Loewy- and Primary-Decompositions of D-Modules

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Abstract

Starting from the well-known factorization of linear ordinary differential equations, we define the generalized Loewy and primary decomposition for a $\mathcal{D}$-module. To this end, for any module I, overmodules $J \supseteq I$ are constructed. They subsume the conventional factorization as special cases. Furthermore, the new concept of the module of relative syzygies $Syz(I, J)$ is introduced. The invariance of this module and its solution space w.r.t. the set of generators is shown. We design an algorithm which constructs the Loewy and primary decompositions for finite-dimensional and some kinds of more general $\mathcal{D}$-modules. Also an algorithm is exhibited which describes all isomorphisms of finite-dimensional $\mathcal{D}$-modules. These results are applied for solving various second- and third-order linear partial differential equations.

Introduction

The concept of factorization of a linear ordinary differential equation (lode) originally goes back to Beke [1] and Schlesinger [28]. Loewy [19] extended it and introduced a unique decomposition of any lode into largest completely reducible factors, i. e. factors which are the least common multiple of irreducible right factors. Similar as in the algebraic case, if such a nontrivial decomposition may be found, the solution procedure is faciliated because the order of the equations to be solved is lowered. Algorithms for factoring a lode have also been described by Schwarz [30] and, with an improved complexity bound, by Grigoriev [9]. A survey on factorization of lode's may be found in the book by Singer and van der Put [23].

Factoring linear partial differential equations (lpde's) is much more difficult. So far there has been no common agreement on what to understand by factoring lpde's in general. A first attempt to generalize the above theory was undertaken by Tsarev [36]. The paper by Li et al. [18] considered factoring those lpde's which have a finite-dimensional solution space, it is achieved by a fairly straightforward extension of the factorization of lode's. Recently in [13] the problem of factoring a single lpde was studied. An algorithm was designed for factoring so-called separable lpde's, but the general factorization problem remained open, see also [37].

Here an algebraic approach is suggested which subsumes the conventional factorizations and its corresponding decompositions as special cases. Any given linear differential equation is considered as the result of applying a differential operator to a differential indeterminate. This operator or, if a system of equations is involved, this set of operators, are considered as generators of a left $\mathcal{D}$-module over an appropriate ring of differential operators. Some background on $\mathcal{D}$-modules may be found e. g.

in the books by Sabbah [26] or Coutinho [5]. Certain algorithms for $\mathcal{D}$-modules may be found in [10], [11], [12] and [20]. In our algebraic approach decomposing a $\mathcal{D}$-module means finding overmodules which describe various parts of the solution of the original problem. There are two possibilites for constructing these overmodules.

- A set of new generators is searched for such that the original module may be reduced to zero wrt. to them. This stands for the conventional factorization like factoring linear ode's [30], factoring linear pde's with a finite-dimensional solution space [18], or the factorizations that have been described in [13].

- It may be possible to construct new generators forming a Janet base of an overmodule in combination with the given ones, which are not necessarily of lower order.

In either case, the result is a set of operators generating an overmodule of the given one. The further proceeding depends on the result of this construction. It may occur that several over-modules have been obtained such that their intersection is identical to the given one. If this is true, solving the original problem is reduced to solving several, possibly simpler problems, each of which describes some part of the desired solution. In Loewy's terminology [19] such a module is called *completely reducible.*

If this case does not apply, for each over-module the module of *relative syzygies* is constructed as defined in Section 2 below. Then the same procedure is applied to it as for the originally given module. This process terminates until no further over-modules may be constructed. The result is the natural generalization of Loewy's decomposition of ordinary differential operators.

From this decomposition the solution of the originally given equation may be obtained iteratively. At first all homogeneous problems have to be solved. The solutions of the rightmost factors are already part of the solution of the full problem. In the next step the solutions of the module of relative syzygies are taken as inhomogeneity of the respective rightmost factor. Solving these problems yields additional parts of the solution of the full problem. This process is repeated until the last module of relative syzygies has been reached. If all equations that occur in this decomposition may be solved, the general solution of the original problem has been obtained or, if this is not true, at least some part of it.

In Section 1 we show that the space of solutions of a module is determined by its isomorphism class (Proposition 1.1), up to an equivalence $\simeq_{\mathcal{D}}$ which is called $\mathcal{D}$-isomorphism. In Section 2 we introduce the new concept of the module of relative syzygies $Syz(I,J)$ of two modules I and J with $I \subseteq J$. It extends the one given in [18] for finite-dimensional modules. It is shown that it is essentially invariant w.r.t. to the set of generators. We also show that the space $V_{Syz(I,J)}$ of solutions of $Syz(I,J)$ is isomorphic to V_I/V_J (Lemma 2.4). In addition we describe a procedure to calculate the module of relative syzygies. Finally, the relation $a_\tau(Syz(I,J)) = a_\tau(I) - a_\tau(J)$ (Theorem 2.6) is proved for the leading coefficients a_τ of the Hilbert-Kolchin polynomials; τ is the differential type of the module I, see [16] and [17].

In Section 3, at first we define a unique Loewy decomposition of a finite-dimensional module I. The crucial role here plays the intersection $R(I)$ of all maximal overmodules of I. Instead of I the modules $R(I)$ and $Syz(I,R(I))$ with smaller differential type or smaller typical differential dimension (see e.g., [16], [17]) are considered in the inductive definition. After that the Loewy decomposition is generalized to infinite-dimensional modules I of differential type $\tau > 0$. It relies on the intersection $R_\tau(I)$ of the classes of maximal overmodules of I with differential type τ, considered up to modules of differential types less than τ. In Section 4 a unique primary decomposition of a finite-dimensional module I is introduced. The intersection $N(I)$ of all proper overmodules of I plays a crucial role here. Similar to Section 3, we replace I in the inductive definition by $N(I)$ and $Syz(I,N(I))$. Then the primary decomposition is extended to an infinite-dimensional module I of differential type $\tau > 0$. Similar to Section 3 the intersection $N_\tau(I)$ of the classes of overmodules, up to modules of differential type less than τ is introduced. In Section 5 we introduce the formal concept of a parametric-algebraic family of $\mathcal{D}$-modules. Its significance is justified by the algorithms from [18] which generates all the overmodules of a finite-dimensional module I as a parametric-algebraic family. Based on it and on the algorithms manipulating algebraic varieties, we design an algorithm which accepts two such families as input and returns a family of pairs of modules such that one of the modules in a pair contains the other. In particular, this allows one to produce the family of all maximal modules in a given family. Relying on these algorithms one can construct Loewy- and primary decompositions of a finite-dimensional module, see Section 6. In Section 7 an algorithm is exhibited which allows one to yield a parametric-linear family of all $\mathcal{D}$-homomorphisms of two finite-dimensional $\mathcal{D}$-modules and furthermore, a parametric-algebraic family of all their $\mathcal{D}$-isomorphisms. The results of Section 5 are applied in Section 8 for the discussion of algorithms. In particular, the theory outlined in the preceding sections is applied to certain classes of second- and third-order linear pde's with rational function coefficients. An algorithm is presented that accomplishes its Loewy decomposition whenever possible. If it succeeds the solution may be obtained from it.

A short version of this paper has been presented in [14]. Many details and examples have been included in the version at hand, e. g. the complete Section 4 and 5.

1 Invariance of the Space of Solutions of a $\mathcal{D}$-Module

Let F be a universal differential field [16] with commuting derivatives $d_1, \dots, d_m$ and $\mathcal{D} = F[d_1, \dots, d_m]$ be the ring of partial differential operators. Denote by $C \subset F$ its subfield of constants. Introduce differential indeterminates $y_1, \dots, y_n$ over F. By Θ denote the commutative monoid generated by $d_1, \dots, d_m$ and by Γ the set of all the derivatives θy_i for $\theta \in \Theta, 1 \le i \le n$. We fix also an admissible total ordering $\prec$ on the derivatives [17, 27]. A background in differential algebra may be found in [16, 3, 33, 34].

Let $I \subset \mathcal{D}^n$ be a left $\mathcal{D}$-module. For vectors $g = (g_1, \dots, g_n) \in \mathcal{D}^n$, $v = (v_1, \dots, v_n) \in F^n$ we denote the inner product $gv = (g, v^T) = \sum g_i v_i \in F$. By $V_I = \{v \in F^n : Iv = 0\} \subset F^n$ we denote the space of solutions of I being a C-vector space. A priori V_I depends on the embedding $I \subset \mathcal{D}^n$. The purpose of this section is to show that actually V_I depends up to an isomorphism just on the factor $\mathcal{D}^n/I$, considered as well up to an isomorphism.

Now let $I_1 \subset \mathcal{D}^{n_1}$, $I_2 \subset \mathcal{D}^{n_2}$. We say that a $n_1 \times n_2$ matrix $A = (a_{ij})$ with $a_{ij} \in \mathcal{D}$ provides a $\mathcal{D}$-homomorphism from $\mathcal{D}^{n_1}/I_1$ to $\mathcal{D}^{n_2}/I_2$ if $(\mathcal{D}^{n_1}/I_1)A \subset (\mathcal{D}^{n_2}/I_2)$, i.e. $I_1 A \subset I_2$. Clearly one gets a homomorphism of $\mathcal{D}$-modules.

We call $\mathcal{D}^{n_1}/I_1$ and $\mathcal{D}^{n_2}/I_2$ to be $\mathcal{D}$-isomorphic if in addition there exists a $n_2 \times n_1$ matrix $B = (b_{ij})$ with $b_{ij} \in \mathcal{D}$ such that $(\mathcal{D}^{n_2}/I_2)B \subset \mathcal{D}^{n_1}/I_1$ and

$$AB|_{(\mathcal{D}^{n_1}/I_1)} = id, \quad BA|_{(\mathcal{D}^{n_2}/I_2)} = id. \tag{1}$$

For the spaces of solutions $V_{I_1} \subset F^{n_1}$, $V_{I_2} \subset F^{n_2}$ we say that a matrix A provides a $\mathcal{D}$-*homomorphism* if $A(V_{I_2})^T \subset (V_{I_1})^T$ (more precisely, one should talk about a $\mathcal{D}$-homomorphism of the embeddings $V_{I_1} \subset F^{n_1}$, $V_{I_2} \subset F^{n_2}$). In a similar way, if there exists a $n_2 \times n_1$ matrix B such that $B(V_{I_1})^T \subset (V_{I_2})^T$ and

$$AB|_{V_{I_1}^T} = id, \quad BA|_{V_{I_2}^T} = id \tag{2}$$

we call V_{I_1}, V_{I_2} to be $\mathcal{D}$-*isomorphic* and denote this by $V_{I_1} \simeq_{\mathcal{D}} V_{I_2}$. The following proposition extends Lemma 2.5 [32] (established for the ordinary case $m = 1$) to finite-dimensional modules.

Proposition 1.1 *i) A matrix A provides a $\mathcal{D}$-homomorphism of $\mathcal{D}^{n_1}/I_1$ to $\mathcal{D}^{n_2}/I_2$ if and only if it provides $\mathcal{D}$-homomorphisms of V_{I_2} to V_{I_1}.*

ii) $\mathcal{D}^{n_1}/I_1$ and $\mathcal{D}^{n_2}/I_2$ are $\mathcal{D}$-isomorphic if and only if V_{I_1} and V_{I_2} are $\mathcal{D}$-isomorphic.

Proof. *i*) Assume that $(\mathcal{D}^{n_1}/I_1)A \subset (\mathcal{D}^{n_2}/I_2)$. We need to verify that $A(V_{I_2})^T \subset (V_{I_1})^T$. The latter is equivalent to the equality $I_1A(V_{I_2})^T = 0$ which holds because of the inclusion $I_1A \subset I_2$.

Conversely, assume that $A(V_{I_1})^T \subset (V_{I_1})^T$, then as above $I_1A(V_{I_2})^T = 0$ which implies $I_1A \subset I_2$ due to the duality in the differential Zariski topology (see Corollary 1, page 148 in [16], also [33]). Hence $(\mathcal{D}^{n_1}/I_1)A \subset (\mathcal{D}^{n_2}/I_2)$.

ii) Assume that (1) holds. One has to verify (2), i. e. for any $v \in V_{I_1}$ to show that $ABv^T = v^T$. The latter holds if and only if for any $g \in \mathcal{D}^{n_1}$ the equality $gABv^T = gv^T$ is true. Equation (1) entails that $gABv^T = (g + g_0)v^T = gv^T$ for a certain vector $g_0 \in I_1$.

We mention that $\mathcal{D}$-isomorphism of $\mathcal{D}$-modules implies isomorphism of the spaces of their solutions in a more general setting, see e.g. [22], [24] (while the converse essentially uses that we deal with a universal differential field).

Conversely, assume (2) is valid. For any $g \in \mathcal{D}^{n_1}$ (2) implies the equality $(gAB - g)(V_{I_1})^T = 0$, therefore $gAB - g \in I_1$ again due to Corollary 1, page 148 of [16]. This establishes (1). □

Remark 1.2 *We observe that for any two $\mathcal{D}$-modules $I_1 \subset \mathcal{D}^{n_1}, I_2 \subset \mathcal{D}^{n_2}$ such that $\dim_F(\mathcal{D}^{n_1}/I_1) = \dim_F(\mathcal{D}^{n_2}/I_2) < \infty$ we have $\mathcal{D}^{n_1}/I_1 \simeq_{\mathcal{D}} \mathcal{D}^{n_2}/I_2$. On the other hand, in case of infinite-dimensional modules the isomorphism does not always hold, e.g., in case $m = 2$ the modules $\mathcal{D}/(d_1)$ and $\mathcal{D}/(d_2)$ are not $\mathcal{D}$-isomorphic.*

2 Relative Syzygies of $\mathcal{D}$-modules

In Loewy's original decomposition scheme, the largest completely reducible right factors are removed by exact division. This is a valid procedure because all ideals of ordinary differential operators are principal. In the ring of partial differential operators this is not true any more. In addition to the relations following from the division there are the integrability conditions which guarantee that an ideal or module is generated by a Janet base. The proper generalization of the exact quotient is given by the following

Definition 2.1 *(Relative syzygies module) Let $I \subseteq J \subseteq \mathcal{D}^n$ be two $\mathcal{D}$-modules, and let $J =< g_1, \dots, g_t >$. The relative syzygies $\mathcal{D}$-module $Syz(I, J)$ of I and J is $Syz(I, J) = \{(h_1, \dots, h_t) \in \mathcal{D}^t | \sum h_i g_i \in I\}$.*

This definition is more general than the definition of the quotient of $\mathcal{D}$-modules in [18] because we do not require $g_1, \dots, g_t$ to be a Janet basis of J (for a background on Janet basis see e.g. [16, 17, 29, 27]) and in addition it takes into account all relations among $g_1, \dots, g_t$ which put them in I. We notice that in case when $I = 0$ the module $Syz(0, J)$ coincides with the usual syzygies module $Syz(J)$. Our next goal is to show that Definition 2.1 does not depend on the choice of generators $g_1, \dots, g_t$. Another proof may be obtained applying the methods of [24] and [25].

Lemma 2.2 *Let $I \subseteq I_1 \subseteq J$ be $\mathcal{D}$-modules. Then $Syz(I_1, J)/Syz(I, J) \simeq I_1/I$.*

Proof. First we verify that the mapping $\varphi(h_1, \dots, h_t) = \sum h_i g_i$ provides a homomorphism $\varphi : Syz(I_1, J)/Syz(I, J) \to I_1/I$ being a monomorphism according to Definition 2.1. Finally, for any representative $g \in I_1$ of a class $\bar{g} \in I_1/I$ one can write $g = \sum h_i g_i$, then $\varphi(h_1, \dots, h_t) = g$. □

Corollary 2.3 *i) $\mathcal{D}^t/Syz(I, J) \simeq J/I$;*
ii) $Syz(I, J)/Syz(J) \simeq I$.

The main goal for introducing the relative syzygies module according to Definition 2.1 is the following statement proved in [18] in case when $g_1, \dots, g_t$ being a Janet basis of J, one can find in [25] another proof of it.

Lemma 2.4 *With the notation above the C-linear spaces $V_{Syz(I,J)}$ and V_I/V_J are isomorphic.*

Proof. The mapping $\psi : v \to (g_1, \dots, g_t)^T v$ assures the monomorphism $V_I/V_J \hookrightarrow V_{Syz(I,J)} \subset F^t$. To establish that it is an epimorphism, suppose first that $g_1, \dots, g_t$ constitute a Janet basis of J. Let $y = (y_1, \dots, y_n)$ be a vector of differential indeterminates. For any vector $(w_1, \dots, w_t)^T \in V_{Syz(I,J)}$ the system of linear pde's $g_i\overline{y} = w_i$, $1 \le i \le t$ is solvable since $\{g_1 y - w_1, \dots, g_t y - w_t\}$ is a linear coherent autoreduced set, see page 136 in [16], also Theorem 5.5.6, page 247 [17] and [18]. Taking any $f \in I$ one can represent $f = \sum h_i g_i$, then $(h_1, \dots, h_t) \in Syz(I,J)$ and $0 = \sum h_i w_i = f\overline{y}$, thus $\overline{y} \in V_I$. This completes the proof that $\psi : V_I/V_J \simeq V_{Syz(I,J)}$ is an isomorphism.

To get rid of the supposition that $g_1, \dots, g_t$ constitute a Janet basis take an arbitrary set $g_1^{(1)}, \dots, g_{t_1}^{(1)}$ of generators of J and construct the syzygies module $Syz(I,J)^{(1)} \subset \mathcal{D}^{t_1}$; the notation $Syz(I,J)^{(1)}$ is used to distinguish it from the syzygies module $Syz(I,J)$ constructed from a Janet basis $g_1, \dots, g_t$. Take a $t \times t_1$ matrix $W = (w_{i,j})$ over $\mathcal{D}$ such that $(g_1, \dots, g_t)^T = W(g_1^{(1)}, \dots, g_{t_1}^{(1)})^T$ and a $t_1 \times t$ matrix $W^{(1)} = (w_{i,j}^{(1)})$ such that $(g_1^{(1)}, \dots, g_{t_1}^{(1)})^T = W^{(1)}(g_1, \dots, g_t)^T$. We claim that the matrices $W, W^{(1)}$ provide a $\mathcal{D}$-isomorphism from $\mathcal{D}^t/Syz(I,J)$ to $\mathcal{D}^{t_1}/Syz(I,J)^{(1)}$ (see Section 1). Indeed, for the element $(h_1, \dots, h_t) \in \mathcal{D}^t$ we have

$$(h_1, \dots, h_t)WW^{(1)} = (\dots, \sum_{1 \le i \le t, 1 \le j \le t_1} h_i w_{i,j} w_{j,k}^{(1)}, \dots)$$

where the expression is given for the k-th coordinate of the vector. Hence

$$((h_1, \dots, h_t)WW^{(1)}, (g_1, \dots, g_t)^T) = ((h_1, \dots, h_t), (g_1, \dots, g_t)^T),$$

in particular, $(h_1, \dots, h_t)WW^{(1)} - (h_1, \dots, h_t) \in Syz(I,J)$. The dual calculation for the product $W^{(1)}W$ proves the claim. Proposition 1.1 entails that $V_{Syz(I,J)} \simeq_{\mathcal{D}} V_{Syz(I,J)^{(1)}}$. Together with the isomorphism ψ this completes the proof. $\square$

Remark 2.5 *As usual, having Janet bases of $I =< f_1, \dots, f_s >$ and of $J =< g_1, \dots, g_t >$ one can construct a Janet basis of $Syz(I,J)$, e. g. cf. Theorem 5.3.7 in [17], also [18]. Briefly to remind, for each f_j there holds $f_j = \sum h_{i,j} g_i$, $1 \le j \le s$ for certain $h_{i,j} \in \mathcal{D}$. Furthermore, for each pair (k,j) with $1 \le k < j \le t$ we represent the Δ-polynomial of g_k and g_j as $lc(g_j)\theta_1 g_k - lc(g_k)\theta_2 g_j = \sum h_{ijk} g_i$ such that the operators $lc(g_j)\theta_1 g_k$ and $lc(g_k)\theta_2 g_j$ have the same leading terms with the minimal possible leading derivative w.r.t. the applied term ordering $\prec$. Then the basis of $Syz(I,J)$ consists of the vectors $(h_{1,j}, \dots, h_{t,j})$, $1 \le j \le s$, and of the vectors*

$$(h_{1jk}, \dots, h_{kjk} - lc(g_j)\theta_1, \dots, h_{jjk} - lc(g_k)\theta_2, \dots, h_{tjk}), \quad 1 \le k < j \le t. \tag{3}$$

In the special case $I = 0$, the relative syzygies module $Syz(0,J)$ reduces to the syzygies module of J. Then as in Schreyer's theorem, page 212 of [4], one can show that the constructed basis of $Syz(0,J)$ which consists of vectors of the form (3), constitutes in fact, a Janet basis.

We mention also that relying on the algorithm from [10] one can produce a basis of $Syz(I,J)$ starting with arbitrary, not necessarily Janet bases, of I and J, with double-exponential complexity.

Let us denote by H_I the Hilbert-Kolchin polynomial of I w.r.t. the usual filtration by order of derivatives, so $(\mathcal{D}^n)_r = \{f \in \mathcal{D}^n : ord\, f \le r\}$ (cf. page 223 of [17]). The degree $deg(H_I)$ of H_I is called the *differential type* of I [16], page 130 and [17], page 229, and its leading coefficient $lc(H_I)$ is called the *typical differential dimension* of I ibid.

The next theorem can be deduced directly from Theorem 5.2.9 of [17], but we give an independent proof following the arguments from [17], cf. also Theorem 4.1 in [33].

Theorem 2.6 *Let again $I \subseteq J \subseteq \mathcal{D}^n$. Then deg $(H_J) \leq$ deg (H_I), deg $(H_{Syz(I,J)}) \leq$ deg (H_I) and deg $(H_{Syz(I,J)}) =$ deg $(H_I - H_J)$, lc $(H_{Syz(I,J)}) =$ lc $(H_I - H_J)$.*

Proof. We recall that the isomorphism $\varphi : \mathcal{D}^t/Syz(I,J) \hookrightarrow J/I$ in Lemma 2.2 (putting $I_1 = J$) maps $h_1, \ldots, h_t$ to $\sum h_i g_i$. Let $ord(g_i) \leq p$, $1 \leq i \leq t$. Since we have in the filtration $(J/I)_r = J_r/I_r$, $r \geq 0$ (cf. Theorem 5.1.8 of [17]) we obtain that $\varphi((\mathcal{D}^t/Syz(I,J))_r) \subseteq (J/I)_{r+p}$ and thereby

$$H_{Syz(I,J)}(r) = dim_F(\mathcal{D}^t/Syz(I,J))_r \leq dim_F(J/I)_{r+p} = H_I(r+p) - H_J(r+p)$$

for sufficiently large r.

Conversely, assuming w.l.o.g that $g_1, \ldots, g_t$ constitute a Janet basis of J we conclude that for any $g \in (J/I)_r$ one can represent $g = \sum h_i g_i$ with $ord(h_i g_i) \leq r$, $1 \leq i \leq t$ and hence

$$H_I(r) - H_J(r) = dim_F V_{(J/I)_r} \leq dim_F V_{(\mathcal{D}^t/Syz(I,J))_r} = H_{Syz(I,J)}(r)$$

for sufficiently large r. □

Definition 2.7 *(Gauge of a module) Let I be a $\mathcal{D}$-module. We call the pair $(deg(H_I), lc(H_I))$ the gauge of I. We say that a module I_1 is of lower gauge than another one I_2 if the pair $(deg(H_{I_1}), lc(H_{I_1}))$ is less than $(deg(H_{I_2}), lc(H_{I_2}))$ in the lexicographic ordering.*

Remark 2.8 *The construction of the relative syzygies allows one to reduce finding a basis of V_I to finding a basis of V_J and joining it with any solution y of the system $g_i y = w_i$, $1 \leq i \leq t$ (see the proof of Lemma 2.4) for each element $(w_1, \ldots, w_t)$ of a basis of $V_{Syz(I,J)}$. An algorithm for solving the inhomogeneous system $g_i y = w_i$ may be obtained by a proper generalization of Lagrange's variation of constants, see e. g. the textbook [35], page 193-195 if the homogeneous system is known to have a finite-dimensional solution space which will always be the case in our applications. Theorem 2.6 implies that both J and $Syz(I,J)$ have gauges not greater than the gauge of I. Moreover, in the applications in the next section, the gauges of J and $Syz(I,J)$ will be actually lower than the gauge of I. In case of a finite-dimensional ideal I this reduction was exploited in [18].*

3 Loewy Decompositions

Let us first study the case of a finite-dimensional module $I \subset \mathcal{D}^n$, i. e. modules of differential type 0. Consider the intersection $R(I) = J^{(0)} = \cap J$ of all maximal modules $J \supseteq I$. Any intersection of maximal modules will be called a *complete intersection*. $R(I)$ plays a role similar to the role of the radical of two-sided ideals in a ring. Note that there exists a finite number of maximal modules $J_1, \ldots, J_q$ for which $J_1 \cap \cdots \cap J_q = R(I)$. Indeed, keep taking $J_1, J_2, \ldots$ while it is possible to have $dim_C V_{J_1 \cap \cdots \cap J_{i+1}} > dim_C V_{J_1 \cap \cdots \cap J_i}$ for every $i \geq 1$. Since $dim_C V_I < \infty$ we arrive finally at $J_1, \ldots, J_q$ such that $dim_C V_{J_1 \cap \cdots \cap J_q \cap J} = dim_C V_{J_1 \cap \cdots \cap J_q}$ for any maximal module $J \supseteq I$. Then $J_1 \cap \cdots \cap J_q = R(I)$. Applying this procedure to the relative syzygies module $I^{(1)} = Syz(I, J^{(0)})$, replacing the role of I, which one can compute making use of Remark 2.5, this yields a complete intersection $J^{(1)}$ such that $J^{(1)} = R(I^{(1)}) \supseteq I^{(1)}$. Continuing this way, one obtains successively the complete intersections $J^{(0)}, J^{(1)}, \ldots, J^{(s)}$ and the modules $I^{(1)}, \ldots, I^{(s)}$ such that $J^{(l)} = R(I^{(l)})$ and $I^{(l+1)} = Syz(I^{(l)}, J^{(l)})$ for $0 \leq l \leq s-1$, defining $I^{(0)} = I$. In the last step there holds $J^{(s)} = I^{(s)}$. We have $\dim_C V_{I^{(l)}} - \dim_C V_{I^{(l+1)}} = \dim_C V_{J^{(l)}}$ for $0 \leq l \leq s$, defining $V_{I^{(s+1)}} = \{0\}$. Thus, $\dim_C V_I = \sum_{0 \leq l \leq s} \dim_C V_{J^{(l)}}$, which provides an upper bound $s < \dim_C V_I$ on the number of steps of the described procedure. The uniquely defined sequences $J^{(0)}, J^{(1)}, \ldots, J^{(s)}$ and $I^{(1)}, \ldots, I^{(s)}$ can be viewed as a *Loewy decomposition* of I. To get the spaces of solutions $V_{J^{(l)}}$, $0 \leq l \leq s$ of the complete intersections $J^{(l)} = \cap_q J_q^{(l)}$ where $J_q^{(l)}$ are maximal modules, we apply proposition 3.1 [33] (see also the beginning of the proof of theorem 4.1 [33], p.483 and [3]) which entails that $V_{J^{(l)}} = \sum_q V_{J_q^{(l)}}$. Thus, we have proved the following proposition.

Proposition 3.1 *Any finite-dimensional module $I \subset \mathcal{D}^n$ has the unique Loewy decomposition.*

Now we proceed to a Loewy decomposition of an infinite-dimensional module $I \subset \mathcal{D}^n$ of differential type $\tau > 0$. To this end, we introduce another concept first.

Definition 3.2 *(Gauge-equivalence) We say that two modules $J_1, J_2 \subset \mathcal{D}^n$ are gauge-equivalent if J_1, J_2 and $J_1 \cap J_2$ are of the same gauge.*

If J_1 and J_2 are gauge-equivalent, then by Theorem 4.1 in [33] also $J_1 + J_2$ is of the same gauge.

Lemma 3.3 *i) Two modules $J_1 \subseteq J_2$ of differential type τ are gauge-equivalent if and only if $deg(H_{J_1} - H_{J_2}) < \tau$;*

ii) if each of two modules $J_1, J_2 \subseteq J$ is gauge-equivalent to J then $J_1 \cap J_2$ is also gauge-equivalent to J;

iii) gauge-equivalence is an equivalence relation.

Proof. i) follows from Definition 2.7.

ii) Since we have in the filtration $(J_1 \cap J_2)_r = (J_1)_r \cap (J_2)_r$ (cf. Section 2) we get for Hilbert-Kolchin polynomials that $H_{J_1 \cap J_2} - H_J \le (H_{J_1} - H_J) + (H_{J_2} - H_J)$ (the inequality for polynomials means the inequality for their values at sufficiently big integer points), which proves ii).

To prove iii) assume that each of two modules J_1, J_3 of differential type τ is gauge-equivalent to J_2. Then each of two modules $J_1 \cap J_2, J_3 \cap J_2$ is gauge-equivalent to J_2. From ii) we deduce that $J_1 \cap J_2 \cap J_3$ is gauge-equivalent to J_2. Hence i) entails that $deg(H_{J_1 \cap J_2 \cap J_3} - H_{J_2}) < \tau$. On the other hand, the assumption and i) imply that $deg(H_{J_1 \cap J_2} - H_{J_1}) < \tau$, $deg(H_{J_1 \cap J_2} - H_{J_2}) < \tau$, therefore $deg(H_{J_1 \cap J_2 \cap J_3} - H_{J_1}) < \tau$. The latter inequality and the inclusions $J_1 \cap J_2 \cap J_3 \subseteq J_1 \cap J_3 \subseteq J_1$ entail that $deg(H_{J_1 \cap J_3} - H_{J_1}) < \tau$. Together with a similar inequality $deg(H_{J_1 \cap J_3} - H_{J_3}) < \tau$ this completes the proof of iii). □

The equivalence class of gauge-equivalent modules of a module J is denoted by $[J]$. If the actual value of the differential type of the elements of a class $[J]$ equals to τ, any two members of it are called *τ-equivalent* (below τ is fixed and $|J|$ means a class of τ-equivalence).

Example 3.4 *Let $J_1 =< \partial_x >$, $J_2 =< \partial_{xx}, \partial_{xy} >$ and $J_3 =< \partial_y >$. Then $J_1 \cap J_2 = J_2$, $J_1 + J_2 = J_1$ all of which are of gauge $(1,1)$. Consequently J_1 and J_2 are gauge-equivalent. Notice that although J_3 is also of gauge (1,1), it is not gauge-equivalent to J_1 because $J_1 \cap J_3 =< \partial_{xy} >$ which is of gauge (1,2).*

The generic solution of J_1 is $F(y)$, where F is an "undetermined function", whereas J_2 has generic solution $Cx + F(y)$, C being a generic constant. The generic solution here and below is defined with the help of the defining ideal (see e. g. [16], page 146 and [17], page 132) as follows. For a set of elements of a differential field its defining ideal consists of all lpdo's which annihilate them. A solution of an ideal J is generic if its defining ideal coincides with J. Then above C is a generic constant, i. e. the defining ideal of C coincides with $< \partial_x, \partial_y >$, the defining ideal of F is J_1 and the defining ideal of $Cx + F$ coincides with J_2.

We say that $[J_1]$ is *subordinated* to $[J_2]$ if $J_1 \cap J_2$ is τ-equivalent to J_1.

Lemma 3.5 *i) If modules J_1, J_1' are τ-equivalent, J_2, J_2' are also τ-equivalent and moreover, $J_1 \cap J_2$ has differential type τ, then $J_1 \cap J_2$, $J_1' \cap J_2'$ are τ-equivalent as well;*

ii) under the same assumption $J_1 + J_2$, $J_1' + J_2'$ are τ-equivalent;

iii) the relation of subordination is independent of a choice of representatives J_1, J_2 of the classes of τ-equivalence.

Proof. i) From the inclusions $J_1 \cap J_1' \cap J_2 \cap J_2' \subseteq J_1 \cap J_1' \cap J_2 \subset J_1 \cap J_2$ and Lemma 3.3 i), taking into account that $H_{J_2 \cap J_2' \cap J} - H_{J_2 \cap J} \le H_{J_2 \cap J_2'} - H_{J_2}$ for any module J, we conclude (cf. the proof of Lemma 3.3 iii)) that $deg(H_{J_1 \cap J_1' \cap J_2 \cap J_2'} - H_{J_1 \cap J_2}) < \tau$. Therefore, $J_1 \cap J_1' \cap J_2 \cap J_2'$ has differential type τ, as well as $J_1' \cap J_2'$. In a similar way one obtains that $deg(H_{J_1 \cap J_1' \cap J_2 \cap J_2'} - H_{J_1' \cap J_2'}) < \tau$. Then i) follows from Lemma 3.3 i),iii).

ii) From the inclusions $J_1 + J_2 \subseteq J_1 + J_1' + J_2 \subseteq J_1 + J_1' + J_2 + J_2'$ and Lemma 3.3 i), taking into account the inequality $H_{J_1+J} - H_{J_1+J_1'+J} \le H_{J_1} - H_{J_1+J_1'}$ for any module J, we conclude (as in the proof of i)) that $deg(H_{J_1+J_2} - H_{J_1+J_1'+J_2+J_2'}) < \tau$. Since $J_1 + J_2$ has differential type τ due to Theorem 4.1 in [33], $J_1 + J_1' + J_2 + J_2'$ has also differential type τ, as well as $J_1' + J_2'$. In a similar way one obtains that $deg(H_{J_1'+J_2'} - H_{J_1+J_1'+J_2+J_2'}) < \tau$. Then ii) follows from Lemma 3.3 i),iii).

iii) Under the assumption of i) and making use of that J_1 is τ-equivalent to $J_1 \cap J_2$ (thereby, the assumption that $J_1 \cap J_2$ has differential type τ, is fulfilled automatically), we obtain iii) due to Lemma 3.3 iii). □

Remark 3.6 *The proof of i) shows that $J_1 \cap J_2$, $J_1' \cap J_2'$ are gauge-equivalent without the assumption that $J_1 \cap J_2$ has differential type τ because the differential type of $J_1 \cap J_2$ is greater or equal to τ.*

We denote the relation of subordination by $[J_1] \trianglelefteq [J_2]$. Then $lc(H_{J_1}) \ge lc(H_{J_2})$. If in addition $[J_1] \ne [J_2]$ (we denote this by $[J_1] \triangleleft [J_2]$) then $lc(H_{J_1}) > lc(H_{J_2})$. Hence any increasing chain of τ-equivalence classes stops and one can consider maximal τ-equivalence classes.

For any τ-equivalence classes $[J_1], [J_2]$ satisfying $[J] \trianglelefteq [J_1]$, $[J] \trianglelefteq [J_2]$ one can uniquely define the class $[J_1 \cap J_2]$ such that $[J] \trianglelefteq [J_1 \cap J_2]$. One can verify that $\deg(H_{J_1 \cap J_2}) = \tau$ and the class $[J_1 \cap J_2]$ does not depend on the representatives J_1, J_2.

Example 3.7 *Let $J =< \partial_{xyy} >$ with gauge $(1,3)$, $J_1 =< \partial_x >$ and $J_2 =< \partial_y >$, both with gauge $(1,1)$. Because $J \cap J_1 = J \cap J_2 = J$ there holds $[J] \trianglelefteq [J_1]$ and $[J] \trianglelefteq [J_2]$. Furthermore $J_1 \cap J_2 =< \partial_{xy} > \equiv J_3$ with gauge $(1,2)$ and $[J] \trianglelefteq [J_3]$. Because $lc(H_J) = 3$, $lc(H_{J_3}) = 2$ and $lc(H_{J_1}) = lc(H_{J_2}) = 1$, both $[J_1]$ and $[J_2]$ are maximal.*

Now take all τ-maximal classes $[J]$ such that $[I] \trianglelefteq [J]$. Since $J + I$ is τ-equivalent to J (again due to Theorem 4.1 [33]) we can assume without loss of generality that the representatives are chosen in such a way that $I \subseteq J$. We choose consecutively such classes $[J_1], [J_2], \dots, [J_p]$ while it is possible to have

$$[J_1] \triangleright [J_1 \cap J_2] \triangleright \cdots \triangleright [J^{(0)} = J_1 \cap J_2 \cap \cdots \cap J_p].$$

Clearly, $p \le lc(H_I)$. Then for any maximal class $[J]$ for which $[I] \trianglelefteq [J]$, we obtain $[J^{(0)}] \trianglelefteq [J]$. Hence for any finite family $[J_1'], \dots, [J_q']$ of τ-maximal classes for which $[I] \trianglelefteq [J_l']$, $1 \le l \le q$, we conclude that $[J^{(0)}] \trianglelefteq [J_1' \cap \cdots \cap J_q']$. Therefore, the class $[J^{(0)}]$ is defined uniquely and in addition $I \subseteq J^{(0)}$ holds. We say that $J^{(0)} = J_1 \cap J_2 \cap \cdots \cap J_p$ is *completely τ-reducible.*

We define a Loewy decomposition of I by induction on the gauge of I. As a base of induction when the τ-class $[I]$ is maximal then I provides a Loewy decomposition of itself. When $[I]$ is not maximal one can further apply the described inductive definition of a Loewy decomposition (thereby, replacing the role of I) to the relative syzygies module $I^{(1)} = Syz(I, J^{(0)})$ (see Section 2) taking into account that either $\deg(H_{I^{(1)}}) < \tau$ or $\deg(H_{I^{(1)}}) = \tau$, and in the latter case $lc(H_{I^{(1)}}) = lc(H_I) - lc(H_{J^{(0)}}) < lc(H_J)$ due to Theorem 2.6; in other words, $I^{(1)}$ is of a lower gauge than I. In case when $\deg(H_{I^{(1)}}) < \tau$ we have $[I] = [J^{(0)}]$ again due to Theorem 2.6 and $[I]$ being completely τ-reducible.

Continuing this way we arrive at a sequence of modules $J^{(0)}, J^{(1)}, \dots, J^{(q)}$ with non-decreasing differential types such that each module $J^{(l)}$, $0 \le l \le q$ is completely $\deg(H_{J^{(l)}})$-reducible. We notice that this sequence is not necessarily unique unlike the Loewy decomposition of a finite-dimensional module. The obtained sequence could be called a *generalized Loewy decomposition* of I. At present we don't possess an algorithm to construct it in general. The following example due to Blumberg [2] shows the non-uniqueness of the decomposition described above.

Example 3.8 *In his dissertation Blumberg [2] gave the following example of a non-unique factorization of a third-order partial differential operator.*

$$\partial_{xxx} + x\partial_{xxy} + 2\partial_{xx} + 2(x+1)\partial_{xy} + \partial_x + (x+2)\partial_y = \begin{cases} (\partial_{xx} + x\partial_{xy} + \partial_x + (x+2)\partial_y)(\partial_x + 1) \\ (\partial_{xx} + 2\partial_x + 1)(\partial_x + x\partial_y). \end{cases}$$

The second-order factor in the second line at the right hand side may be written as $\partial_{xx} + 2\partial_x + 1 = lclm(\partial_x + 1, \partial_x + 1 - \frac{1}{x})$, *i. e. it is completely reducible, the remaining operators are absolutely irreducible. Consequently in the notation introduced above there holds*

$$J_1^{(0)} =< \partial_x + 1 >, \quad J_1^{(1)} =< \partial_{xx} + x\partial_{xy} + \partial_x + (x+2)\partial_y >,$$
$$J_2^{(0)} =< \partial_x + x\partial_y >, \quad J_2^{(1)} =< \partial_{xx} + 2\partial_x + 1 > .$$

This result may also be obtained by applying the algorithm given in Section 8.

4 Primary Decompositions

At first let $I \subset \mathcal{D}^n$ be a finite-dimensional module. Denote by $J^{(0)} = N(I) = \cap_{J \supset I} J$ the intersection of all modules J properly containing I (we mention that $N(I)$ plays a role similar to the role of the nil-radical of two-sided ideals in a ring). We call I *primary* if $N(I) \neq I$. In the latter case $N(I)$ is the minimal module which properly contains I.

Lemma 4.1 *The relative syzygies module* $Syz(I, N(I))$ *is a maximal module in* $\mathcal{D}^t$ *(see Definition 2.1).*

Proof. Assume the contrary. According to Corollary 2.3 there is an isomorphism $\phi : \mathcal{D}^t/Syz(I, J^{(0)}) \to J^{(0)}/I$. Due to the assumption there exists a $\mathcal{D}$-module M such that

$$Syz(I, J^{(0)}) \subset M \subset \mathcal{D}^t$$

and both quotient modules $M/Syz(I, J^{(0)})$ and $\mathcal{D}^t/M$ are non-zero. Then the quotient modules are the same in the sequence

$$\{0\} \subset \phi(M/Syz(I, J^{(0)})) \subset J^{(0)}/I$$

Considering the epimorphism $\psi : J^{(0)} \to J^{(0)}/I$, we get one more sequence

$$I \subset \psi^{-1}(\phi(M/Syz(I, J^{(0)})) \subset J^{(0)}$$

with the same quotient modules, but this contradicts to the choice of $J^{(0)} = N(I)$, which proves the lemma. □

Lemma 4.2 *Any finite-dimensional module I is an intersection of a finite number of primary modules.*

Proof goes by induction on $\dim_C(I)$. The base of induction for a maximal module is obvious because it is primary. For the inductive step in case when I is not primary one can represent it as a finite intersection $I = J_1 \cap \cdots \cap J_q$ with $J_i \supset I$ (cf. Section 3). Then lemma follows from the inductive hypothesis applied to J_i. $\square$

Therefore, by recursion on $\dim_C(I)$ one can define a *primary decomposition* of I. If I is not primary then one takes $I = J_1 \cap \cdots \cap J_q$ from Lemma 4.2 and the primary decomposition of I is defined as the collection of primary decompositions of $J_1, \dots, J_q$ by the recursive hypothesis. For a primary module I its primary decomposition consists of a pair of the relative syzygies module $Syz(I, N(I))$ (being a maximal module) and a primary decomposition of $N(I)$ by the recursive hypothesis. One can make a primary decomposition unique taking in the intersection $I = J_1 \cap \cdots \cap J_q$ all the primary modules $J_i \supset I$ being minimal with respect to the inclusion. Although, it is unclear whether one could take $I = J_{i_1} \cap \cdots \cap J_{i_{q'}}$ uniquely for a proper subfamily $1 \leq i_1 < \cdots < i_{q'} \leq q$. Thus, we have proved the following proposition.

Proposition 4.3 *Any finite-dimensional module $I \subset \mathcal{D}^n$ has the unique primary decomposition.*

One can view as an advantage of a primary decomposition versus the Loewy decomposition from Section 3 that the expensive operation of taking the relative syzygies module leads to a maximal module, and so taking relative syzygies modules do not iterate each other. On the other hand, the primary decomposition still allows one to find the space V_I by combining the already cited result from [33, 3], and also obtaining V_I from V_J and the space V_I/V_J (see Remark 2.8).

Now let $I \subset \mathcal{D}^n$ be a $\mathcal{D}$-module of differential type τ. We follow the notations from Section 3. We choose consecutively classes $[J_1], [J_2], \dots, [J_p]$ such that $[J_i] \rhd [I]$ for each i (again one can assume that $I \subseteq J_i$) while it is possible such that

$$[J_1] \rhd [J_1 \cap J_2] \rhd \cdots \rhd [J^{(0)} = J_1 \cap J_2 \cap \cdots \cap J_p].$$

Then for any class $[J]$ for which $[I] \lhd [J]$ we have $[J^{(0)}] \unlhd [J]$. We denote $[J^{(0)}] = N_\tau([I])$. If $N_\tau([I]) \rhd [I]$ we call $[I]$ *τ-primary.*

We define a *primary decomposition* of I by induction on the gauge of I. For the base of induction when $[I]$ is a maximal τ-equivalence class then I constitutes its own primary decomposition. For the inductive step a primary decomposition of I consists of the ones of the modules $J_1, \dots, J_p$ and in addition of the relative syzygies module $Syz(I, J^{(0)})$ which has a gauge less than the gauge of I (due to Theorem 2.6, cf. also Section 3). We observe that when the differential type $\dim(Syz(I, J^{(0)})) = \tau$ then $Syz(I, J^{(0)})$ is τ-maximal and provides its own primary decomposition (one can prove this similar to Lemma 4.1). Else $\dim(Syz(I, J^{(0)})) < \tau$ and one deals further in the induction with modules of differential types less than τ.

As a result we arrive at a set of modules $\{J\}$ such that each $[J]$ is a $\dim(J)$-maximal class, which one can view as a *primary decomposition* of I. It would be interesting to design an algorithm which constructs a primary decomposition.

5 Parametric-Algebraic Families of $\mathcal{D}$-Modules

For the rest of the paper, dealing with the design of algorithms, we assume that the coefficients of the input operators belong to the differential field $F_0 = \overline{\mathbf{Q}}(X_1, \dots, X_m)$ (cf. Remark 1.2) with derivatives $d_k = \partial/\partial X_k, 1 \leq k \leq m$ and $\mathcal{D}_0 = F_0[d_1, \dots, d_m], \mathcal{D} = F[d_1, \dots, d_m]$ where F is a universal extension of F_0.

In the sequel we suppose that all the considered algebraic (affine) varieties $W \subset \overline{\mathbf{Q}}^N$ are given in an efficient way, say as in [8]. Namely, $W = \cup W_j$ where W_j are irreducible over $\mathbf{Q}$ components of W, and the algorithms from [8] represent each W_j (of dimension s) in two following ways.

First, we represent W_j by means of a *generic point*, i.e. an isomorphism

$$\mathbf{Q}(t_1, \dots, t_s)[\alpha] \simeq \mathbf{Q}(W_j) \tag{4}$$

where $\mathbf{Q}(W_j)$ is the field of rational functions on W_j. The elements $t_1, \dots, t_s \subset \{Z_1, \dots, Z_N\}$ constitute a basis of transcendency of $\mathbf{Q}(W_j)$ over $\mathbf{Q}$ which can be taken among the coordinates $Z_1, \dots, Z_N$ of the affine space $\overline{\mathbf{Q}}^N$. The element $\alpha = \sum_{1 \le l \le N} \alpha_l Z_l$ for suitable integers α_l is algebraic over the field $\mathbf{Q}(t_1, \dots, t_s)$ with a minimal polynomial $\phi \in \mathbf{Q}(t_1, \dots, t_s)[Z]$. The algorithms from [8] yield the ingredients of (4) explicitly, in other words, $t_1, \dots, t_s$; $\alpha_1, \dots, \alpha_N$; ϕ and the rational expressions of Z_l via $t_1, \dots, t_s, \alpha$, i.e. the rational functions of the form $g_l(t_1, \dots, t_s, Z)/g(t_1, \dots, t_s)$ where the polynomials $g(t_1, \dots, t_s), g_l(t_1, \dots, t_s, Z) \in \mathbf{Q}[t_1, \dots, t_s, Z]$ being such that the equality $Z_l = g_l(t_1, \dots, t_s, Z)/g(t_1, \dots, t_s)$ holds everywhere on W_j.

Second, the algorithms from [8] yield polynomials $h_1, \dots, h_M \in \mathbf{Q}[Z_1, \dots, Z_N]$ such that W_j coincides with the variety of all the points from $\overline{\mathbf{Q}}^N$ which satisfy the system of equations $h_1 = \cdots = h_M = 0$.

The algorithms from [8] allow one to produce the union, intersection, complement of varieties, to get the dimension of W_j, to project a variety (in other words, to eliminate quantifiers), to find all the points of W_j in case when it is finite (i.e. zero-dimensional) or to yield as many points as one wishes in case when W_j is infinite (positive-dimensional). Moreover, one extends these algorithms from varieties to constructive sets , i.e. the unions of the sets of the form $W' \setminus W''$ where W', W'' are varieties (in other terms, constructive sets constitute the boolean algebra generated by all the varieties).

Definition 5.1 *(Parametric-algebraic $\mathcal{D}$-modules) We say that a family of $\mathcal{D}$-modules $\mathcal{J} = \{J\} \subset \mathcal{D}^n$ is parametric-algebraic if there is a constructive set $V = \cup V_j \subset \overline{\mathbf{Q}}^N$ for an appropriate N such that $\mathcal{J} = \cup \mathcal{J}_j$ and for any fixed j the following holds. A Janet basis of any $J \in \mathcal{J}_j$ has fixed leading derivatives $lder(J) = lder_j$ and the parametric derivatives $pder(J) = pder_j$, see [18]. Moreover, any element of the Janet basis of J has the form*

$$\gamma_0 + \sum_{\gamma \in pder_j} A_\gamma(Z_1, \dots, Z_N)\gamma \tag{5}$$

where $\gamma_0 \in lder_j$ and $A_\gamma \in \mathbf{Q}(Z_1, \dots, Z_N)(X_1, \dots, X_m)$.

When $(Z_1, \dots, Z_N)$ ranges over the constructive set V_j, the set of linear differential operators of the form (5) for all $\gamma_0 \in lder_j$ ranges over the Janet basis for all modules J from $\mathcal{J}_j$. Thus, we have a bijective correspondance between the points of V_j and the modules, or rather their Janet basis) from $\mathcal{J}_j$.

We rephrase in our terms the following proposition which was actually proved in [18].

Proposition 5.2 *([18]). One can design an algorithm which for any finite-dimensional $\mathcal{D}$-module $I \subset \mathcal{D}^n$ finds a parametric-algebraic family of all the factors of I, i.e. the modules $J \subset \mathcal{D}^n$ such that $I \subset J$.*

Lemma 5.3 *One can design an algorithm which for a pair of parametric-algebraic families $\mathcal{I}, \mathcal{J}$ of D-modules yields the parametric-algebraic family of all the pairs (I, J) where $I \in \mathcal{I}, J \in \mathcal{J}$ such that $I \subseteq J$.*

Proof. Let

$$\{\gamma_0 + \sum_{\gamma \in pder_j} A_\gamma(Z_1, \dots, Z_N)\gamma\}_{\gamma_0 \in lder_j}$$

be a Janet basis of $\mathcal{J}_j$ and

$$\{\lambda_0 + \sum_{\lambda \in pder_s} B_\lambda(Z_1, \dots, Z_N)\lambda\}_{\lambda_0 \in lder_s}$$

be a Janet basis of $\mathcal{I}_s$. Then the condition that $I \subseteq J$ for $I \in \mathcal{I}_s, J \in \mathcal{J}_j$ can be expressed as the existence for each $\lambda_0 \in lder_s$ of operators of the form $\sum_\theta C_{\theta,\gamma_0,\lambda_0}\theta \in \mathcal{D}$ where $\theta \prec \theta_0$ and $\lambda_0 = \theta_0 y_i$ for a certain $1 \le i \le n$ such that

$$\lambda_0 + \sum_{\lambda \in pder_s} B_\lambda(Z_1, \dots, Z_N)\lambda = \sum_{\gamma_0 \in lder_j} (\sum_\theta C_{\theta,\gamma_0,\lambda_0}\theta)(\gamma_0 + \sum_{\gamma \in pder_j} A_\gamma(Z_1, \dots, Z_N)\gamma) \tag{6}$$

where the external summation in the right-hand side ranges over the elements of the Janet basis of $\mathcal{J}_j$.

Clearly, one can rewrite (6) as a system of linear (algebraic) equations in the unknowns $C_{\theta,\gamma_0,\lambda_0}$, while the entries of this system are the rational functions from $\mathbf{Q}(X_1, \dots, X_m)(Z_1, \dots, Z_N)$. One can find the constructive set $U = U_{j,s} \subset \overline{\mathbf{Q}}^N$ such that just for $(Z_1, \dots, Z_N) \in U$ this linear system is solvable. Combining this for all pairs l, s completes the proof of the lemma. □

Corollary 5.4 *For a finite-dimensional $\mathcal{D}$-module $I \subset \mathcal{D}^n$ one can find a parametric-algebraic family $\mathcal{I}_{max}$ of all maximal $\mathcal{D}$-modules J which contain I.*

Proof. Among the family of all the factors J of I produced in proposition 5.2 one can relying on Lemma 5.3 distinguish all J_0 such that if $J_0 \subseteq J$ then $J_0 = J$ holds. □

6 Constructing Loewy- and Primary Decompositions

Now we are able for a finite-dimensional D-module $I \subset \mathcal{D}_0^n$ to construct its Loewy (see section 3) and primary decompositions (see section 4). First, in order to obtain Loewy decomposition we apply corollary 5.4. After that the purpose is to find the intersection $R(I)$ of all the maximal modules from $\mathcal{I}_{max}$. To this end we conduct the (internal) recursion on $\dim(R(I))$. Assume that a current (complete) intersection J_0 of several maximal modules from $\mathcal{I}_{max}$ is already constructed. Applying lemma 5.3 we test whether there exists a maximal module $J \in \mathcal{I}_{max}$ which does not contain J_0. Then we replace J_0 by the (complete) intersection $J \cap J_0$ and continue the (internal) recursion. Finally, we arrive at $R(I)$ and thereupon (by the external recursion) proceed to the relative syzygies module $Syz(I, R(I))$ (see section 2), provided that the latter is not zero, else halt.

In order to construct a primary decomposition of I we use Proposition 5.2 and Lemma 5.3 in a similar way and (by the internal recursion) compute the intersection $N(I)$ of all the modules strictly containing I in the form $N(I) = \cap J$ where the latter intersection is finite. Thereupon we proceed (by the external recursion) to primary decompositions of all the non-maximal J from this intersection joined by the relative syzygies module $Syz(I, N(I))$ (provided that the latter does not vanish). If all J are maximal then halt.

Thus, we have shown the following

Corollary 6.1 *For a finite-dimensional D-module $I \subset \mathcal{D}_0^n$ one can construct its Loewy and primary decompositions*

7 Testing Isomorphism of Finite-Dimensional $\mathcal{D}$-Modules

We follow the notations of Section 1. We assume that the field of constants $C \subset F$ coincides with $\bar{\mathbf{Q}}$ and the modules $I_1 \subset \mathcal{D}^{n_1}$, $I_2 \subset \mathcal{D}^{n_2}$ are defined over the field $F_0 = \bar{\mathbf{Q}}(X_1, \dots, X_m)$. We design an algorithm to test whether $\mathcal{D}^{n_1}/I_1 \simeq_{\mathcal{D}_0} \mathcal{D}^{n_2}/I_2$. W.l.o.g. one can suppose that $dim(\mathcal{D}^{n_1}/I_1) = dim(\mathcal{D}^{n_2}/I_2) = l$. Then $dim_C V_{I_1} = dim_C V_{I_2} = l$ ([16], page 151). Let $I_1 =< g_1, \dots, g_q >$, $I_2 = < f_1, \dots, f_p >$ be Janet bases of I_1 and I_2 respectively. The condition that a matrix $A = (a_{i,j})$ with $a_{i,j} \in \mathcal{D}$ provides a $\mathcal{D}$-homomorphism of $\mathcal{D}^{n_1}/I_1$ and $\mathcal{D}^{n_2}/I_2$ can be expressed as a system

$$\sum_{1\le i\le n_1} g_{s,i}a_{i,j} = \sum_{1\le t\le p} h_{s,t}f_{t,j}, \;\; 1 \le s \le q, \;\; 1 \le j \le n_2 \tag{7}$$

of lpde's with unknowns $a_{i,j}, h_{s,t} \in \mathcal{D}$. Since $a_{i,j}$ are taken modulo f_j, $1 \le j \le p$ one can assume $a_{i,j}$ to be reduced modulo f_j, $1 \le j \le p$. Let $ord(g_s), ord(f_j) \le r$, $1 \le s \le q$, $1 \le j \le p$, then $ord(a_{i,j}) \le r$, $ord(g_{s,i}a_{i,j}) \le 2r$ and therefore $ord(h_{s,t}f_{t,j}) \le 2r$ as well because $f_1, \dots, f_p$ is a Janet basis. Thus writing $a_{i,j} = \sum_K a_{i,j,K}d^K$, $h_{s,t} = \sum_K h_{s,t,K}d^K$ with the weights of multiindices $|K| \le 2r$, one can treat (7) as a system of lpde's in the indeterminates $a_{i,j,K}$ and $h_{s,t,K}$.

By virtue of Proposition 1.1 the matrix A provides a C-linear transformation of l-dimensional C-vector spaces $(V_{I_2})^T$, $(V_{I_1})^T$. If A provides a zero transformation then $\mathcal{D}^{n_1}A \subset I_2$ due to the duality in the Zariski topology (cf. the proof of Proposition 1.1). Hence the $a_{i,j}$-components of all solutions of (7) constitute a C-linear subspace of $l \times l$ matrices representing C-linear transformations between $(V_{I_2})^T$ and $(V_{I_1})^T$. In other words, the space of all $\mathcal{D}$-isomorphisms $Hom_{\mathcal{D}}(\mathcal{D}^{n_1}/I_1, \mathcal{D}^{n_2}/I_2)$ can be viewed as a C-linear subspace of $l \times l$ matrices over C (this generalizes the considerations of the ordinary case $m = 1$, see pages 42-44 [23]).

In terms close to Definition 5.1 $Hom_{\mathcal{D}}(\mathcal{D}^{n_1}/I_1, \mathcal{D}^{n_2}/I_2)$ can be represented as *parametric-linear* family $A(Z) = (a_{i,j}(Z))$ where the parameters $Z = (\{Z_u\}_{1\le u\le N})$ range over the space C^N, and $a_{i,j}(Z)$ depend on Z linearly.

The algorithm finds this parametric-linear family $A(Z)$ by producing a Janet basis of system (7). We have already established that $A(Z)$ lies in a finite-dimensional C-vector space of dimension at most $N \le l^2$, therefore one obtains from the Janet basis an ideal of $A(Z)$ and thereupon making use of [18], page 448, finds a basis of all rational solutions $A(Z)$ over the field F_0. Slightly changing the notation, we keep the notation $A(Z)$ for the parametric-linear family of all elements from $Hom_{\mathcal{D}_0}(\mathcal{D}_0^{n_1}/I_1, \mathcal{D}_0^{n_2}/I_2)$ with rational coefficients, in other words, $\mathcal{D}_0$-homomorphisms.

In a similar way the algorithm yields a parametric-linear family $B(Z')$ of all the elements from $Hom_{\mathcal{D}_0}(\mathcal{D}_0^{n_2}/I_2, \mathcal{D}_0^{n_1}/I_1)$ with rational coefficients. Then $\mathcal{D}_0^{n_1}/I_1$ and $\mathcal{D}_0^{n_2}/I_2$ are $\mathcal{D}_0$-isomorphic if and only if there exist elements of the form $A = A(Z)$, $B = B(Z')$ such that $AB|_{\mathcal{D}_0^{n_2}/I_2} = id$, $BA|_{\mathcal{D}_0^{n_1}/I_1} = id$ that can be rewritten as a system

$$BAe_i - e_i = \sum_{1\le t\le q} h_t g_t, \;\; 1 \le i \le n_1, \;\; ABe'_j - e'_j = \sum_{1\le s\le p} h'_s f_s, \;\; 1 \le j \le n_2 \tag{8}$$

with unknowns h_t, h'_s, where $e_1, \dots, e_{n_1}$ (respectively $e'_1, \dots e'_{n_2}$) form a basis of the free module $\mathcal{D}_0^{n_1}$ (respectively $\mathcal{D}_0^{n_2}$).

We have already seen that $ord(A), ord(B) \le r$, hence $ord(h_t g_t), ord(h'_s f_s) \le 2r$, taking into account that $g_1, \dots, g_q$ and $f_1, \dots, f_p$ constitute Janet bases. Denote $h_t = \sum h_{t,K}d^K$, $h'_s = \sum h'_{s,K}d^K$ where $|K| \le 2r$. Thus one can treat (8) as a parametric linear *algebraic* system in the indeterminates $h_{t,K}$, $h'_{s,K}$ with parameters Z, Z'. One can solve such a parametric system using an algorithm described e.g. in [9]. The algorithm outputs the constructive set of all parameters Z, Z' for which system (8) is solvable, i.e. which provide an isomorphism $A(Z)$, $B(Z')$, in particular this constructive set is not empty if an only if $\mathcal{D}_0^{n_1}/I_1$ and $\mathcal{D}_0^{n_2}/I_2$ are $\mathcal{D}_0$-isomorphic. We summarize the results of the present section in the following theorem.

Theorem 7.1 *There is an algorithm which finds for any pair of finite-dimensional $\mathcal{D}_0$-modules all $\mathcal{D}_0$-homomorphisms (respectively isomorphisms) of $\mathcal{D}_0^{n_1}/I_1$ and $\mathcal{D}_0^{n_2}/I_2$ as a parametric-linear (respectively* *parametric-algebraic) family. By the same token the algorithm can yield the (algebraic) groups of all $\mathcal{D}_0$-automorphisms of the $\mathcal{D}_0$-module $\mathcal{D}_0^{n_1}/I_1$.*

It would be interesting to design an algorithm to test $\mathcal{D}_0$-isomorphism or even $\mathcal{D}$-isomorphism of infinite-dimensional $\mathcal{D}$-modules.

8 Calculations and Examples

The construction from Corollary 6.1 is the basis in [18] for decomposing finite-dimensional modules. An algorithm has been given there which applies these steps. It has been applied to various examples, an implementation may also be found in the ALLTYPES system [31].

For general modules the answer is less complete. In [13] proper factorizations and the corresponding decompositions have been considered for second- and third-order operators. Here this approach is extended to the case where genuine factors of such operators do not exist. To put this into proper perspective, a short review of the history of these problems is given first.

Most of the research on finding closed-form solutions of lpde's has been restricted to second-order equations for an unknown function z depending on two arguments x and y. The general linear equation of this kind may be written as

$$Rz_{xx} + Sz_{xy} + Tz_{yy} + Uz_x + Vz_y + Wz = 0 \tag{9}$$

where $R, S, \ldots, W$ are from some function differential field which is usually called the base field. Under fairly general constraints for its coefficients it can be shown that it may be transformed either to

$$z_{xy} + A_1 z_x + A_2 z_y + A_3 z = 0 \tag{10}$$

or to

$$z_{xx} + A_1 z_x + A_2 z_y + A_3 z = 0. \tag{11}$$

In this section it is always assumed that all $A_k \in \mathbf{Q}(x, y)$. Any solution scheme is closely related to the question what type of solutions are searched for, which in turn raises the question what kind of solutions do exist at all. For linear ode's the answer is well known. The general solution is a linear combination of a fundamental system over the constants, i. e. the arbitrary elements are n constants if the order of the equation is n. For pde's in general the answer is much more involved. There are equations of the form (10) allowing solutions

$$f_0(x, y)F(x) + f_1(x, y)F'(x) + \ldots + f_m(x, y)F^{(m)}(x) \tag{12}$$

or

$$g_0(x, y)G(y) + g_1(x, y)G'(y) + \ldots + g_n(x, y)G^{(n)}(y) \tag{13}$$

where the f_k, g_k are determined by the given equation, and $F(x)$ and $G(y)$ are generic functions of the respective argument (cf. Example 3.4). The existence of either type of solution, or of both types, depends on the values of the coefficients A_k. To decide their existence is already highly nontrivial. Moreover there may be solutions with integrals involving the "undetermined elements".

An algorithm is described now which performs these steps for certain pde's of second or third order. An auxiliary problem that occurs as part of the proceeding described above is considered first.

Lemma 8.1 *Let the q generic functions $C_j(x)$, $j = 1, \ldots, q$ satisfy the linear homogenenous system of r ode's $\sum_{j=1}^{q} \sum_{k=0}^{p_j} a_{ijk} C_j^{(k)} = 0$, $i = 1, \ldots, r$, $p_j \geq 0$ for $j = 1, \ldots, q$. It can always be decided whether the C_j can be represented as*

$$C_j = f_{j,0}F + f_{j,1}F' + \ldots + f_{j,s}F^{(s)} \tag{14}$$

for a given value of s and F an generic function of x If the answer is affirmative, such a representation can be found.

Proof. In the representation (14) the $f_{i,k}$ are considered as undetermined coefficients of the C_j. They are substituted into the given system of ode's. Because F is considered as a generic function of x, its derivatives are algebraically independent. Therefore the system of conditions obtained can only be satisfied if the coefficients of each derivative of F vanish. This leads to a linear homogeneous system of ode's for the $f_{i,k}$. By autoreduction it can always be decided whether a nontrivial solution exists and, if it is true, a special solution can be found. In [11] a polynomial-time algorithm with the additional property of being of the logarithmic parallel complexity was designed for this problem. □

Pommaret and Quadrat [21] have described a different method for dealing with systems of this kind. Although their method is of lower complexity, it is extremely simple to implement the above scheme if a Janet base algorithm is available.

Example 8.2 *Let the system $C_{2,x} + C_2 - x(C_{1,x} + C_1) = 0$, $C_{3,x} + C_3 - C_{1,x} - C_1 = 0$ be given. Substituting the ansatz $C_j = f_{j,0}F + f_{j,1}F' + f_{j,2}F''$ for $j = 1, 2, 3$, yields a linear homogeneous system of ode's with the Janet base $f_{2,2} = xf_{3,2}$, $f_{1,2} = f_{3,2}$, $f_{2,1} = xf_{3,1} - f_{3,2}$, $f_{1,1} = f_{3,1}$, $f_{3,0} = f'_{3,1} + f_{3,1} - f''_{3,2} - 2f'_{3,2} - f_{3,2}$, $f_{2,0} = xf'_{3,1} + (x-1)f_{3,1} - xf''_{3,2} - (2x-1)f'_{3,2} - (x-1)f_{3,2}$, $f_{1,0} = f'_{3,1} + f_{3,1} - f''_{3,2} - 2f'_{3,2} - f_{3,2}$. Choosing $f_{3,1} = f_{3,2} = 1$ yields $C_1 = C_3 = F''(x) + F'(x)$ and $C_2 = xF''(x) + (x-1)F'(x)$.*

In our algebraic approach equation (10) is written as $D_{xy}z = 0$ where

$$D_{xy} \equiv \partial_{xy} + A_1\partial_x + A_2\partial_y + A_3. \tag{15}$$

This case has been studied most thorougly in the literature. It will be discussed first. The principal ideal $< D_{xy} >$ is of gauge $(1, 2)$. There may exist operators forming a Janet base in combination with (15) which are of the form

$$D_{x^m} \equiv \partial_{x^m} + a_1\partial_{x^{m-1}} + \ldots + a_{m-1}\partial_x + a_m \tag{16}$$

or

$$D_{y^n} \equiv \partial_{y^n} + b_1\partial_{y^{n-1}} + \ldots + b_{n-1}\partial_y + b_n \tag{17}$$

with m and n positive integers. Usually it is a difficult problem to construct new operators which extend a set of given ones to form the Janet base of a larger ideal. However, due to the special structure of the problem, the auxiliary systems for the unknown coefficients a_j and b_j in (16) and (17) may always be solved as is shown next.

Proposition 8.3 *Let an operator of the form (15) be given. The following types of overideals may be constructed.*

a) If $n \geq 2$ is a natural number, it may be decided whether there exists an operator (17) such that (15) and (17) combined form a Janet base. If the answer is affirmative, the operator (17) may be constructed explicitly with coefficients $b_i \in \mathbf{Q}(x, y)$, the ideal $< D_{xy}, D_{y^n} >$ is of gauge (1,1).

b) If $m \geq 2$ is a natural number, it may be decided whether there exists an operator (16) such that (15) and (16) combined form a Janet base. If the answer is affirmative, the operator (16) may be constructed explicitly with coefficients $a_i \in \mathbf{Q}(x, y)$, the ideal $< D_{xy}, D_{x^m} >$ is of gauge (1,1).

Proof. The proof will be given for case a). If the operator (15) is derived repeatedly wrt. y, and the reductum is reduced in each step wrt. (15), $n-2$ equations of the form

$$\partial_{xy^k} + R_k\partial_x + P_{k,k}\partial_{y^k} + P_{k,k-1}\partial_{y^{k-1}} + \ldots + P_{k,0} \tag{18}$$

for $2 \le k \le n-1$ may be obtained. All coefficients R_k and $P_{i,j}$ are differential polynomials in the ring $\mathbf{Q}\{A_1, A_2, A_3\}$. There is no reduction wrt. (17) possible. Deriving the last expression once more wrt. y and reducing the reductum wrt. both (10) and (17) yields

$$\begin{aligned}\partial_{xy^n} + R_n\partial_x + (P_{n,n-1} - P_{n,n}b_1)\partial_{y^{n-1}} + (P_{n,n-2} - P_{n,n}b_2)\partial_{y^{n-2}} + \ldots\\ +(P_{n,1} - P_{n,n}b_{n-1})\partial_y + P_{n,0} - P_{n,n}b_n.\end{aligned} \tag{19}$$

In the first derivative of (17) wrt. x

$$\begin{aligned}\partial_{xy^n} + b_{1,x}\partial_{y^{n-1}} + b_{2,x}\partial_{y^{n-2}} + \ldots + b_{n-1,x}\partial_y + b_{n,x}\\ +b_1\partial_{xy^{n-1}} + b_2\partial_{xy^{n-2}} + \ldots + b_{n-1}\partial_{xy} + b_n\partial_x\end{aligned}$$

the terms containing derivatives of the form ∂_{xy^k} for $1 \le k \le n-1$ may be reduced wrt. (18) or (10) with the result

$$\begin{aligned}&\partial_{xy^n} + (b_{1,x} - P_{n-1,n-1}b_1)\partial_{y^{n-1}}\\ &\quad +(b_{2,x} - P_{n-1,n-2}b_1 - P_{n-2,n-2}b_2)\partial_{y^{n-2}}\\ &\qquad \vdots \qquad\qquad \vdots\\ &\quad +(b_{n-1,x} - P_{n-1,1}b_1 - P_{n-2,1}b_2 \ldots - P_{2,1}b_{n-2} - A_2b_{n-1})\partial_y\\ &\quad +b_{n,x} - P_{n-1,0}b_1 - P_{n-2,0}b_2 - \ldots - P_{2,0}b_{n-2} - A_3b_{n-1}\\ &\quad +(b_n - R_{n-1}b_1 - R_{n-2}b_2 - \ldots - R_2b_{n-2} - A_1b_{n-1})\partial_x.\end{aligned} \tag{20}$$

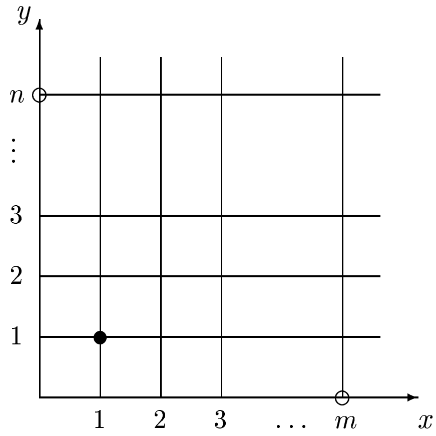
If this expression is subtracted from (19), the coefficients of the derivatives must vanish in order that (10) and (17) form a Janet base. The resulting system of equations is

$$\begin{gathered}b_{1,x} + (P_{n,n} - P_{n-1,n-1})b_1 - P_{n,n-1} = 0,\\ b_{2,x} - P_{n-1,n-2}b_1 + (P_{n,n} - P_{n-2,n-2})b_2 - P_{n,n-2} = 0,\\ \vdots \qquad\qquad \vdots\\ b_{n-1,x} - P_{n-1,1}b_1 - \ldots + (P_{n,n} - A_2)b_{n-1} - P_{n,1} = 0,\\ b_{n,x} - P_{n-1,0}b_1 - \ldots - A_3b_{n-1} + P_{n,n}b_n - P_{n,0} = 0,\\ b_n - R_{n-1}b_1 - R_{n-2}b_2 - \ldots - R_2b_{n-2} - A_1b_{n-1} = 0.\end{gathered} \tag{21}$$

The last equation may be solved for b_n. Substituting it into the equation with leading term $b_{n,x}$, and eliminating the first derivatives $b_{j,x}$ for $j = 1, \ldots, n-1$ by means of the preceding equations, it may be solved for b_{n-1}. Proceeding in this way, due to the triangular structure, finally b_1 is obtained from the equation with leading term $b_{2,x}$. Backsubstituting these results, all b_k are explicitly known. Substituting them into the first equation, a constraint for the coefficients A_1, A_2 and A_3 expressing the condition for the existence of a Janet base comprising (10) and (17), i. e. a solution of the type specified above in case a) is obtained. The proof for case b) is similar and is therefore omitted. □

Goursat [7], Section 110, describes a method for constructing a linear ode which is in involution with a given second order equation $z_{xy} + az_x + bz_y + cz = 0$. As is well known, this an extremely difficult problem in general. The basic achievement of the method applied above is the construction of a triangular system from which the unknown coefficients are guaranteed to be obtained by elimination. They are supplemented by a set of conditions on the coefficients a, b and c which make the problem feasible. Exactly the same strategy works for the third-order equations discussed below. It is not obvious how to generalize Goursat's scheme to any other case beyond the second-order equation considered by him.

Case a), $n = 1$ and case b), $m = 1$, have been discussed in detail in [13]. The corresponding ideals are maximal and principal, because they are generated by $\partial_y + a_1$ and $\partial_x + b_1$ respectively. The term *factorization* is applied in these cases in the proper sense because the obvious analogy to ordinary differential operators where all ideals are principal. For any value $m > 1$ or $n > 1$ the overideals are $J_m =< D_{xy}, D_{x^m} >$ or $J_n =< D_{xy}, D_{y^n} >$. For any fixed values $m_1 < m_2$, the corresponding ideals obey $J_{m_2} \subset J_{m_1}$, and similary for values of n. This situation becomes particularly clear from the following graph.



The heavy dot at $(1, 1)$ represents the leading derivative ∂_{xy} of the given equation. If a second equation with leading derivative ∂_{x^m} represented by the circle at $(m, 0)$ exists, the ideal is enlarged by the corresponding operator. For $m = 1$ this ideal contains the original operator with leading derivative ∂_{xy}, i. e. this operator is redundant. This shows clearly how the conventional factorization corresponding to a first-order operator is obtained as a special case for any m. A similar discussion applies to additional equations with leading derivative ∂_{x^n}.

A similar scheme for equations (11) is not known. First of all, searching for an equation (16) does not make sense because by reduction wrt. (11) it would mean searching for a first-order factor which is covered by Theorem 3.1 of [13]. A Janet base comprising equations (11) and (17) has a four-dimensional solution space. Even if a basis can be determined, it does not really bring forward the solution procedure as is shown by the following example.

Example 8.4 *Let the equation be* $(\partial_{xx} - \partial_y)z = 0$*. In conjunction with* $\partial_{yy}z = 0$ *it forms a Janet base. The ideals* $I =< \partial_{xx} - \partial_y >$ *and* $J =< \partial_{xx} - \partial_y, \partial_{yy} >$ *generate the relative syzygy module* $Syz(I, J) =< (1, 0), (\partial_{yy}, \partial_{xx} - \partial_y) >=< (1, 0), (0, \partial_{xx} - \partial_y) >\in \mathcal{D}^2$*. Determining a solution for the latter comes down to solving the original equation and therefore is not helpful.*

Next the algebraic approach will be applied third to order equations of the form $D_{xyy}z = 0$ where

$$D_{xyy} \equiv \partial_{xyy} + A_1\partial_{xy} + A_2\partial_{yy} + A_3\partial_x + A_4\partial_y + A_5. \tag{22}$$

The ideal $< D_{xyy} >$ is of gauge $(1, 3)$. Proper right factors of differential type 1 and of first or second order may be obtained by Corollary 4.3 of [13]. For completeness they are given next without proof.

Proposition 8.5 *An operator of the form (22) generates an ideal* $< D_{xyy} >$ *of gauge* $(1,3)$. *It may have the following proper right factors of order two or one.*

a) If $2A_{2,y} + A_1A_2 - A_4 \neq 0$ *and* $b_{1,y} - b_1^2 + A_1b_1 - A_3 = 0$ *where*

$$b_1 = \frac{1}{2A_{2,y} + A_1A_2 - A_4}(A_{2,yy} + 2A_{2,y}A_1 + A_2A_{1,y} - A_{4,y} - A_1A_4 - A_2A_3 + A_1^2A_2)$$

a right factor $\partial_{xy} + b_1\partial_x + b_2\partial_y + b_3$ *exists,* $b_2 = A_2$, $b_3 = A_2b_1 + A_4 - A_{2,y} - A_1A_2$.

b) If $2A_{2,y} + A_1A_2 - A_4 = 0$ *and* $A_5 - A_{2,yy} - A_{2,y}A_1 - A_2A_3 = 0$, *a right factor* $\partial_{xy} + b_1\partial_x + b_2\partial_y + b_3$ *exists where* b_1 *is a solution of* $b_{1,y} - b_1^2 + A_1b_1 - A_3 = 0$, *and* $b_2 = A_2$, $b_3 = A_2b_1 + A_{2,y}$.

c) If $A_4 - 2A_{2,y} - A_1A_2 = 0$ *and* $A_5 - A_{2,yy} - A_{2,y}A_1 - A_2A_3 = 0$, *a right factor* $\partial_x + b$ *exists with* $b = A_2$.

d) If $A_4 - A_1A_2 - A_{1,x} \neq 0$ *and* $b_y - b^2 + A_1b - A_3 = 0$ *where* $b = \dfrac{A_5 - A_2A_3 - A_{3,x}}{A_4 - A_1A_2 - A_{1,x}}$, *a right factor* $\partial_y + b$ *exists.*

e) If $A_4 - A_1A_2 - A_{1,x} = 0$ *and* $A_5 - A_2A_3 - A_{3,x} = 0$, *a right factor* $\partial_y + b$ *exists where* b *is a solution of* $b_{1,y} - b_1^2 + A_1b - A_3 = 0$.

The ideals generated in case a) and b) are of gauge (1,2), in the remaining cases they are of gauge (1,1).

If such a factor does not exist, over-ideals of the form $< D_{xyy}, D_{x^m} >$ or $< D_{xyy}, D_{y^n} >$ may be searched for. This is considered next.

Proposition 8.6 *Let an operator of the form (22) be given. The following types of overideals of differential type 1 may be constructed.*

a) If $n \geq 2$ *is a natural number, it may be decided whether there exists an operator (17) such that (22) and (17) combined form a Janet base. If the answer is affirmative, the operator (17) may be constructed explicitly with coefficients* $b_i \in \mathbf{Q}(x,y)$.

b) If $m \geq 2$ *is a natural number, it may be decided whether there exists an operator (16) such that (22) and (16) combined form a Janet base. If the answer is affirmative, the operator (16) may be constructed explicitly with coefficients* $a_i \in \mathbf{Q}(x,y)$.

The proofs of Propositions 8.3 and 8.6 are constructive. They allow determining any overideal with generators of the form (16) or (17) for given m and n for the equations under consideration. These results are combined now to produce the following algorithm *DecomposeLpde* which returns the most complete decomposition for any operator of the form (10) or (22) if the order of the additional operator is limited.

Algorithm *DecomposeLpde*(L,d). Given an operator L of the form (10) or (22) generating $I =< L >$, its decomposition into overideals of differential type 1 and with leading derivative of order not higher than d is returned.

$S1$: *Proper factorization.* Determine right factors $f_1, f_2, \ldots$ of L as described in Corollary 3.3 of [13] or Proposition 8.5. If any are found, collect them as $F := \{f_1, f_2, \ldots\}$.

$S2:$ *Extend ideal.* If step $S1$ failed, apply Proposition 8.3 or 8.6 in order to construct operators $g_1, g_2, \ldots$ of the form (16) or (17) with $m \leq d$ and $n \leq d$, beginning with $m = n = 2$ and increasing its value stepwise by 1 until d is reached. If any are found, assign them to $G := \{g_1, g_2, \ldots\}$. If F and G are empty return L.

$S3:$ *Completely reducible?* If $J := Lclm(F) =< L >$ return F, else if for the elements on G there holds $J := Lclm(< L, g_1 >, < L, g_2 >, \ldots) =< L >$, return G.

$S4:$ *Relative syzygies.* Determine generators of $S := Syz(I, J)$ and transform S it into a Janet base. If F is not empty return (S, F) else return (S, G).

This algorithm has been implemented in ALLTYPES, a computer algebra type system which may be accessed over the internet on the website `www.alltypes.de` [31]. From this decomposition large classes of solutions of an equation $Lz = 0$ may be obtained. In the completely reducible case, from the operators returned in step $S3$ solutions may be constructed as described in [13], or by solving the returned linear ode and applying Lemma 8.1. If L is not completely reducible, the result of step $S4$ is applied as follows. From F or G a partial solution is obtained similar as in the previous case. Solving the equations corresponding to S and taking the result as inhomogeneity for F or G respectively yields an additional part of the solution. This proceeding may fail if not all of the equations which occur during this proceeding can be solved. In these cases only a partial solution is obtained. The following examples have been treated according to this proceeding. Most of them are taken from the literature quoted at the beginning of this section. The first example due to Goursat leads to proper factorization, but is not completely reducible.

Example 8.7 *(Goursat 1906) The equation $(\partial_{xy} - y\partial_y)z = 0$ has been considered in [7], vol II, page 212. The ideal $< \partial_{xy} - y\partial_y >$ is of gauge (1,2). In step $S1$ the single right factor ∂_y is obtained. It generates the over-ideal $< \partial_y >$ of gauge (1,1) which contributes a generic function $G(x)$ (cf. Example 3.4) to the solution. Step $S4$ yields the module of relative syzygies $< \partial_x - y >$ with solution $F(y)\exp(xy)$. Taking it as inhomogeneity of the right factor equation, the generic solution $G(x) + \int F(y)\exp(xy)dy$ is obtained.*

The next example taken from Forsyth shows how complete reducibility has its straightforward generalization if there are no proper factors.

Example 8.8 *(Forsyth 1906) Define $D_{xy} \equiv \partial_{xy} + \frac{2}{x-y}\partial_x - \frac{2}{x-y}\partial_y - \frac{4}{(x-y)^2}$ which generates the principal ideal $I =< D_{xy} >$ of gauge (1,2). The equation $D_{xy}z = 0$ has been considered in [6], vol. VI, page 80. In step $S1$ no first-order factor is obtained. Step $S2$ shows that there exist generators*

$$D_{xx} \equiv \partial_{xx} - \frac{2}{x-y}\partial_x + \frac{2}{(x-y)^2} \text{ and } D_{yy} \equiv \partial_{yy} + \frac{2}{x-y}\partial_y + \frac{2}{(x-y)^2}$$

such that the ideals $J_1 =< D_{xy}, D_{xx} >$ and $J_2 =< D_{xy}, D_{yy} >$, each of gauge (1,1), are generated by a Janet base. In step $S3$ it is found that $I = Lclm(J_1, J_2)$, i.e. I is completely reducible, and the sum ideal is $J_1 + J_2 =< D_{xy}, D_{xx}, D_{yy} >$. The generic solution of $D_{xx}z = 0$ is $C_1(x-y) + C_2x(x-y)$ where $C_{1,2}$ are generic functions of y. Substitution into $D_{xy}z = 0$ yields $C_{1,y} + yC_{2,y} - C_2 = 0$. By Lemma 8.1 they may be represented as $C_1 = 2F(y) - yF'(y)$ and $C_2 = F'(y)$. Consequently the solution $z_1 = 2(x-y)F(y) + (x-y)^2F'(y)$ is obtained. The equation $D_{yy}z = 0$ has generic solution $C_1(y-x) + C_2y(y-x)$ where $C_{1,2}$ are generic functions of x now. By a similar procedure as above, the solution $z_2 = 2(y-x)G(x) + (y-x)^2G'(x)$ is obtained. The generic solution of $D_{xy}z = 0$ is $z_1 + z_2$.

The following example by Imschenetzky has been reproduced in many places in the literature. Unfortunately a misprint from the original has been reproduced by all quotations.

Example 8.9 *(Imschenetzky 1872) The equation* $(\partial_{xy} + xy\partial_x - 2y)z = 0$ *has been considered in [15]. Step* $S1$ *shows again that there are no first-order right factors. According to step* $S2$*, an operator of the form (17) with* $n \le 3$ *does not exist. However, for* $m = 3$ *there is an operator* ∂_{xxx} *such that the ideal* $< \partial_{xy} + xy\partial_x - 2y, \partial_{xxx} >$ *of gauge (1,1) is generated by a Janet base. The equation* $z_{xxx} = 0$ *has the generic solution* $C_1 + C_2 x + C_3 x^2$ *where the* C_i*,* $i = 1, 2, 3$ *are generic functions of* y*. Substituting it into the above equation and equating the coefficients of* x *to zero leads to the system* $C_{2,y} - 2yC_1 = 0$*,* $C_{3,y} - \frac{1}{2}yC_2 = 0$*. By Lemma 8.1, the* C_i *may be represented as* $C_1 = \frac{1}{y^2}F'' - \frac{1}{y^3}F'$*,* $C_2 = \frac{2}{y}F'$*,* $C_3 = F$*,* F *is a generic function of* y*,* $F' \equiv dF/dy$*. It yields the solution* $z_1 = x^2F(y) + \frac{2xy^2 - 1}{y^3}F'(y) + \frac{1}{y^2}F''(y)$ *of the given equation. In step* $S4$*, from the ideals* $I =< \partial_{xy} + xy\partial_x - 2y >$ *and* $J =< \partial_{xy} + xy\partial_x - 2y, \partial_{xxx} >$ *the relative syzygy module* $Syz(I, J) =< (1, 0), (\partial_{xx}, -\partial_y - xy) >=< (1, 0), (0, \partial_y + xy) >$ *of gauge (1,1) is constructed, the latter being generated by a Janet base. Its solution* $(0, G(x)s(x, y))$ *with* $s(x, y) = \exp(-\frac{1}{2}xy^2)$ *and* $G(x)$ *an generic function of* x *yields the solution*

$$z_2 = \frac{1}{2}\int G(x)s(x,y)x^2dx - x\int G(x)s(x,y)xdx + \frac{1}{2}x^2\int G(x)s(x,y)dx$$

of the original equation, its generic solution is $z_1 + z_2$*.*

The last example is a third-order equation which allows a single over-ideal generated by ∂_{xxx} if the order is limited to five.

Example 8.10 *Let the third-order operator*

$$D_{xyy} \equiv \partial_{xyy} + (x+y)\partial_{xy} + (x+y)\partial_x - 2\partial_y - 2$$

be given. It generates the principal ideal $I =< D_{xyy} >$ *of gauge (1,3). Step* $S1$ *does not yield any right factors of order one or two. In step* $S2$ *an operator of the form (17) and* $n \le 5$*, or an operator of the form (16) for* $m \le 2$ *is not found. However, for* $m = 3$ *there is an operator* $D_{xxx} \equiv \partial_{xxx}$ *such that the ideal* $J =< D_{xyy}, D_{xxx} >$ *of gauge (1,1) is generated by a Janet base. The equations* $D_{xyy}z = 0$ *and* $D_{xxx}z = 0$ *yield the solution*

$$z_1 = [(x+y)^2 - 2(x+y) + 2]F(y) + 2(x+y-1)F'(y) + F''(y)$$

where F *is a generic function of* y*. In step* $S4$*,* I *and* J *yield the relative syzygy module* $Syz(I, J) =< (1, 0), (\partial_{xx}, -\partial_{yy} - (x+y)\partial_y - x - y) >=< (1, 0), (0, \partial_{yy} + (x+y)\partial_y + x + y) >$ *of gauge (1,2). Its solution* $G(x)s(x,y) + H(x)s(x,y)\int e^{-y}\frac{dy}{s(x,y)}$*, where* $s(x,y) = \exp(-\frac{1}{2}(x+y-2)^2 - y)$ *and* G, H *are generic functions of* x*. According to the discussion in the Introduction on page 2 one finally obtains*

$$z_2 = \frac{1}{2}\int G(x)s(x,y)x^2dx - x\int G(x)s(x,y)xdx + \frac{1}{2}x^2\int G(x)s(x,y)dx$$

and for z_3 *an identical expression with* $G(x)$ *replaced by* $H(x)$ *and* $s(x,y)$ *by* $s(x,y)\int e^{-y}\frac{dy}{s(x,y)}$*. The generic solution of the given equation* $D_{xyy}z = 0$ *is* $z_1 + z_2 + z_3$*.*

9 Conclusion

The results presented in this article allow decomposing partial differential operators of the form (10) or (22) into components of lower gauge. If such a decomposition is found, it may be applied to determine the general solution of the corresponding pde, or at least some parts of it.

It is highly desirable to develop a similar scheme to large classes of modules of partial differential operators. The possible types of overmodules can always be determined. The hard part is to identify those for which generators may be constructed algorithmically. If this is not possible for a particular type, this overmodule has to be discarded. An important field of application could be the symmetry analysis of nonlinear pde's, because the determining equations of these symmetries are linear homogeneous pde's. A different type of problem is to find an upper bound for the order d of possible operators in algorithm *DecomposeLpde*. Such a bound would mean that full classes of over-modules could be excluded from the decomposition. On the other hand, a negative answer would be an evidence that this problem could be undecidable

Acknowledgments. The paper was done during a stay of the first author at the institute SCAI of the Fraunhofer Gesellschaft, supported by the Forschungspreis of the Humboldt Foundation. The hospitality at SCAI and the support of the Humboldt Foundation are gratefully acknowledged. The authors also appreciate the valuable remarks of an anonymous referee.

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