in place of ν from assertion (A)) satisfying the inequality

$$\min_{(i,r)\in\mathcal{I}}\{\operatorname{ord} z_{i,r}^*\} \geqslant \min_{(j,s)\in\mathcal{J}}\{\operatorname{ord} u_{j,s}'\} - \nu_1, \tag{48}$$

and a system of generators

$$(\, z_{\alpha,i,r}\,)_{(i,r)\in\mathcal{I}}, \quad 1\leqslant\alpha\leqslant\beta, \tag{49}$$

of the module J' of solutions of the homogeneous system corresponding to (47). Notice that if the coefficients of (39) do not depend on X_n then J' is a module over the homogenization $F[X_0, X_1, \ldots, X_{n-1}, D_1, \ldots, D_{n-1}]$ of the Weyl algebra of $X_1, \ldots, X_{n-1}, D_1, \ldots, D_{n-1}$. But obviously in the last case (49) gives also a system of generators of the ${}^hA$-module $J'' = {}^hAJ'$ of solutions of the homogeneous system corresponding to (47). Put

$$z_j^* = -\sum_{(i,r)\in\mathcal{I}} z_{i,r}^*\delta_{i,r,j} + u_j'', \quad 1\leqslant j\leqslant l_1,$$

$$z_i^* = \sum_{0 \leqslant r < \deg g_{i,i}} z_{i,r}^* D_n^r, \quad l_1 + 1 \leqslant i \leqslant k,$$
$$z^* = (z_1^*, \ldots, z_k^*).$$

Then z^* is a particular solution of (39). Put

$$z_{\alpha,j} = -\sum_{(i,r) \in \mathcal{I}} z_{\alpha,i,r} \delta_{i,r,j}, \quad 1 \leqslant j \leqslant l_1,\ 1 \leqslant \alpha \leqslant \beta,$$
$$z_{\alpha,i} = \sum_{0 \leqslant s < \deg g_{i,i}} z_{\alpha,i,s} D_n^s, \quad l_1 + 1 \leqslant i \leqslant k,\ 1 \leqslant \alpha \leqslant \beta,$$
$$z_{\beta - l_1 + i, j} = 0, \quad l_1 + 1 \leqslant i, j \leqslant k,\ j \neq i,$$
$$z_{\beta - l_1 + i, i} = g_{i,i}, \quad l_1 + 1 \leqslant i \leqslant k,$$
$$z_{\beta - l_1 + i, j} = -g_{i,j}, \quad 1 \leqslant j \leqslant l_1,\ l_1 + 1 \leqslant i \leqslant k.$$

Then $J = \sum_{1 \leqslant \alpha \leqslant \beta + k - l_1} {}^h A(z_{\alpha,1}, \ldots, z_{\alpha,k})$. Hence $(z_{\alpha,1}, \ldots, z_{\alpha,k})$, $1 \leqslant \alpha \leqslant \beta + k - l_1$, is a system of generators of the module J. By (48) and the definitions of u', u''_j and $u'_{j,s}$ we have $\operatorname{ord} z^* \geqslant \operatorname{ord}(u) - \nu_0 - \nu_1$. Put $\nu = \nu_0 + \nu_1$.

LEMMA 10 *All the degrees* $\deg \delta_{i,j}$, $\deg g_{i,i}$, $\deg g_{i,j}$, $\deg \delta_{i,r,j}$, $\deg \delta_{i,r,j,s}$ *and* ν_0*, see above, are bounded from above by* $(nld)^{O(1)}$*, the degrees* $\deg u'_j$, $\deg u''_j$, $\deg u'_{j,s}$ *are bounded from above* $d' + (nld)^{O(1)}$*. Further, all* $\operatorname{ord} u''_j$, $\operatorname{ord} u'_{j,s}$ *are bounded from below by* $\operatorname{ord} u - \nu_0$*. Finally, in system (47) the number of equations* $\#\mathcal{J}$ *is bounded from above by* $(nld)^{O(1)}$ *and the number of unknowns* $\#\mathcal{I}$ *is bounded from above by* $k(nld)^{O(1)}$.

PROOF This follows immediately from the described construction.

Let us return to the proof of Theorem 2. Applying Lemma 10 and recursively assertions (A) and (B) for the formulas giving z^* and J we get (A) and (B) from the statement of the theorem. The last assertion (related to the case when all $b_{i,j}$ and u_j do not depend on D_n) has been already proved. The theorem is proved.

8 Proof of Theorem 1 for Weyl algebra

Let us show that it is sufficient to prove the theorem for an infinite field F. Indeed, let F_1 be an infinite field and $F_1 \supset F$. Let $f_1, \ldots, f_m$ be a Janet basis of the module $I \otimes_F F_1$ with all the degrees $\deg f_w$, $1 \leqslant w \leqslant m$, bounded from above by $d^{2^{O(n)}}$. There is a finite extension $F_2 \supset F$ such that for all v, i, j for all $1 \leqslant w \leqslant m$ the coefficient of f_w at the monomial $e_{v,i,j}$ belongs to the field F_2. Let a_α, $1 \leqslant \alpha \leqslant \mu$, be the basis of the field F_2 over F. Then one can represent $f_w = \sum_{1 \leqslant \alpha \leqslant \mu} a_\alpha f_{\alpha,w}$ where all $f_{\alpha,w} \in I$. Now $\deg f_{\alpha,w} \leqslant \deg f_w$ and $f_{\alpha,w}$, $1 \leqslant w \leqslant m$, $1 \leqslant \alpha \leqslant \mu$ is a Janet basis of the module I. Moreover, the reduced Janet basis of the module I remains the same after an arbitrary extension of scalars. The required assertion is proved. So extending the ground field F we shall suppose without loss of generality that the field F is infinite.

Let a be the matrix from Section 1. We shall suppose without loss of generality that the vectors $(a_{i,1}, \ldots, a_{i,l})$, $1 \leqslant i \leqslant k$, are linearly independent over the field F. We have $\deg a_{i,j} < d$. This implies $k \leqslant l\binom{d+2n}{2n}$.

Put the matrix $b = {}^h a$. Let us define the graded submodules of ${}^h I$

$$J_0 = {}^h A(b_{1,1}, \dots, b_{1,l}) + \ldots + {}^h A(b_{k,1}, \dots, b_{k,l}),$$
$$J_\nu = J_0 : (X_0^\nu) = \{z \in {}^h A^l \,:\, z X_0^\nu \in J_0\}, \quad \nu \geqslant 1.$$

We have the exact sequence of graded ${}^h A$-modules

$${}^h A^k \to J_0 \to 0.$$

Further, $J_\nu \subset J_{\nu+1} \subset {}^h I$ for every $\nu \geqslant 0$ and ${}^h I = \bigcup_{\nu \geqslant 0} J_\nu$. Since ${}^h A$ is Noetherian there is $N \geqslant 0$ such that ${}^h I = J_N$. So to construct a system of generators of ${}^h I$ it is sufficient to compute the least N such that ${}^h I = J_N$ and to find a system of generators of J_N.

LEMMA 11 *${}^h I = J_N$ for some N bounded from above by $(dl)^{2^{O(n)}}$. There is a system of generators $b_1, \dots, b_s$ of the module J_N such that s and all the degrees $\deg b_v$, $1 \leqslant v \leqslant s$, are bounded from above by $(dl)^{2^{O(n)}}$.*

PROOF Let us show that the module $J_{N+1} \subset J_N$ for $N \geqslant \nu$. Let $u \in J_{N+1}$. Consider system (39). By assertion (A) of Theorem 2 there is a particular solution z^* of (39) such that $\operatorname{ord} z^* \geqslant 1$. Hence $u \in X_0 J_N \subset J_N$. The required assertion is proved. Hence ${}^h I = J_\nu$.

Let us replace in (39) $(u_1, \dots, u_l)$ by $(U_1 X_0^\nu, \dots, U_l X_0^\nu)$, where $U_1, \dots, U_l$ are new unknowns. Then applying (B) from Theorem 2 to this new homogeneous linear system with respect to all unknowns $U_1, \dots, U_l$, $Z_1, \dots, Z_k$ we get the required estimations for the number of generators of J_ν and the degrees of these generators. The lemma is proved.

COROLLARY 1 *Let $(a_{i,1}, \dots, a_{i,l})$, $1 \leqslant i \leqslant l$, be from the beginning of the section and the integer N be from Lemma 11. Then for every integer $m \geqslant 0$ the F–linear space*

$$A_{m+N}(a_{1,1}, \dots, a_{1,l}) + \ldots + A_{m+N}(a_{k,1}, \dots, a_{k,l}) \supset I_m. \tag{50}$$

PROOF By Lemma 11 we have $(J_0)_{m+N} \supset X_0^N (J_N)_m = X_0^N ({}^h I)_m$. Taking the affine parts we get (50). The corollary is proved.

Now everything is ready for the proof of Theorem 1. By Lemma 11 and Lemma 1 there is a system of generators of the module $\mathrm{gr}(I)$ with degrees bounded from above by $(dl)^{2^{O(n)}}$. By Lemma 12 from Appendix 1 the Hilbert function $H(\mathrm{gr}(I), m)$ is stable for $m \geqslant (dl)^{2^{O(n)}}$. By (11) Section 2 the Hilbert function $H(I, m)$ is stable for all $m \geqslant (dl)^{2^{O(n)}}$.

Consider the linear order $<$ on the monomials from ${}^h A^l$ which is induced by the linear order $<$ on the monomials from A^l, see Section 4. Then the monomial (i.e., generated by monomials) submodule ${}^c I \subset {}^c A^l$ is defined, see Section 4, where ${}^c A = F[X_0, \dots, X_n, D_1, \dots, D_n]$ is the polynomial ring. By (24) Section 4 the Hilbert function $H({}^c I, m)$ is stable for all $m \geqslant (dl)^{2^{O(n)}}$. Hence all the coefficients of the Hilbert polynomial of ${}^c I$ are bounded from above $(dl)^{2^{O(n)}}$. Therefore, according to Lemma 13 the module ${}^c I$ has a system of generators with degrees $(dl)^{2^{O(n)}}$. We can suppose without loss of generality that the last system of generators of ${}^c I$ consists of monomials. The sets of monomials from ${}^cI$ and from $\mathrm{Hdt}({}^hI)$ are in the natural degree–preserving one–to–one correspondence, see Section 4. Therefore, see Section 4, the degrees of all the elements of a Janet basis of ${}^hI$ with respect to the induced linear order $<$ are bounded from above by $(dl)^{2^{O(n)}}$. Since the ideal ${}^hI$ is homogeneous the same bound holds for the degrees of all the elements (they are homogeneous) of the reduced Janet basis of ${}^hI$. Hence by Lemma 3 (iii) Section 4 the same is true for some Janet basis $f_1, \dots, f_m$ (respectively by Lemma 3 (ii) for the reduced Janet basis in the case when the initial order $<$ is degree-compatible) of the module I with respect to the linear order $<$ on the monomials from A^l.

It remains to consider the case $l = 1$ and an arbitrary admissible linear order $<$. We need to obtain the estimates for the reduced Janet basis of I in this case. In the considered case the linear order $<$ is given on the set of pairs of multiindices (i, j), $i, j \in \mathbb{Z}_+^n$. Now, see, for example, [13] p. 58, there is a real ordered field R and a linear form $L \in R[Y_1, \dots, Y_n, Z_1, \dots, Z_n]$ with all positive coefficients such that for all pairs (i, j), (i', j') of multiindices $(i', j') < (i, j)$ if and only if

$$L(i - i', j - j') = L(i_1 - i'_1, \dots, i_n - i'_n, j_1 - j'_1, \dots, j_n - j'_n) > 0$$

in the real ordered field R.

Let $\psi_1 < \ldots < \psi_a$ be all the monomials in $X_1, \dots, X_n, D_1, \dots, D_n$ with nonzero coefficients of the elements $f_1, \dots, f_m$ and $(i^{(1)}, j^{(1)}) < \ldots < (i^{(a)}, j^{(a)})$ the corresponding pairs of multiindices. Let $\varepsilon > 0$ be an infinitesimal with respect to the field R. Now

$$L(i^{(s+1)} - i^{(s)}, j^{(s+1)} - j^{(s)}) \geqslant \varepsilon, \quad 1 \leqslant s \leqslant a - 1 \tag{51}$$

in the field $R(\varepsilon)$. Let $U = \sum_{1 \leqslant w \leqslant n}(u_w Y_w + v_w Z_w)$ be a generic linear form in $Y_1, \dots, Y_n, Z_1, \dots, Z_n$, i.e., the family $\{u_w, v_w\}_{1 \leqslant w \leqslant n}$ of coefficients of U has transcendency degree $2n$ over $R(\varepsilon)$. Consider the system of linear inequalities with coefficients from $\mathbb{Q}[\varepsilon]$ with respect to u_w, v_w, $1 \leqslant w \leqslant n$,

$$\left\{ \begin{array}{ll} U(i^{(s+1)} - i^{(s)}, j^{(s+1)} - j^{(s)}) \geqslant \varepsilon, & 1 \leqslant s \leqslant a - 1, \\ u_w \geqslant \varepsilon, & 1 \leqslant w \leqslant n, \\ v_w \geqslant \varepsilon, & 1 \leqslant w \leqslant n. \end{array} \right. \tag{52}$$

Denote by K_ε the set of solutions of system (52) from $R(\varepsilon)^{2n}$. By (51) and since all the coefficients of the linear form L are positive system (52) has a solution in $R(\varepsilon)^{2n}$. The left parts of the inequalities from system (52) are linear forms in u_w, v_w, $1 \leqslant w \leqslant n$, with integer coefficients. Denote them by $Q_1, \dots, Q_\mu$, $\mu = a - 1 + 2n$. Notice that the absolute values of the coefficients of the linear forms $Q_1, \dots, Q_\mu$ are bounded from above by $d^{2^{O(n)}}$.

Let us show that there are indices $1 \leqslant w_1 < \ldots < w_s \leqslant \mu$, $s \leqslant 2n$, such that $\mathcal{Z}(Q_{w_1} - \varepsilon, \dots, Q_{w_s} - \varepsilon) \subset K_\varepsilon$ (here $\mathcal{Z}(Q_{w_1} - \varepsilon, \dots, Q_{w_s} - \varepsilon)$ is the set of all common zeroes of the polynomials $Q_{w_1} - \varepsilon, \dots, Q_{w_s} - \varepsilon$ in $R(\varepsilon)^{2n}$) and the linear forms $Q_{w_1}, \dots, Q_{w_s}$ are linearly independent over $\mathbb{Q}$. Indeed, one can construct $Q_{w_1}, \dots, Q_{w_s}$ recursively choosing subsequently Q_{w_α}, $\alpha \geqslant 1$, such that $\mathcal{Z}(Q_{w_\alpha} - \varepsilon)$ has a nonempty intersection with the boundary of $\mathcal{Z}(Q_{w_1} - \varepsilon, \dots, Q_{w_{\alpha-1}} - \varepsilon) \cap K_\varepsilon$ (we leave the details to the reader).

Solving the linear system $Q_{w_1} - \varepsilon = \ldots = Q_{w_s} - \varepsilon = 0$ we see that there is a point $(u'_w, v'_w)_{1 \leqslant w \leqslant n} \in K_\varepsilon$, such that $u'_w = a_w \varepsilon / c$, $v'_w = b_w \varepsilon / c$ where all a_w, b_w, c are positive integers with absolute values bounded from above by $d^{2^{O(n)}}$. Put $\varepsilon^* = 1$, $u_w^* = a_w/c$ and $v_w^* = b_w/c$, $1 \leqslant w \leqslant n$. Consider (52) as a linear system with respect to all u_w, v_w and ε. Then u_w^*, v_w^* and $\varepsilon^* > 0$ is a solution of (52) from $\mathbb{Q}^{2n+1}$. Set $L^* = c\sum_{1\leqslant w\leqslant n}(u_w^* Y_w + v_w^* Z_w)$.

Now $L^* \in \mathbb{Z}[Y_1, \dots, Y_n, Z_1, \dots, Z_n]$ is linear form with all positive integer coefficients bounded from above by $d^{2^{O(n)}}$ such that

$$L^*(i^{(s+1)} - i^{(s)}, j^{(s+1)} - j^{(s)}) > 0, \quad 1 \leqslant s \leqslant a-1. \tag{53}$$

We shall suppose without loss of generality that $\mathrm{Hdt}(f_1), \dots, \mathrm{Hdt}(f_m)$ is the family of leading monomials of the reduced Janet basis $f_1', \dots, f_m'$ of the module I with respect to the linear order $<$, and $\mathrm{Hdt}(f_1) > \ldots > \mathrm{Hdt}(f_m)$. For any $g \in A$ put $\lambda(g) = L^*(i, j)$ where $\mathrm{Hdt}(g) = g_{i,j} X^i D^j$, $0 \neq g_{i,j} \in F$. Then (53) and the definitions imply that $\lambda(f_w) = \lambda(f_w')$ for all $1 \leqslant w \leqslant m$. Hence all $\lambda(f_w')$ are bounded from above by $d^{2^{O(n)}}$. But obviously $\deg f_w' \leqslant \lambda(f_w')$, $1 \leqslant w \leqslant m$. Theorem 1 is proved for Weyl algebra.

9 The case of algebra of differential operators

Extending the ground field F we shall suppose without loss of generality that the field F is infinite. Denote by $B = F(X_1, \dots, X_n)[D_1, \dots, D_n]$ the algebra of differential operators. Recall that $A \subset B$ and hence relations (2) are satisfied. Further, each element $f \in B$ can be uniquely represented in the form

$$f = \sum_{j_1,\dots,j_n \geqslant 0} f_{j_1,\dots,j_n} D_1^{j_1} \dots D_n^{j_n} = \sum_{j \in \mathbb{Z}_+^n} f_j D^j,$$

where all $f_{j_1,\dots,j_n} = f_j \in F(X_1, \dots, X_n)$ and $F(X_1, \dots, X_n)$ is a field of rational functions over F. Let us replace everywhere in Section 1 and Section 2 A, $X^i D^j$, $\deg f = \deg_{X_1,\dots,X_n,D_1,\dots,D_n} f$, $\dim_F M$, $e_{v,i,j}$, $f_{v,i,j} \in F$, (v, i, j), (i, j), (i', j'), (i'', j'') by B, D^j, $\deg f = \deg_{D_1,\dots,D_n} f$, $\dim_{F(X_1,\dots,X_n)} M$, $e_{v,j}$, $f_{v,j} \in F(X_1, \dots, X_n)$, (v, j), j, j', j'' respectively. Thus, we get the definition of the Janet basis and all other objects from Section 1 for the case of the algebra of differential operators.

We define the homogenization ${}^hB$ of B similarly to ${}^hA$, see Section 3. Namely, ${}^hB = F(X_1, \dots, X_n)[X_0, D_1, \dots, D_n]$ given by the relations

$$\begin{array}{ll} X_i X_j = X_j X_i, \ D_i D_j = D_j D_i, & \text{for all} \quad i, j, \\ D_i X_i - X_i D_i = X_0, \ 1 \leqslant i \leqslant n, & X_i D_j = D_j X_i \quad \text{for all} \quad i \neq j. \end{array} \tag{54}$$

Further, the considerations are similar to the case of the Weyl algebra A with minor changes. We leave them to the reader. For example, Theorem 2 for the case of the algebra of differential operators is the same. One need only to replace everywhere in its statement A, ${}^hA$ and X_n by B, ${}^hB$ and D_n respectively. Thus, one can prove Theorem 1 for the case when A is an algebra of differential operators (but now it is B). Theorem 1 is proved completely.

One can consider a more general algebra of differential operators. Let $\mathcal{F}$ be a field with n derivatives $D_1, \dots, D_n$. Then $K_n = \mathcal{F}[D_1, \dots, D_n]$ is the algebra of differential operators. Similarly one can define its homogenization ${}^hK_n$ by means of adding the variable X_0 satisfying the relations

$$D_i D_j = D_j D_i, \quad X_0 D_i = D_i X_0, \quad D_i f - f D_i = f_{D_i} X_0$$

for all i, j and all elements $f \in \mathcal{F}$ where $f_{D_i} \in \mathcal{F}$ denotes the result of the application of D_i to f. Following the proof of Theorem 1 one can deduce the following statement.

REMARK 8 *Bounds similar to the one from Theorem 1 hold for K_n (in place of the algebra of differential operators A).*

Appendix 1: Degrees of generators of a graded module over a polynomial ring and its Hilbert function.

We give a short proof of the following result, cf. [1], [12], [6], [4]. Let $\mathcal{A} = F[X_0, \dots, X_n]$ be a graded polynomial ring. Homogeneous elements of $\mathcal{A}$ are homogeneous with respect to $X_0, \dots, X_n$ polynomials.

LEMMA 12 *Let $I \subset \mathcal{A}^l$ be a graded $\mathcal{A}$-module, and I is given by a system of generators $f_1, \dots, f_m$ of degrees less than d where $d \geqslant 2$. Then the Hilbert function $H(\mathcal{A}^l/I, m) = \dim_F(\mathcal{A}^l/I)_m$ is stable for $m \geqslant (dl)^{2^{O(n+1)}}$. Further, all the coefficients of the Hilbert polynomial of $\mathcal{A}^l/I$ are bounded from above by $(dl)^{2^{O(n+1)}}$.*

PROOF Extending the ground field F we shall suppose without loss of generality that the field F is infinite. Denote $M = \mathcal{A}^l/I$. Let $L \in F[X_0, \dots, X_n]$ be a linear form in general position. Denote by K the kernel of the morphism $M \to M$ of multiplication to L. We have $K = \{z \in \mathcal{A}^l : Lz = \sum_{1 \leqslant i \leqslant m} f_i z_i, \& \, z_i \in \mathcal{A}\}$. Hence solving a linear system over $\mathcal{A}$, we get that K has a system of generators $g_1, \dots, g_\mu$ with degrees bounded from above by $(dl)^{2^{O(n+1)}}$. Let $\mathfrak{P}$ be an arbitrary associated prime ideal of the module M such that $\mathfrak{P} \neq (X_0, \dots, X_n)$. Since L is in general position we have $L \notin \mathfrak{P}$. Hence $\mathfrak{P}$ is not an associated prime ideal of K. Therefore, $K_N = 0$ for all sufficiently big N. So $X_i^N g_j \in I$ for sufficiently big N and all i, j. Hence $g_j = \sum_{1 \leqslant i \leqslant m} y_{j,i} f_i$ where $y_{j,i} \in F(X_i)[X_0, \dots, X_n]$. Solving a linear system over the ring $F(X_i)[X_0, \dots, X_n]$ we get an estimation for denominators from $F[X_i]$ of all $y_{j,i}$. Since all g_j and f_i are homogeneous we can suppose without loss of generality that all the denominators are X_i^N. Thus, we get an upper bound for N. Namely, N is bounded from above by $(dl)^{2^{O(n+1)}}$.

Therefore, the sequence

$$0 \to M_m \to M_{m+1} \to (M/LM)_{m+1} \to 0 \tag{55}$$

is exact for $m \geqslant (dl)^{2^{O(n+1)}}$. But $M/LM = \mathcal{A}^l/(I + L\mathcal{A}^l)$ is a module over a polynomial ring of $F[X_0, \dots, X_n]/(L) \simeq F[X_0, \dots, X_{n-1}]$. Hence by the inductive assumption the Hilbert function $H(\mathcal{A}^l/(I + L\mathcal{A}^l), m)$ is stable for $m \geqslant (dl)^{2^{O(n)}}$. Now (55) implies that the Hilbert function $H(\mathcal{A}^l/I, m)$ is stable for $m \geqslant (dl)^{2^{O(n+1)}}$.

Obviously for $m < (dl)^{2^{O(n+1)}}$ the values $H(\mathcal{A}^l/I, m)$ are bounded from above by $(dl)^{2^{O(n+1)}}$. Hence by the Newton interpolation all the coefficients of the Hilbert polynomial of $\mathcal{A}^l/I$ are bounded from above by $(dl)^{2^{O(n+1)}}$. The lemma is proved.

We need also a conversion of Lemma 12.

LEMMA 13 *Let $I \subset \mathcal{A}^l$ be a graded $\mathcal{A}$-module. Assume that the Hilbert function $H(\mathcal{A}^l/I, m) = \dim_F(\mathcal{A}^l/I)_m$ is stable for $m \geqslant D$ and all absolute values of the coefficients of the Hilbert polynomial of the module $\mathcal{A}^l/I$ are bounded from above by D for some integer $D > 1$. Then I has a system of generators $f_1, \ldots, f_m$ with degrees $D^{2^{O(n+1)}}$.*

PROOF Let us choose $f_1, \ldots, f_m$ to be the reduced Gröbner basis of I with respect to an admissible linear order $<$ on the monomials from $\mathcal{A}^l$, cf. the definitions from Section 1 and Section 4. The degree of a monomial from $\mathcal{A}^l$ is defined similarly to Section 1 and Section 4. We shall suppose additionally that the considered linear order is degree compatible, i.e., for any two monomials z_1, z_2 if $\deg z_1 < \deg z_2$ then $z_1 < z_2$. For every $z \in \mathcal{A}^l$ the greatest monomial $\mathrm{Hdt}(z)$ is defined. Further the monomial module $\mathrm{Hdt}(I)$ is generated by all $\mathrm{Hdt}(z)$, $z \in I$. Now $\mathrm{Hdt}(f_1), \ldots, \mathrm{Hdt}(f_m)$ is a minimal system of generators of $\mathrm{Hdt}(I)$ and $\deg f_i = \deg \mathrm{Hdt}(f_i)$ for every $1 \leqslant i \leqslant m$. The values of Hilbert functions $H(\mathcal{A}^l/\mathrm{Hdt}(I), m) = H(\mathcal{A}^l/I, m)$ coincide for all $m \geqslant 0$, cf. Section 4. Thus, replacing I by $\mathrm{Hdt}(I)$ we shall assume in what follows in the proof that I is a monomial module.

For every $1 \leqslant i \leqslant l$ denote by $\mathcal{A}_i \subset \mathcal{A}^l$ the i-th direct summand of $\mathcal{A}^l$. Put $I_i = I \cap \mathcal{A}_i$, $1 \leqslant i \leqslant l$. Then $I \simeq \oplus_{1 \leqslant i \leqslant l} I_i$ since I is a monomial module. Further, for every $1 \leqslant \alpha \leqslant m$ there is $1 \leqslant i \leqslant l$ such that $f_\alpha \in I_i$. Let us identify $\mathcal{A}_i = \mathcal{A}$. Then $I_i \subset \mathcal{A}$ is a homogeneous monomial ideal. The case $I_i = \mathcal{A}$ is not excluded for some i. For the Hilbert functions we have

$$H(\mathcal{A}^l/I, m) = \sum_{1 \leqslant i \leqslant l} H(\mathcal{A}/I_i, m), \quad m \geqslant 0. \tag{56}$$

If $(\mathcal{A}/I_i)_D = 0$ for some i then $(\mathcal{A}/I_i)_m = 0$ for every $m \geqslant D$. In this case the ideal I_i is generated by $\sum_{0 \leqslant m \leqslant D} (I_i)_m$. Hence in (56) for the values $m \geqslant D$ one can omit this index i in the sum from the right part. Therefore, in this case the proof is reduced to a smaller l. So we shall assume without loss of generality that $(\mathcal{A}/I_i)_D \neq 0$, $1 \leqslant i \leqslant l$.

Further, we use the exact description of the Hilbert function of a homogeneous ideal, see [4] Section 7. Namely there are the unique integers $b_{i,0} \geqslant b_{i,1} \geqslant \ldots \geqslant b_{i,n+2} = 0$ such that

$$H(\mathcal{A}/I_i, m) = \binom{m+n+1}{n+1} - 1 - \sum_{1 \leqslant j \leqslant n+1} \binom{m - b_{i,j} + j - 1}{j} \tag{57}$$

for all sufficiently big m and

$$b_{i,0} = \min\{d : d \geqslant b_{i,1} \,\&\, \forall\, m \geqslant d \quad (57) \quad \text{holds}\}. \tag{58}$$

This description (without constants $b_{i,0}$) is originated from the classical paper [11]. The integers $b_{i,0}, \ldots, b_{i,n+2}$ are called the Macaulay constants of the ideal I_i. Besides that,

$$h(i,m) = H(\mathcal{A}/I_i, m) - \binom{m+n+1}{n+1} + 1 + \sum_{1 \leqslant j \leqslant n+1} \binom{m - b_{i,j} + j - 1}{j} \geqslant 0 \tag{59}$$

for every $m \geqslant b_{i,1}$, see [4] Section 7. By Lemma 7.2 [4] for all $1 \leqslant \alpha \leqslant m$ if $f_\alpha \in I_i$ then $\deg f_\alpha \leqslant b_{i,0}$. Hence it is sufficient to prove that all $b_{i,0}$, $1 \leqslant i \leqslant l$, are bounded from above by $D^{2^{O(n+1)}}$.

By (56) and (57) the coefficient at m^{n-j}, $0 \leqslant j \leqslant n$, of the Hilbert polynomial of $\mathcal{A}^l/I$ is

$$\frac{\mu_j}{(n+1-j)!} \sum_{1 \leqslant i \leqslant l} b_{i,n+1-j} + \sum_{0 \leqslant v \leqslant j-1} \sum_{1 \leqslant i \leqslant l} \frac{1}{(n+1-v)!} \mu_{j,v}(b_{i,n+1-v}), \qquad (60)$$

where $0 \neq \mu_j$ is an integer and $\mu_{j,v} \in \mathbb{Z}[Z]$, $0 \leqslant v \leqslant j-1$, is a polynomial with integer coefficients and the degree $\deg \mu_{j,v} = j - v + 1$. Moreover, $|\mu_j|$ and absolute values of all the coefficients of all the polynomials $\mu_{j,v}$ are bounded from above by, say, $2^{O(n^2)}$. Denote $b_j = \sum_{1 \leqslant i \leqslant l} b_{i,j}$, $0 \leqslant j \leqslant n+2$. By the condition of the lemma all the coefficients of the Hilbert polynomial of $\mathcal{A}^l/I$ are bounded from above by D. Hence from (60) one can recursively estimate $b_{n+1}, b_n, \ldots, b_1$. Namely, $b_{n+1-j} = (2^{n^2} lD)^{2^{O(j+1)}}$, $0 \leqslant j \leqslant n$. Hence $b_1 = (lD)^{2^{O(n+1)}}$. Notice that $b_{i,1} \leqslant \max_{1 \leqslant i \leqslant l} b_{i,1} \leqslant b_1$ for every $1 \leqslant i \leqslant m$.

Now let $m \geqslant \max_{1 \leqslant i \leqslant l} b_{i,1}$. By (59) if $h(i,m) \neq 0$ for some $1 \leqslant i \leqslant l$ then $m < D$, i.e., m is less than the bound D for the stabilization of the Hilbert function of $\mathcal{A}^l/I$. Thus, $b_{i,0} \leqslant \max\{b_{i,1}, D\}$ by (58). Hence $b_{i,0}$ is bounded from above by $(lD)^{2^{O(n+1)}}$.

We have $(\mathcal{A}/I_i)_D \neq 0$ for every $1 \leqslant i \leqslant l$. This implies $H(\mathcal{A}^l/I, D) \geqslant l$. Denote by c_j the j-th coefficient of the Hilbert polynomial of the module $\mathcal{A}^l/I$. Now $|c_j| D^j \geqslant l/(n+1)$ for at least one j. Hence $D^{n+1}(n+1) \geqslant l$ by the condition of the lemma. This implies that $l^{2^{O(n+1)}}$ is bounded from above by $D^{2^{O(n+1)}}$. Therefore, $b_{i,0}$ is bounded from above by $D^{2^{O(n+1)}}$. The lemma is proved.

Appendix 2: Bound on the Gröbner basis of a monomial module via the coefficients of its Hilbert polynomial

Denote by $C_l = \mathbb{Z}_+^n \cup \cdots \cup \mathbb{Z}_+^n$ the disjoint union of l copies of the semigroup $\mathbb{Z}_+^n = \{(i_1, \ldots, i_n) \in \mathbb{Z}^n : i_j \geqslant 0,\ 1 \leqslant j \leqslant n\}$. A subset of C_l which intersects each disjoint copy of $\mathbb{Z}_+^n$ by a semigroup closed with respect to addition of elements from $\mathbb{Z}_+^n$ is called an ideal of C_l. Clearly, I corresponds to a monomial submodule M_I in the free module $(F[X_1, \ldots, X_n])^l$. Any ideal I in C_l has a unique finite Gröbner basis $V = V_I$ corresponding to the Gröbner basis of M_I. Denote $T = C_l \setminus I$. The degree of an element $u = (k; i_1, \ldots, i_n) \in C_l, 1 \leqslant k \leqslant l$ is defined as $|u| = i_1 + \cdots + i_n$. The degree of a subset in C_l is defined as the supremum of the degrees of its elements. The Hilbert function $H_T(z)$ equals to the number of vectors $u \in T$ such that $|u| \leqslant z$. Hence $H_T(z) = \sum_{0 \leqslant s \leqslant m} c_s z^s, \quad z \geqslant z_0$ for suitable z_0 and integers $c_0, \ldots, c_m$ where the degree $m \leqslant n$. Denote $c = \max_{0 \leqslant s \leqslant m} |c_s| s! + 1$.

PROPOSITION 1 *(cf. [6], [12], [4]). The degree of V does not exceed $(cn)^{2^{O(m)}}$.*

PROOF An *s-cone*, $0 \leqslant s \leqslant n$, we call a subset of a k-th copy of $\mathbb{Z}_+^n$ in C_l for a certain $1 \leqslant k \leqslant l$ of the form

$$P = \{X_{j_1} = i_1, \dots, X_{j_{n-s}} = i_{n-s}\} \tag{61}$$

for suitable $1 \leqslant j_1, \dots, j_{n-s} \leqslant n$. We define the degree of s-cone (61) as $|P| = i_1 + \dots + i_{n-s}$ (note that this definition is different from the one in [4]). By a *predecessor* of (61) we mean each s-cone in the same k-th copy of $\mathbb{Z}_+^n$ of the type

$$\{X_{j_1} = i_1, \dots, X_{j_{p-1}} = i_{p-1}, X_{j_p} = i_p - 1, X_{j_{p+1}} = i_{p+1}, \dots, X_{j_{n-s}} = i_{n-s}\} \tag{62}$$

for some $1 \leqslant p \leqslant n - s$, provided that $i_p \geqslant 1$. Fix an arbitrary linear order on s-cones compatible with the relation of predecessors.

By inverse recursion on s we fill gradually T (as a union) by s-cones with $0 \leqslant s \leqslant m$. For the base we start with $s = m$. Assume that a current union $T_0 \subset T$ of m-cones is already constructed (at the very beginning we put $T_0 = \emptyset$) and an m-cone of the form (61) with $s = m$ is the least one (with respect to the fixed linear order on m-cones) which is contained in T and not contained in T_0. Observe that each predecessor of this m-cone was added to T_0 at earlier steps of its construction. Since the total number of m-cones added to T_0 does not exceed $c_m m! < c$ we deduce that the degree of every such m-cone is less than $c_m m!$ (taking into account that the very first m-cone added to T_0 has the degree 0).

For the recursive step assume that the current T_0 is a union of all possible m-cones, $(m-1)$-cones, …, $(s+1)$-cones and perhaps, some s-cones. This can be expressed as $\deg(H_T - H_{T_0}) \leqslant s$. Again as in the base take the least s-cone of the form (61) which is contained in T and not contained in T_0. Observe that each predecessor of the type (62) of this s-cone is contained in an appropriate r-cone Q, $r \geqslant s$, such that Q was added to T_0 at earlier steps of its constructing and $Q \subset \{X_{j_p} = i_p - 1\}$. Hence

$$|Q| \geqslant i_p - 1. \tag{63}$$

The described construction terminates when $T_0 = T$. Denote by t_s the number of s-cones added to T_0 and by k_s the maximum of their degrees. We have seen already that $t_m, k_m < c$.

Now by inverse induction on s we prove that $t_s, k_s \leqslant (cn)^{2^{O(m-s)}}$. To this end we introduce a relevant semilattice on the set of cones. Let $\mathcal{C} = \{C_{\alpha,\beta}\}_{\alpha,\beta}$, $0 \leqslant \beta \leqslant \gamma_\alpha$ be a family of cones of the form (61) where $\dim C_{\alpha,\beta} = \alpha$. By an α-piece we call an α-cone being the intersection of a few cones from $\mathcal{C}$. All the pieces constitute a semilattice $\mathcal{L}$ with respect to the intersection with maximal elements from $\mathcal{C}$. We treat $\mathcal{L}$ also as a partially ordered set with respect to the inclusion relation. Clearly, the depth of $\mathcal{L}$ is at most $n+1$. Our nearest purpose is to bound from above the size of $\mathcal{L}$. For the sake of simplifying the bound we assume (and this will suffice for our goal in the sequel) that $\gamma_\alpha \leqslant (cn)^{2^{O(m-\alpha)}}$ for $s \leqslant \alpha \leqslant m$ and $\gamma_\alpha = 0$ when $\alpha < s$, although one could write a bound in general in the same way. Besides that we assume that the constant in $O(\dots)$ is sufficiently big. In what follows all the constants in $O(\dots)$ coincide.

LEMMA 14 *Suppose that for all $s \leqslant \alpha \leqslant m$ the number $\gamma_\alpha \leqslant (cn)^{2^{O(m-\alpha)}}$,* *see above. Then the number of α-pieces in $\mathcal{L}$ does not exceed $(cn)^{2^{O(m-\alpha)}+1}$ for $s \leqslant \alpha \leqslant m$ or $(cn)^{2^{O(m-s)}(s-\alpha+1)+1}$ when $\alpha < s$.*

PROOF For each α-piece choose its arbitrary irredundant representation as the intersection of the cones from $\mathcal{C}$. Let δ be the minimal dimension of these cones. Then this intersection contains at most $\delta - \alpha + 1$ cones. Therefore, the number of possible α-pieces does not exceed

$$\sum_{\max\{\alpha,s\}\leqslant\delta\leqslant m} (cn)^{2^{O(m-\delta)}(\delta-\alpha+1)},$$

that proves the lemma.

Now we come back to estimating t_s, k_s by inverse induction on s. Let in the described above construction the current T_0 is the union of all added m-cones, $(m-1)$-cones, ..., s-cones. Denote this family of cones by $\mathcal{C}$ and consider the corresponding semilattice $\mathcal{L}$ (see above). Our next purpose is to represent T_0 as a $\mathbb{Z}$-linear combination of the pieces from $\mathcal{L}$ by means of a kind of the inclusion-exclusion formula. We assign the coefficients of this combination by recursion in $\mathcal{L}$. As a base we assign 1 to each maximal piece, so to the elements of $\mathcal{C}$. As a recursive step, if for a certain piece $P \in \mathcal{L}$ the coefficients are already assigned to all the pieces greater than P, we assign to P the coefficient ϵ_P in such a way that the sum of the assigned coefficients to P and to all the greater pieces equals to 1. Therefore

$$T_0 = \sum_{P\in\mathcal{L}} \epsilon_P P$$

where the sum is understood in the sense of multisets. Hence

$$H_{T_0}(z) = \sum_{P\in\mathcal{L}} \epsilon_P \binom{z-|P|+\dim P}{\dim P} \tag{64}$$

for large enough z. We recall that $\deg(H_T - H_{T_0}) \leqslant s-1$.

Now we majorate the coefficients $|\epsilon_P|$ by induction in the semilattice $\mathcal{L}$. The inductive hypothesis on $t_\alpha \leqslant (cn)^{2^{O(m-\alpha)}}, s \leqslant \alpha \leqslant m$ and Lemma 14 imply that

$$\sum_{\dim P=\lambda} |\epsilon_P| \leqslant (cn)^{2^{O(m-\lambda)}}, \quad s-1 \leqslant \lambda \leqslant m.$$

by inverse induction on λ and the definition of ϵ_P. In fact, one could majorate in a similar way also $\sum_{\dim P=\lambda} |\epsilon_P|$ when $\lambda < s-1$, but we don't need it. The inductive hypothesis on $k_\alpha \leqslant (cn)^{2^{O(m-\alpha)}}, \quad s \leqslant \alpha \leqslant m$ and (64) entail that the coefficient of $H_{T_0}(z)$ at the power z^α does not exceed $(cn)^{2^{O(m-\alpha)}}, \quad s-1 \leqslant \alpha \leqslant m$ (actually, due to the inequality $\deg(H_T - H_{T_0}) \leqslant s-1$ the coefficients at the powers z^α for $s \leqslant \alpha \leqslant m$ are less than c). In particular, the coefficient at the power z^{s-1} does not exceed $(cn)^{2^{O(m-s+1)}}$. Denote $H_T - H_{T_0} = \eta z^{s-1} + \cdots$. By constructing T_0 we add to it $t_{s-1} = \eta(s-1)!$ of $(s-1)$-cones. Hence $t_{s-1} \leqslant (cn)^{2^{O(m-s+1)}}$. This justifies the inductive step for t_{s-1}.

Let us prove that $k_{s-1} \leqslant (cn)^{2^{O(m-s+1)}}$. We observe that for each $(s-1)$-cone P added to T_0 either every its predecessor is contained in a cone of dimension at least s, or some its predecessor is an $(s-1)$-cone as well. In the former case $|P| \leqslant (\max_{s\leqslant\alpha\leqslant m} k_\alpha + 1)(n-s+1)$ (due to (63)), while in the latter case $|P|$ is greater by 1 than the degree of this predecessor Hence $k_{s-1} \leqslant (\max_{s\leqslant\alpha\leqslant m} k_\alpha + 1)(n-s+1)+t_{s-1}$. Finally, we exploit the inductive hypothesis for $k_m, \dots, k_s$ and the just obtained inequality on t_{s-1}.

To complete the proof of the proposition it suffices to notice that for any vector from the basis V treated as an 0-cone, each its predecessor of the type (62) for $s=0$ is contained in an appropriate r-cone from the described construction, whence the degree of V does not exceed $(\max_{0\leqslant\alpha\leqslant m} k_\alpha + 1)n$ again due to (63) (cf. above).

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