Nash resolution for binomial varieties as Euclidean division. Apriori termination bound, polynomial complexity in dim 2

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Abstract. We establish a (novel for desingularization algorithms) apriori bound on the length of resolution of singularities by means of the compositions of the normalizations with Nash blowings up, albeit that only for affine binomial varieties of (essential) dimension 2 . Contrary to a common belief the latter algorithm turns out to be of a very small complexity (in fact polynomial).

To that end we prove a structure theorem for binomial varieties and, consequently, the equivalence of the Nash algorithm to a combinatorial algorithm that resembles Euclidean division in dimension ≥ 2 and, perhaps, makes the Nash termination conjecture of the Nash algorithm particularly interesting.

A bound on the length of the normalized Nash resolution of a minimal surface singularity via the size of the dual graph of its minimal desingularization is in the Appendix (by M. Spivakovsky).

Contents

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1. Introduction.

1.1. **Nash blow ups and normalizations: conjectures.** For a reduced equidimensional algebraic variety X , say of $\dim X = n$, over an algebraically closed field $\mathbb{K}$ of zero characteristic (this requirement is relaxed in Sections 3, 5) the Gauss map G_X is defined off singular points $\operatorname{Sing} X$ of X and sends every point $\mathcal{P} \in \operatorname{Reg} X := X \setminus \operatorname{Sing} X$ to the tangent space $T_{\mathcal{P}}X$ (to X at $\mathcal{P}$) as points of the respective Grassmanian bundle restricted over X . (Using embeddings of affine charts of X in $\mathbb{K}^N$ consider the Grassmanian variety of n-dimensional subspaces of $\mathbb{K}^N$. The latter naturally embeds into projective space $\mathbb{P}(\wedge^n \mathbb{K}^N)$ by means of Plücker coordinates, i. e. the homogeneous coordinates in $\wedge^n \mathbb{K}^N$.) The Nash blow up $N(X)$ of X is the closure of the graph of G_X together with the natural projection $N_X : N(X) \to X$.

Nash conjecture. The sequence of Nash blowings up starting with any algebraic variety stabilizes, resulting in a desingularization.

Over affine charts the ring of 'regular functions' $\mathbb{K}[\mathcal{N}(X)]$ on the normalization $\mathcal{N}_X : \mathcal{N}(X) \to X$ of a variety X is the integral closure of $\mathbb{K}[X]$ in its field of fractions. When Z is nonsingular and $X \simeq Z \times Y$ (locally) it follows that $\mathcal{N}(X) \simeq Z \times \mathcal{N}(Y)$ (of course also only locally). Normalization separates irreducible components with respect to the étale topology (their ideals in the completions of the local rings are the minimal prime ideals). We refer to the compositions of normalizations with Nash blowings up as *normalized Nash blowings up.*

Normalized Nash conjecture. Normalized Nash blowings up starting with any algebraic variety result in a desingularization.

Remark 1.1. So far Nash and normalized Nash desingularizations remain elusive in respective dimensions larger than one and two. Moreover, in dimension larger than one an apriori estimate for the length of the normalized Nash desingularization is novel (no such estimates are known for other desingularizations).

(i) If the Nash blow up $N_X : N(X) \to X$ is an isomorphism then X is nonsingular, see [12] and [13].

(ii) The Nash conjecture is true when $\dim X = 1$ and there is a simple estimate for the length of sequences by Nash blowings up leading to a desingularization (e. g. by means of Newton-Puiseux expansions).

(iii) M. Spivakovsky proved that the sequence of normalized Nash blowings up terminates when $\dim X = 2$, see [15] and [10]. In fact $1 + \log_2(\#\Gamma)$ is an upper bound on the length of normalized Nash desingularization of a minimal surface singularity, where $\#\Gamma$ is the number of vertices of the *dual graph* Γ of its *minimal desingularization*; see below in the Appendix authored by M. Spivakovsky.

1.2. **Summary of results and the structure of the paper.** In Section 4 we establish the equivalence of the Nash algorithm for desingularization of binomial varieties with a simple combinatorial algorithm that resembles classical Euclidean division. The main first step (proved in Sections 3 , 5) is our structure theorem for binomial varieties: we establish isomorphisms of the irreducible components of any affine binomial variety $\hat{V} \subset \mathbb{A}^N$ with its toric component V and, also, the isomorphisms of the étale germs of the latter with those of the product of a suitable subtorus of V and of the binomial subvariety $\hat{Y}_V := \{w \in V : w_i = 1 \ , \ i \in I\}$, where w_i's are the coordinates on $\mathbb{A}^N$ and $I := \{i : V \cap \{w_i = 0\} = \emptyset\}$ (Theorem 3.7). The toric component Y_V of $\hat{Y}_V \hookrightarrow \mathbb{A}^{N-\#I}$ contains $\mathbf{0}$ (Corollary 3.5). An affine toric variety $Y \ni \mathbf{0}$ iff $\mathbf{0} \notin \mathrm{Conv}(\mathcal{E})$:= the convex hull of the set $\mathcal{E} \subset \mathbb{Z}^m$ of the exponents of a monomial parametrization of the torus of Y (Claim 3.2). The latter **essential** property is preserved (Claim 4.6) in the affine charts of a suitable covering of the Nash blowing up and implies that Y is nonsingular iff the set $\mathcal{E}$ is spanned over nonnegative integers by a subset of $\mathcal{E}$ of size $m = \dim Y$ (Criterion 3.16). As a consequence and by following the changes in the exponents $\mathcal{E}$ under the Nash blowings up we establish in Section 4 the equivalence of the algorithm for desingularization of binomial varieties by means of Nash blowings up to an *'Euclidean m-dimensional division'*. However, its termination predicted by the Nash conjecture remains open even when $m = 2$.

On the other hand a combinatorial version of the composition of normalizations with Nash blowings up for $m = 2$ yields (unexpectedly for any desingularization) a sharp apriori bound $2 \cdot \log_2 D$ on the length of desingularizing sequences, where D is the area of the parallelogram on the shortest integral generators of the cone spanned by the exponents $\mathcal{E}$, see Theorems 2.1 , 6.8 and Corollary 6.9 (and a 'guidance' remark at the beginning of Section 6.1). Consequently, when $m = 2$ we establish a polynomial complexity of the algorithm in the binary size of the input, see Corollaries 7.5 , 7.6 , 7.7 . Also, every affine chart is covered after the normalized Nash blowing up by at most 5 affine charts with at most 3 among them being singular, cf. Claim 6.10 . In Section 8 we establish (local) invariance of $D = D_o$, $o \in Y$, with respect to local isomorphisms that preserve hypersurfaces invariant under the action of the torus of Y and contain o .

An earlier version of this work appeared in [8] .

1.3. **Euclidean m-dimensional division algorithm, $m \geq 1$.** Let $\mathbb{R}_+ \subset \mathbb{R}$ and $\mathbb{Q}_+ \subset \mathbb{Q}$ denote nonnegative real and rational numbers respectively; let $\mathbb{Z}_+ \subset \mathbb{Z}$ be the set of positive integers. For a finite set $\mathcal{E} \subset \mathbb{Z}^m$ let $\mathbb{Z}_+(\mathcal{E})$ denote the additive semigroup spanned by $\mathcal{E}$. Let $\mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E})$ be the convex cone spanned by $\mathcal{E}$ over $\mathbb{Q}_+$ in $\mathbb{Q}^m$. Then

The input is a finite $\mathcal{E} := \{\vec{\Delta}_j\}_j \subset \mathbb{Z}^m = \mathrm{Span}_{\mathbb{Z}}\mathcal{E}$ with $\mathbf{0} \notin \mathrm{Conv}(\mathcal{E})$ and $\mathcal{E} = \mathcal{E}xtreme(\mathbb{Z}_+(\mathcal{E})) :=$ the minimal set of generators of $\mathbb{Z}_+(\mathcal{E})$.

Let $\mathcal{S} := \{J \subset \mathcal{E} : \#J = m \ , \ det\{\vec{\Delta}_j\}_{j\in J} \neq 0\}$, $\{\vec{\Delta}_J := \sum_{j\in J} \vec{\Delta}_j\}_{J\in\mathcal{S}}$ and say that $h \in (\mathbb{R}^m)^{dual}$ is 'irrational' iff $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}\{h_i\}_i = m$.
The branching set For each irrational h such that $h|_{\mathcal{E}} > 0$, there exists a unique $J \in \mathcal{S}$ which minimizes the quantity $h(\vec{\Delta}_J)$. By definition, the branching set $\mathcal{B}$ consists of all such J as h runs over the irrational elements of $(\mathbb{R}^m)^{dual}$ satisfying $h|_{\mathcal{E}} > 0$. In other words, let $\mathcal{E}_J := \mathcal{E} \cup \{\vec{\Delta}_{J'} - \vec{\Delta}_J\}_{J\neq J'\in\mathcal{S}}$. Then $\mathcal{B} = \mathcal{S}' := \{J \in \mathcal{S} : \mathbf{0} \notin \mathrm{Conv}(\mathcal{E}_J)\}$. See Claim 4.6 and its proof for details.

The output corresponding to $J \in \mathcal{B}$ is $N_J(\mathcal{E}) := \mathcal{E}xtreme(\mathbb{Z}_+(\mathcal{E}_J))$.

A branch terminates at $\mathcal{E}$ when $\#\mathcal{E} = m$, cf. Claim 4.9.

Normalized Euclidean m-dimensional division is the alternating composition of Euclidean m-dimensional division with a combinatorial version of **normalization:** $\mathcal{E} \mapsto \mathcal{N}(\mathcal{E})$, where $\mathcal{N}(\mathcal{E}) := \mathcal{E}xtreme(\mathbb{Z}^m \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) \setminus \mathbf{0})$.

1.4. **Desingularization reductions briefly.** A **binomial** is the difference of two monomials in a given set of variables, which is fixed once and for all. *Affine binomial* (shortly $\mathcal{AB}$-)varieties (see e. g. [3]) are defined as the closures in $\mathbb{K}^N$ of the zeroes (off coordinate hyperplanes) of collections of binomials. Theorem 3.7 provides a reduction of Nash and of normalized Nash desingularizations of $\mathcal{AB}$-varieties to that of **essential** varieties, i. e. affine toric varieties containing the origin. All the isomorphism types of singularities of an essential variety occur in every neighbourhood of its origin, see Claim 3.13.

Remark 1.2. Of course, if X consists of several irreducible components $X = \cup_i X_i$ then $N(X) = \cup_i N(X_i)$ and $N(X_i)$ are the irreducible components of $N(X)$. When the variety X has an étale open set U which is a product of a nonsingular variety Z with a (possibly singular) variety Y , then $N(X)$ over U is isomorphic to the product $Z \times N(Y)$ of Z with $N(Y)$.

Let X be an algebraic variety and let X_1 , X_2 denote two smooth germs of irreducible components of a certain étale open neighourhood

U of a point $\xi \in X$. Let I_j , $j = 1, 2$, denote the defining ideals of X_1 and X_2 in the completion $\hat{\mathcal{O}}$ of the local ring $\mathcal{O}$ of the ambient manifold at ξ . The **contact** between X_1 and X_2 is defined to be the largest integer l such that $I_1 + \hat{\mathfrak{m}}^l = I_2 + \hat{\mathfrak{m}}^l$, where $\hat{\mathfrak{m}}$ is the maximal ideal of $\hat{\mathcal{O}}$. Nash blow up either separates X_1 and X_2 or reduces the *contact* between them. Thus the sequence of Nash blowings up of a variety all of whose étale irreducible components are smooth at every point terminates, separating 'Nash liftings' of these components.

1.5. **Singularities vis-a-vis structure of binomial varieties.** Let $\mathbb{K}[w]$ denote the ring of polynomials in N variables w_j with coefficients in a field $\mathbb{K}$. Affine binomial (or $\mathcal{AB}$-)varieties are defined to be the closures $\hat{V}$ in $\mathbb{A}^N$ of the sets of common zeroes on the standard torus $\mathbb{T}^N := \cap_j \{w_j \neq 0\} \subset \mathbb{A}^N := \operatorname{Spec} \mathbb{K}[w]$ of collections of binomials. In Sections 3 , 5 we state and prove a structure theorem for affine binomial (shortly $\mathcal{AB}$-)varieties (assuming that the numbers defined in Theorem 3.7 **C**, namely $\underline{d}(\hat{V}) := \underline{d} \neq \mathbf{0}$ in $\mathbb{K}$, e. g. whenever $\mathbb{K}$ is of characteristic zero), see Theorem 3.7 **C**. For an algebraically closed field $\mathbb{K}$ one may replace $\mathbb{A}^N$ by $\mathbb{K}^N$. Let I be the set defined in Section 1.2 . We split all w-coordinates on $\mathbb{A}^N$ into y- and z-coordinates, $w = (y \ , \ z)$, with $y = \{w_i\}_{i \notin I}$ and $z = \{w_i\}_{i \in I}$. Let $\pi : \mathbb{A}^N \ni (y \ , \ z) \mapsto z \in \mathbb{A}^{N-L}$. We refer to varieties of the form $(\pi)^{-1}(W)$ for a nonsingular $W \hookrightarrow \pi(\hat{V})$ as $\hat{V}$*-admissible.* Due to Theorem 3.7 the singularities of the irreducible components of the étale germs of a variety $\hat{V}$ in the $\mathcal{AB}$ class and of the $\hat{V}$-admissible varieties are essentially 'the same', see Claim 3.14 and Remark 3.15 (the latter class of *generalized* $\mathcal{AB}$ or shortly $\mathcal{GAB}$-varieties includes all *quasi-binomial varieties*, i. e. varieties such that every defining equation is a linear combination of two monomials).

Other consequences of Theorem 3.7 include a reduction of Nash (respectively normalized Nash) desingularizations of $\mathcal{GAB}$-varieties to the respective desingularizations of irreducible binomial varieties passing through the origins of the (appropriate) ambient affine coordinate charts. We also obtain simple criteria of nonsingularity for all toric varieties in terms of the exponents of monomial parametrizations of their dense tori and, as a consequence, for blowings up of smooth affine spaces at the ideals generated by monomials, see Criteria 3.18 and 3.19.

Affine toric varieties are the closures in $\mathbb{A}^N$ of the images - $\phi_{\mathcal{E}}(\mathbb{T}^m)$ of the standard tori $\mathbb{T}^m \simeq \underline{\mathbb{T}}^m := \cap_j \{x_j \cdot \tilde{x}_j = 1\} \hookrightarrow \mathbb{A}^{2m}$ under monomial bijections $\phi_{\mathcal{E}} : \mathbb{T}^m \to \phi_{\mathcal{E}}(\mathbb{T}^m) \hookrightarrow \mathbb{T}^N$ (with $\mathcal{E} \subset \mathbb{Z}^m$ being the set of the exponents of the monomial components of $\phi_{\mathcal{E}}$). Toric varieties are binomial, but not necessarily normal, e. g. the Whitney

Umbrella $\{x^2 - z \cdot y^2 = 0\} \subset \mathbb{C}^3$. Moreover, Nash blowings up of normal varieties with open dense tori may fail to be normal, e. g. the Nash blow up of a surface $S := \overline{\phi_{\mathcal{E}}(\mathbb{T}^2)} \subset \mathbb{C}^3$, where $\phi_{\mathcal{E}} : (x_1, x_2) \mapsto (x_1 \cdot x_2 \ , \ x_1 \cdot x_2^2 \ , \ x_1^3 \cdot x_2^2)$, fails to be normal in spite of the fact that S is a normal surface. Indeed, normality of the latter is a consequence (due to a criterion in Section 2.1 of [5]) of the property of the exponents $\mathcal{E} = \{(1,1) \ , \ (1,2) \ , \ (3,2)\} \subset (\mathbb{Z}_+)^2$ to span over $\mathbb{Z}_+$ all points of its integral lattice within the cone spanned (over $\mathbb{R}_+$) by $\mathcal{E}$ in $\mathbb{R}^2$. See Example 6.3 for the failure of normality for $N(S)$. Consequently we refer to the varieties with a dense torus as *toric* (as in [16] or [1]), while in [5] they are refered to as toric only when normal.

2. A sharp apriori bound in essential dimension 2.

We consider algebraic varieties (so called binomial) that admit (Zariski) open coverings by 'affine binomial' varieties, i. e. closures $\hat{V}$ in $\mathbb{A}^N$ of sets $V^*(\hat{f}) := \{w \in \mathbb{T}^N : \hat{f}_j(w) = 0 \, , \, 1 \leq j \leq M \ \}$, where $(\hat{f})$ are the ideals in the ring $\mathbb{K}[w]$ of polynomials in $w := (w_1, \ldots, w_N)$ with coefficients in a field $\mathbb{K}$ generated by binomials

$$\hat{f}_j := w_1^{\hat{\alpha}_{j1}} \cdots w_N^{\hat{\alpha}_{jN}} - w_1^{\hat{\beta}_{j1}} \cdots w_N^{\hat{\beta}_{jN}} \ . \tag{2.1}$$

Let the *exponent matrix* $\hat{E}$ of $\hat{V}$ have entries $\hat{\alpha}_{ji} - \hat{\beta}_{ji}$ and denote by (j) the vector with the only nonzero coordinate (the j^{th}) equal to one. Let $\{\vec{\delta}_i \times \mathbf{0}\}_i \subset \mathbb{Z}^N$ be a $\mathbb{Z}$-basis of $\operatorname{Ker} \hat{E} \cap (\mathbb{Q}^L \times \mathbf{0}) \subset \mathbb{Q}^N$, where $\{\vec{\delta}_i\}_{1 \leq i \leq m} \subset \mathbb{Z}^L$ (with splittings $w = (y, z)$ and $\mathbb{K}^N = \mathbb{K}^L \oplus \mathbb{K}^{N-L}$ as in the previous section, while $\mathbb{K}$ being an algebraically closed field of zero characteristic, cf. Section 1.1). Our main estimate is

Theorem 2.1. Complexity bound on desingularization when $m = 2$.
(i) The convex hull of $\{((\vec{\delta}_1)_l \ , \ (\vec{\delta}_2)_l)\}_{1 \leq l \leq L}$ *does not contain* $\mathbf{0} \in \mathbb{R}^2$.
(ii) Let D *be the size of the coordinate of* $\vec{\delta}_1 \wedge \vec{\delta}_2$ *at* $(l) \wedge (k)$, $1 \leq l \ , \ k \leq L$ *, for which the cone in* $\mathbb{R}^2$ *spanned over* $\mathbb{R}_+$ *by* $((\vec{\delta}_1)_l \ , \ (\vec{\delta}_2)_l)$ *and* $((\vec{\delta}_1)_k \ , \ (\vec{\delta}_2)_k)$ *contains all vectors* $((\vec{\delta}_1)_j \ , \ (\vec{\delta}_2)_j)$, $1 \leq j \leq L$ *. Then after at most* $2 \cdot \log_2 D$ *normalized Nash blowings up starting with the variety* $\hat{V}$ *the process stabilizes.*

Theorem 2.1 (i) (for any m) is a consequence of Claim 3.2 (cf. Remark 4.1 below), while the second claim is a consequence of Theorem 6.8 proved in Section 7 , see also a 'guidance' remark at the beginning of Section 6.1 .

Remark 2.2. *Finding D of Theorem 2.1 (and Section 1.2)*:
Note that for any integral basis $\{\vec{\delta}_i\}_{1\le i\le m}$, as considered preceding Theorem 2.1, the coordinates of $\vec{\delta}_1 \wedge \cdots \wedge \vec{\delta}_m$ in the standard basis are unique up to a sign and can simply be found by choosing *any* $\mathbb{Q}$-basis $\{\vec{v}_i\}_{1\le i\le m}$ with the same $\mathbb{Q}$-span as that of the $\{\vec{\delta}_i\}_{1\le i\le m}$, then multiplying the respective coordinates of $\vec{v}_1 \wedge \cdots \wedge \vec{v}_m$ by their least common denominator and subsequently dividing the obtained integers by their g.c.d. . For $m = 2$ we may, moreover, determine the bound D of Theorem 2.1 up to a sign by detecting which $(l) \wedge (k)$ coordinate of the resulting sequence of integers to take. To that end the criterion of detecting pairs (l, k) of Theorem 2.1 does not depend on the choice of a basis and can be applied as well with a basis $\{\vec{v}_i\}_{1\le i\le m}$.

We prove the local invariance of the integer D from Theorem 2.1 (and Section 1.2) in Section 8 , cf. Lemma 8.3 and Corollary 8.5 .

Part 1. Arbitrary dimension.

We state the main result of Part 1, namely Theorem 3.7 in Section 3.

3. Reduction to essential toric case.

We adopt the following notations: $\hat{V}$ is the closure of $\hat{V}^* := \hat{V} \cap \mathbb{T}^N = V^*(\hat{f}) = \{w \in \mathbb{T}^N : w^{\hat{E}} = \mathbb{I}_M\}$, $\mathbf{0} \in \mathbb{A}^M$ is the origin, $\mathbb{I}_M := (1, \ldots, 1) \in \mathbb{A}^M$, Id_N denotes the unit matrix of size $N \times N$. We refer to the closure in $\mathbb{A}^N$ of the image of a bijective monomial map $\phi_{\mathcal{E}} : \mathbb{T}^m \to X^*_{\mathcal{E}} := \phi_{\mathcal{E}}(\mathbb{T}^m) \subset \mathbb{A}^N$ (with the exponents in $\mathcal{E} \subset \mathbb{Z}^m$) as an affine toric variety and denote the latter by $X_{\mathcal{E}}$. For the sake of convenience we denote by $(A||B)$ the matrix with columns of A followed by the columns of B and the matrix with rows from the exponents set $\mathcal{E}$ by the same letter, i. e. $\phi_{\mathcal{E}}(x) = x^{\mathcal{E}}$. But we denote both the set of columns and transpose matrix of a matrix T by T^{tr} , e.g. $\pi \circ \phi_{\mathcal{E}} = \phi_{(\pi(\mathcal{E}^{tr}))^{tr}}$ (for π from Section 1.5). We refer to $\Delta \subset \mathbb{Z}^N$ as a $\mathbb{Z}$-basis when $\#(\Delta) = \mathrm{rank}\,(\Delta)$ and $\mathrm{Span}_{\mathbb{Z}}(\Delta) = \mathrm{Span}_{\mathbb{Q}}(\Delta) \cap \mathbb{Z}^N$.

Classical construction: applying 'Gauss elimination' let Λ , λ be square matrices with entries in $\mathbb{Z}$ and $\det(\Lambda) = 1 = \det(\lambda)$ such that matrix $\tau := \Lambda \cdot \hat{E} \cdot \lambda$ has vanishing entries except in the upper-left corner on a 'diagonal' of length $r = \mathrm{rank}\,\hat{E}$ (while for the successive integral entries $d_q \in \mathbb{Z}_+$, $q = 1, \ldots, r$, the ideals generated in $\mathbb{Z}$ by the $q \times q$ minors of matrix $\hat{E}$ and, respectively, by $d_1 \cdot \ldots \cdot d_q$ coincide; it is the so called Smith normal form). Denote $d(\hat{E}) := |d_1 \cdot \ldots \cdot d_r|$.

Remark 3.1. Immediate consequences of this construction include

1. $\dim \hat{V} = N - r$;

2. solutions of $w^{\hat{E}} = \mathbb{I}_M$ and of $\tilde{w}^{\tau} = \mathbb{I}_M$ in $\mathbb{T}^N$ are related by an automorphism ϕ_λ of $\mathbb{T}^N$;

3. $d(\hat{E}) = \#((\mathrm{Span}_{\mathbb{Q}}(\hat{E}^{tr}) \cap \mathbb{Z}^M)/\mathrm{Span}_{\mathbb{Z}}(\hat{E}^{tr}))$;

4. $\hat{V}^*$ has $[d(\hat{E})]$ irreducible components, , where $[d(\hat{E})] := d(\hat{E})$ or $[d(\hat{E})] := d(\hat{E}) \cdot p^{-s} \in \mathbb{Z} \setminus (p \cdot \mathbb{Z})$ (with an appropriate $s \in \mathbb{Z}_+$) depending on whether $\mathbb{K}$ has characteristic $p = 0$ or $p > 0$;

5. when $N = r$ the morphism $\phi_{\hat{E}} : \mathbb{T}^N \ni w \mapsto w^{\hat{E}} \in \mathbb{T}^M$ is a parametrization iff $d(\hat{E}) = 1$, is surjective iff field $\mathbb{K}$ is perfect, is an étale isomorphism iff $[d(\hat{E})] = d(\hat{E})$ and is finite of degree $d(\hat{E})$ when $X^*_{\hat{E}} = X_{\hat{E}}$ (since ϕ_Λ is an automorphism of $\mathbb{T}^M$);

6. the irreducible component $V^* \ni \mathbb{I}_N$ of $\hat{V}^*$ is a torus $V^* = X^*_{\mathcal{E}}$ with the choices for parametrizing V^* consisting of exponents $\mathcal{E} \subset \mathbb{Z}^n$, $n := N - r$, such that the columns of $\mathcal{E}$ as a matrix form a $\mathbb{Z}$-basis of $\mathrm{Ker}\, \hat{E} \cap \mathbb{Z}^N$.

Proof. To prove the last (less immediate) statement we note that for $\{\tilde{w} \in \mathbb{T}^N : \tilde{w}^{\tau} = \mathbb{I}_M\}$ parametrizations $x \mapsto \tilde{w} = x^{\tilde{\mathcal{E}}}$ are determined by the $\mathbb{Z}$-bases of $\mathbb{Z}^n \hookrightarrow \mathbb{Z}^N$, implying the claim by means of the automorphism $w = \tilde{w}^{\lambda}$ of $\mathbb{T}^N$ and the correspondence $\mathcal{E} := \lambda \cdot \tilde{\mathcal{E}}$. □

Consequently for affine binomial varieties the following holds.

Property A. Cosets $[g] \in \Gamma := \hat{V}^*/V^*$ of $g \in \hat{V}^*$ uniquely identify the irreducible components of $\hat{V}$ as $g \cdot \overline{V^*}$, $\hat{V}^* \subset \mathrm{Reg}\, \hat{V}$, cf. Remark 5.2.

Claim 3.2. *The affine toric variety* $X_{\mathcal{E}} \ni \mathbf{0}$ *iff* $\mathrm{Conv}\,(\mathcal{E}) \not\ni \mathbf{0}$.

Proof. Indeed, the 'only if' follows since if $\mathrm{Conv}\,(\mathcal{E}) \ni \mathbf{0}$ then there are $\mathcal{E}' \subset \mathcal{E}$ and $\{p_e \in \mathbb{Z}_+\}_{e \in \mathcal{E}'}$ such that $\sum_{e \in \mathcal{E}'} p_e \cdot e = \mathbf{0}$, which implies that $X_{\mathcal{E}} \subset \{w : \prod_{e \in \mathcal{E}'} w_e^{p_e} = 1\}$. 'If' follows by choosing $\eta \in \mathbb{Z}^m \subset (\mathbb{R}^m)^{dual}$ with $\eta(e) > 0$ for $e \in \mathcal{E}$ since then $\mathbf{0} \in X_{\eta(\mathcal{E})} \subset X_{\mathcal{E}}$. □

The proofs of the claims of this section are in Section 5 or included here.

Claim 3.3. *The torus* $X \cap \mathbb{T}^N$ *of an affine* m*-dimensional toric variety* X *admits a parametrization* $\phi_{\mathcal{E}}$ *with exponents* $\mathcal{E} \subset (\mathbb{Z}_+)^m$ *iff* $\mathbf{0} \in X$.

Lemma 3.4. *Pick* $\vec{\xi} \in \mathrm{Ker}\, \hat{E} \cap (\mathbb{Z}_+ \cup \{\mathbf{0}\})^N$. *There exists* $(\vec{\xi})_i > 0$ *iff* $i \notin I$ *, where* I *is from Section 1.2 (i.e.* w_i *for* $\hat{V}$ *is a 'y-variable').*

Corollary 3.5. *There exists* $\vec{\xi}^{+} \in \operatorname{Ker} \hat{E} \cap (\mathbb{Z}_+ \cup \{\mathbf{0}\})^N$ *such that* $(\vec{\xi}^{+})_i > 0$ *iff* w_i *is a 'y-variable'. Also,* $(\mathbf{0}, \mathbb{I}_{N-L}) \in X_{\mathcal{E}^+} \subset \hat{V}$ *for* $\mathcal{E}^+ := \{(\vec{\xi}^{+})_i\}_i \subset \mathbb{Z}$.

The ideal of $\hat{V} := \overline{V^*(\hat{f})}$ in terms of equations $\hat{f}$ is identified by

Claim 3.6. *A polynomial* $P \in \mathbb{K}[w]$ *vanishes on* $\hat{V}$ *if and only if* $(y_1 \cdot \ \ldots \ \cdot y_L)^l \cdot P \in (\hat{f})$ *for some* $l \in \mathbb{Z}_+$.

Theorem 3.7. *For any affine binomial variety* $\hat{V} \hookrightarrow \mathbb{A}^N$ *and* $V := \overline{V^*}$

B. *The variety* $\pi(\hat{V}) = \overline{\pi(\hat{V}^*)}$ *is binomial and closed in* $\mathbb{A}^{N-L}$ *, while* $\hat{V} \cap (\mathbb{A}^L \times \mathbb{I}_{N-L}) = \overline{V^*(\hat{f}) \cap (\mathbb{A}^L \times \mathbb{I}_{N-L})}$ *and the latter variety has an irreducible toric component* $Y =: X_{\mathcal{E}_Y}$, $\mathcal{E}_Y \subset \mathbb{Z}^{\dim Y}$ *, with* $\hat{Y} := V \cap (\mathbb{A}^L \times \mathbb{I}_{N-L})$.

C. *There exists* $\mathcal{E}_Z \subset \mathbb{Z}^{n-\dim Y}$ *such that* $V^* = X^*_{(\mathcal{E}_Y || \mathcal{E}_Z)}$ *and, moreover,* $\pi(V^*) \hookrightarrow \mathbb{A}^{N-L}$ *and* $Z := X^*_{\mathcal{E}_Z} \hookrightarrow V^*$ *are closed in* $\mathbb{A}^{N-L}$ *and* $\mathbb{A}^N$ *respectively. The morphisms* $\pi|_Z : Z \to \pi(V)$ *and multiplication* $\mu : Z \times \hat{Y} \to V$ *are surjective iff field* $\mathbb{K}$ *is perfect and are finite of degree* $\underline{d} := d(\pi(\mathcal{E}_Z^{tr}))$ *with all fibres of size equal* $[\underline{d}] = \#(\hat{Y}^*/Y^*)$ *. Both morphisms are étale isomorphisms iff* $\underline{d} \neq 0$ *in* $\mathbb{K}$.

Also, $\mu|_{Z \times (g \cdot Y)}$ *for* $g \in \hat{Y}^*$ *are surjective (when* $\mathbb{K}$ *is perfect) and finite of degree* $\underline{d}$.

To connect with the notations of Section 1.2 note that $\hat{Y}_V := \hat{Y}$ is binomial (due to **B.**), while $Y_V := Y$ is an irreducible component of $\hat{Y}$ containing $\mathbb{I}_N$ and is toric due to **A.** .

Remark 3.8. The degree of μ in **C.** is $\dim_{\mu^*(\mathbb{K}(V))} \mathbb{K}[Z \times \hat{Y}] \cdot S^{-1}$, where $\mathbb{K}(V)$ is the field of rational functions on V and $S := \mu^*(\mathbb{K}[V] \setminus \{\mathbf{0}\})$.

Example 3.9. Note that $\mu|_{Z \times Y} : Z \times Y \to V$ need not be an étale isomorphism, e. g. if $V := \{y_1^2 = z_1 \cdot y_2^2 \ , \ z_1 \cdot z_2 = 1\}$ then $Y = \{z_1 = z_2 = 1 \ , \ y_1 = y_2\}$ ($Z := \{z_1 = y_1 = y_2^2 \ , \ z_1 \cdot z_2 = 1\}$ satisfies the assumptions of Theorem 3.7 **C.**) and there are two étale irreducible components of V at the points of $V \cap \{y_1 = y_2 = 0\}$, while $Z \times Y$ is nonsingular, and hence étale irreducible at every point. Nevertheless the local étale irreducible components of an affine binomial variety $\hat{V}$ are isomorphic to the respective étale germs of $Z \times Y$ due to Theorem 3.7.

Note that μ and $\mu|_{Z \times Y}$ are finite since $\mathbb{K}[Z \times Y] \simeq \mathbb{K}[t \ , \ s \ , \ s^{-1}]$ and $\mathbb{K}[Z \times \hat{Y}] \simeq \mathbb{K}[y_1 \ , \ y_2 \ , \ s \ , \ s^{-1}]/(y_1^2 - y_2^2)$ are integral over

$\mathbb{K}[t\cdot s^2\ ,\ t\cdot s\ ,\ s^2\ ,\ s^{-2}]\simeq\mu|^*_{Z\times Y}(\mathbb{K}[V])\hookrightarrow\mathbb{K}[Z\times Y]$ and, respectively, $\mathbb{K}[y_1\cdot s^2\ ,\ y_2\cdot s\ ,\ s^2\ ,\ s^{-2}]/(y_1^2-y_2^2)\simeq\mu|^*_{Z\times\hat{Y}}(\mathbb{K}[V])\hookrightarrow\mathbb{K}[Z\times\hat{Y}]$.

Claim 3.10. *For Z from the second line of Theorem 3.7* **C.** *the distinct irreducible components of $\hat{Y}$ are $g\cdot Y$, where $g\in\tilde{\Gamma}:=(\pi|_Z)^{-1}(\mathbb{I}_{N-L})$ and $\#(\tilde{\Gamma})=\#(\hat{Y}^*/Y^*)$. The map $\mu|_{Z\times(g\cdot Y)}$ is surjective and finite for all $g\in\tilde{\Gamma}$ iff it is the case for some $g\in\tilde{\Gamma}$.*

Proof. Identify V^* with $\mathbb{T}^n$ via the bijection $\phi_{(\mathcal{E}_Y||\mathcal{E}_Z)}$ with $\mathcal{E}_Z\subset\mathbb{Z}^{n-m}$ from Theorem 3.7 **C.** . The equalities $\#(\hat{Y}^*/Y^*)=d(\pi(\mathcal{E}_Z^{tr}))=\#(\pi|_Z^{-1}(\mathbb{I}_{N-L}))$ follow by replacing $\hat{E}$ by $\pi(\mathcal{E}_Z^{tr})^{tr}$ in Remark 3.1 . (Note that by replacing $\hat{E}$ by $E_{\hat{Y}}$ in Remark 3.1 , where $E_{\hat{Y}}$ is any exponent matrix of the equations of $\hat{Y}$, it also follows that $\#(\hat{Y}^*/Y^*)=[d(E_{\hat{Y}})]$; and the matrix with the rows of $\hat{E}$ followed by the rows of $(0||\mathrm{Id}_{N-L})$ could be used as $E_{\hat{Y}}$ if $d(\hat{E})=1$ or, equivalently, if $\hat{V}=V$.)

All irreducible components of $\hat{Y}^*$ are of the form $g\cdot Y^*$ for some $g\in\hat{Y}^*$ (Property **A.**) and the multiplication $\mu|_{Z\times Y^*}:Z\times Y^*\to V^*$ is a bijection (since $\phi_{(\mathcal{E}_Y||\mathcal{E}_Z)}$ is). Therefore $Y^*\cap(\pi|_Z)^{-1}(\mathbb{I}_{N-L})=\{\mathbb{I}_N\}$ and $Z\cap g\cdot Y^*\neq\emptyset$ for any $g\in V^*$. Hence distinct points of the subgroup $\tilde{\Gamma}$ belong to distinct irreducible components of $\hat{Y}$ and, respectively, every irreducible component intersects $\tilde{\Gamma}$ implying the first claim.

The remainder is a consequence of multiplication by $g\in\tilde{\Gamma}\hookrightarrow Z$ being an isomorphism of $Z\to Z$, of $V\to V$ and of $Y\to g\cdot Y$. □

Claim 3.11. *Let $\tilde{\mathcal{E}}^{tr}$ be any $\mathbb{Z}$-basis of* $\operatorname{Ker}\hat{E}\cap\mathbb{Z}^N$. *Then $d(\pi(\tilde{\mathcal{E}}^{tr}))$ (as determined by 3. of Remark 3.1) depends only on $V^*\hookrightarrow\mathbb{T}^N$ and coincides with $d(\pi(\mathcal{E}_Z^{tr}))$, where the choice of $\mathcal{E}_Z$ is as in Theorem 3.7* **C.**

Proof. Note that $X^*_{(\mathcal{E}_Y||\mathcal{E}_Z)}=V^*=X^*_{\tilde{\mathcal{E}}}$ due to the choice of $\mathcal{E}_Z$ and 6. of Remark 3.1 . Also, obviously, $\pi((\mathcal{E}_Y)^{tr})=\{\mathbf{0}\}$. Furthermore, the 'z-coordinates' of $\tilde{\mathcal{E}}^{tr}$, i. e. $\pi(\tilde{\mathcal{E}}^{tr})$, generate over $\mathbb{Z}$ a sublattice

$$\mathrm{Span}_{\mathbb{Z}}(\pi(\tilde{\mathcal{E}}^{tr}))\subset\mathbb{Z}^{N-L}\cap\mathrm{Span}_{\mathbb{Q}}(\pi(\tilde{\mathcal{E}}^{tr}))=\mathbb{Z}^{N-L}\cap\pi(\operatorname{Ker}\hat{E})$$

that depends only on $V^*\hookrightarrow\mathbb{T}^N$, implying that $d(\pi(\tilde{\mathcal{E}}^{tr}))=d(\pi(\mathcal{E}_Z^{tr}))$ and the remainder of the claim. □

Remark 3.12. *An outline of the initial arguments of the proof of Theorem 3.7 in Section 5.* The sets of exponents parametrizing the tori of Y and V are the rows of the matrices whose columns must be $\mathbb{Z}$-bases $\mathcal{E}_Y$ of $\operatorname{Ker}\hat{E}\cap(\mathbb{Z}^L\times\mathbf{0})$ and $\mathcal{E}_V:=(\mathcal{E}_Y||\mathcal{E}_Z)$ of $\operatorname{Ker}\hat{E}$ (Remark 3.1). Hence, $\phi_{(\pi(\mathcal{E}_Z^{tr}))^{tr}}=\pi|_Z\circ\phi_{\mathcal{E}_Z}$ implying (when $\pi(V^*)=\pi(V)$ and Z are closed) that the properties of $\pi|_Z:Z\to\pi(V^*)$ listed in part

C. are equivalent to the analogous properties of $\phi_{(\pi(\mathcal{E}_Z^{tr}))^{tr}}$. Of course $\pi(\mathcal{E}_Z^{tr})$ is a $\mathbb{Q}$-basis of $\pi(\operatorname{Ker}\hat{E})\cap\mathbb{Z}^{N-L}$, but (as in the Example 3.9) it need not be a $\mathbb{Z}$-basis. Respectively $\pi|_Z$ need not be an isomorphism, but is only a finite map of degree $\underline{d}$ as in part **C.**. Finally, the properties of μ listed in **C** follow from the respective properties of $\pi|_Z$ by making use of the coordinatewise multiplication action by Z on V (the missing details are in Section 5).

We refer to $Y \hookrightarrow \hat{V}$ as an *essential* subvariety. With π as above, and with a convention of identifying $\mathbb{A}^L \times \mathbb{I}_{N-L} \simeq \mathbb{A}^L$ and $\mathbf{0} \times \mathbb{I}_{N-L} \simeq \mathbf{0}$ we find that the variety Y itself is *essential* , i. e. $\mathbf{0} \in Y$, due to Corollary 3.5 .

Claim 3.13. *Essential varieties are distinguished by the property of having the origin as their most singular point, i. e. all singularities of these varieties occur in any neighbourhood of their respective origins.*

Proof. Consider the automorphisms of Y induced by the coordinatewise multiplication by $g \in X_{\mathcal{E}^+}$ with $X_{\mathcal{E}^+}$ from Corollary 3.5. Then for any point $\mathcal{P} \in Y \setminus Y^*$ the germs of Y at $g\cdot\mathcal{P}$, $g \in X^*_{\mathcal{E}^+}$, are isomorphic and the origin of Y coincides with $(\mathcal{P}\cdot X_{\mathcal{E}^+}) \setminus (\mathcal{P}\cdot X^*_{\mathcal{E}^+})$, as claimed. □

The remainder of this section contains applications of Theorem 3.7 .

Claim 3.14. *Assume that the morphism μ of Theorem 3.7* **C** *associated with a binomial variety $\hat{V}$ is an étale isomorphism, e. g. true when field $\mathbb{K}$ has characteristic equal to 0 . The irreducible components of the local étale germs of a $\mathcal{GAB}$-variety $\tilde{V}$ that occurs as the $\hat{V}$-admissible subvariety of an $\mathcal{AB}$-variety $\hat{V}$ are isomorphic to the products of nonsingular germs with the respective germs of the subvariety $\hat{Y}_V$ of $\hat{V}$ (from Claim 3.11 , Theorem 3.7* **B***). Hence these components are nonsingular iff $\hat{Y}_V$ is not singular, and the conclusions of Remark 1.2 and of Remark 3.15 apply to all $\mathcal{GAB}$-varieties. Any quasi-binomial variety is in the $\mathcal{GAB}$ class.*

For Nash and normalized Nash blowings up Theorem 3.7 implies

Remark 3.15. The 'towers' of Nash (as well as normalized Nash) blowings up for mutually isomorphic (due to Property **A.**) components $g \cdot V$ of $\hat{V}$, where $g \in \hat{V}^*$ and $[g] \in \Gamma$, are of course themselves mutually isomorphic. Therefore it suffices to study the effect of this process on a single irreducible component V to make them all smooth in the respective 'tower' for $\hat{V}$. Moreover, Remark 1.2 implies that the stabilization of the sequence of Nash blowings up (respectively normalized Nash blowings up) of an affine binomial variety is equivalent to the stabilization of the respective sequence for its essential toric subvariety.

Theorem 3.7 C. also implies a criterion of nonsingularity for an arbitrary affine toric variety $X_{\tilde{\mathcal{E}}}$ in terms of the exponents $\tilde{\mathcal{E}} \subset \mathbb{Z}^n$ of an arbitrary monomial parametrization of the torus $X^*_{\tilde{\mathcal{E}}}$ of $X_{\tilde{\mathcal{E}}}$. In the simpler case of $X_{\tilde{\mathcal{E}}}$ being an essential variety, which in terms of $\tilde{\mathcal{E}}$ means $\mathbf{0} \notin \mathrm{Conv}\,(\tilde{\mathcal{E}})$ (Claim 3.2), the criterion is

Criterion 3.16. *An essential toric variety* $Y := X_{\tilde{\mathcal{E}}}$ *is not singular iff the exponents* $\tilde{\mathcal{E}} \subset \mathbb{Z}^n$ *of an arbitrary monomial parametrization of the torus of* Y *are generated over* $\mathbb{Z}_+$ *by* $\dim Y$ *among them.*

Proof. Of course the 'if' implication is obvious. For the 'only if' implication note that under the nonsingularity assumption, Y near $\mathbf{0}$ coincides with a graph of an étale map-germ, say $\mathcal{G}$, at $\mathbf{0}$ and, also, that Y is the closure of the image under a monomial parametrization, say $\phi_{\mathcal{E}^+}$, of the torus of Y with exponents $\mathcal{E}^+ \subset (\mathbb{Z}_+)^n$ (Claim 3.3). It follows, by making use of the uniqueness of the Taylor series expansion of the composition $\mathcal{G} \circ \phi_{\mathcal{E}^+}$, that the map $\mathcal{G}$ is monomial, which obviously implies the conclusion of the 'only if' implication. $\square$

The latter criterion of $\mathrm{Sing}\, Y = \emptyset$ depends on the assumption $\mathbf{0} \notin \mathrm{Conv}\,(\tilde{\mathcal{E}})$, i. e. on Y being essential, as demonstrated by

Example 3.17. The closure $X_{\tilde{\mathcal{E}}}$ of $\phi_{\tilde{\mathcal{E}}}((\mathbb{C}^*)^4) \subset \mathbb{C}^6$ for a monomial map $(\mathbb{C}^*)^4 \ni x \mapsto \phi_{\tilde{\mathcal{E}}}(x) := (x_1\,,\, x_2\,,\, x_3\,,\, x_4\,,\, x_3^{-1}\,,\, x_3{\cdot}x_4^{-1}) \in (\mathbb{C}^*)^6$ is nonsingular, 4-dimensional and its essential subvariety $Y = \mathbb{C}^2 \times \{\mathbb{I}_4\}$. But $\tilde{\mathcal{E}}$ is not generated over $\mathbb{Z}_+$ by any subset of 4 vectors.

3.1. **Criteria of nonsingularity.** In this section we derive as a consequence of Theorem 3.7 C and Criterion 3.16 simple combinatorial criteria for nonsingularity for arbitrary affine toric varieties and, also, for the blowings up of affine space $\mathbb{A}^n$ at ideals generated by monomials.

We start with a criterion for $\mathrm{Sing}\, X_{\tilde{\mathcal{E}}} = \emptyset$. To that end note that the subset of 'y-coordinates' for $X_{\tilde{\mathcal{E}}}$ among all coordinates w_e , $e \in \tilde{\mathcal{E}} \subset \mathbb{Z}^n$, on $\mathbb{A}^N$ can be identified as

$$\mathcal{E}' = \{e : \exists \eta \in (\mathbb{Q}^n)^{dual}\ ,\ \eta(e) > 0\ ,\ \eta|_{\tilde{\mathcal{E}}} \geq 0\}\ , \tag{3.1}$$

due to Corollary 3.5 . As a straightforward consequence of the definitions the subset of 'z-coordinates' $\tilde{\mathcal{E}} \setminus \mathcal{E}' \supset \mathcal{E}'' := \cup_{l \geq 1} \mathcal{E}_l$, where the subsets $\mathcal{E}_l \setminus \mathcal{E}_{l-1} \subset \tilde{\mathcal{E}} \setminus \mathcal{E}_{l-1}$, $l \geq 1$, are taken to be minimal with respect to $\mathrm{Conv}\,(\mathcal{E}_l \setminus \mathcal{E}_{l-1}) \cap \mathrm{Span}_{\mathbb{Q}}(\mathcal{E}_{l-1}) \neq \emptyset$, $l \geq 2$, and, respectively,

$\mathrm{Conv}\,(\mathcal{E}_1) \supset \mathcal{E}_0 := \{\mathbf{0}\}$. Of course then $\mathrm{Conv}\,(\tilde{\mathcal{E}} \backslash \mathcal{E}'') \cap \mathrm{Span}_{\mathbb{Q}}(\mathcal{E}'') = \emptyset$, implying that exists $\eta \in (\mathbb{Q}^n)^{dual}$ vanishing on set $\mathcal{E}''$ and positive on $\tilde{\mathcal{E}} \setminus \mathcal{E}''$ and then the values of η on the rows of $\tilde{\mathcal{E}}$ provide the $\vec{\xi}^{+}$ of Corollary 3.5 . Consequently, Lemma 3.4 implies

$$\tilde{\mathcal{E}} = \mathcal{E}' \cup \mathcal{E}'' \ . \tag{3.2}$$

(The latter algorithm is single exponential, while that of identifying $\mathcal{E}'$ in $\tilde{\mathcal{E}}$ via formula (3.1) is polynomial, cf. Section 7.2.) Finally

Criterion 3.18. *$V := X_{\tilde{\mathcal{E}}}$ is nonsingular iff all local étale irreducible components of V are nonsingular and V is étale irreducible. Due to Theorem 3.7* **C.** *and Claim 3.11 our criterion for the étale irreducibility of $\hat{Y}$ is ($d(\pi(\tilde{\mathcal{E}}^{tr})) = d(\pi(\mathcal{E}_Z^{tr})) = 1$ or) that the collection of all $\rho \times \rho$ minors of matrix $\mathcal{E}''$ generate the unit ideal, where $\rho := \mathrm{rank}\,(\mathcal{E}'')$. Let $m := n - \rho$.*

Then the étale irreducible components of $X_{\tilde{\mathcal{E}}}$ are nonsingular iff $\mathcal{E}' \subset \mathbb{Z}^n$ is generated over $\mathbb{Z}_+$ (mod $\mathrm{Span}_{\mathbb{Q}}(\mathcal{E}'')$) by a subset consisting of m elements of $\mathcal{E}'$.

Proof. It remains to prove only the claim of the last two lines of the criterion. The case of $\mathcal{E}' = \tilde{\mathcal{E}}$ is fully explained in Criterion 3.16. The reduction to the $\mathcal{E}' = \tilde{\mathcal{E}}$ case follows by means of identifying the torus Y^* of the toric component Y of $V \cap (\cap_{e \in \mathcal{E}''}\{w_e = 1\})$ and by means of a parametrization of $Y^* \overset{\chi}{\hookrightarrow} \mathbb{T}^{\#(\mathcal{E}')} \hookrightarrow \mathbb{T}^N$. (Note that $\phi_{\mathcal{E}'} = \chi \circ \phi_{\tilde{\mathcal{E}}}$.) Let $\mathcal{M}$ be the matrix of size $n \times m$ with entries in $\mathbb{Z}$ and with columns a $\mathbb{Z}$-basis of the orthogonal complement to $\mathrm{Span}_{\mathbb{Q}}(\mathcal{E}'') \subset \mathbb{R}^n$. Then (due to Remark 3.1) the map $\phi_{\mathcal{M}}$ is a parametrization of $\phi_{\tilde{\mathcal{E}}}^{-1}(Y^*) \hookrightarrow \mathbb{T}^n \tilde{\to} V^*$, implying that $\phi_{\tilde{\mathcal{E}}} \circ \phi_{\mathcal{M}}$ is a parametrization of $Y^* \hookrightarrow \mathbb{T}^N$. It follows that the set $\mathcal{E} \subset \mathbb{Z}^m$ of the rows of the product matrix $\mathcal{E}' \cdot \mathcal{M}$ provides the exponents of a monomial parametrization $\phi_{\mathcal{E}}$ of $\chi(Y \cap \mathbb{T}^N)$ (since $\chi \circ \phi_{\tilde{\mathcal{E}}} \circ \phi_{\mathcal{M}} = \phi_{\mathcal{E}'} \circ \phi_{\mathcal{M}} = \phi_{\mathcal{E}' \cdot \mathcal{M}}$). Of course there are m rows of the matrix $\mathcal{E}'$ generating over $\mathbb{Z}_+$ all rows of $\mathcal{E}'$ modulo $\mathrm{Ker}\, \mathcal{M}^{tr} = \mathrm{Span}_{\mathbb{Q}}(\mathcal{E}'')$ iff there are m rows of the matrix $\mathcal{E}' \cdot \mathcal{M}$ generating over $\mathbb{Z}_+$ all rows of $\mathcal{E}' \cdot \mathcal{M}$. But the latter is the Criterion 3.16 of nonsingularity for Y . Also, due to Theorem 3.7 **C.** and Property **A.**, $Y = Y_V$ is nonsingular iff all étale irreducible components of $X_{\tilde{\mathcal{E}}}$ are nonsingular. Combining the equivalences of the last three sentences completes the proof. □

Criterion 3.19. *Let $\mathcal{I} \hookrightarrow \mathbb{K}[x]$ be an ideal generated in $\mathbb{K}[x]$ by monomials $\mathcal{M} = \{x^e\}_{e \in \overline{\mathcal{E}}}$, $\overline{\mathcal{E}} \subset (\mathbb{Z}^n \cap (\mathbb{Q}_+)^n) \setminus \{\mathbf{0}\}$, such that proper subsets of $\mathcal{M}$ do not generate $\mathcal{I}$, and denote by $\Gamma_{\overline{\mathcal{E}}} \subset \overline{\mathcal{E}}$ the set of vertices of $\mathrm{Conv}\,(\cup_{e \in \overline{\mathcal{E}}}(e + \mathbb{R}^n_+))$. Let $\sigma_{\mathcal{I}} : X \to \mathbb{A}^n$ be the blowing up with center* *at ideal $\mathcal{I}$. Then $SingX = \emptyset$ if and only if for all $e \in \Gamma_{\overline{\mathcal{E}}}$ the sets $\mathcal{E}_e := \{e' - e : \ e' \in \overline{\mathcal{E}}\} \cup \{(j)\}_{1\le j\le n}$ are generated over $\mathbb{Z}_+$ by some of their subsets of n elements.*

Proof. By definition of the blowing up X is the closure of the graph of the monomial map $\Psi_{\overline{\mathcal{E}}} := \mathbb{T}^n \ni x \mapsto [\dots : x^e : \dots]_{e\in\overline{\mathcal{E}}} \in \mathbb{P}^N$, where $N := \#\overline{\mathcal{E}} - 1$, and the blow up $\sigma_{\mathcal{I}}$ is the restriction to X of the projection $\mathbb{A}^n \times \mathbb{P}^N \to \mathbb{A}^n$. Let $U_e := \mathbb{P}^N \setminus \{w_e = \mathbf{0}\} \simeq \mathbb{A}^N$, where w_e's are the homogeneous coordinates on $\mathbb{P}^N$. Then $\mathbb{P}^N = \cup_{e\in\overline{\mathcal{E}}} U_e$. Consequently the variety $X = \cup_{e\in\overline{\mathcal{E}}} X_e$ with each $X_e := X \cap (\mathbb{A}^n \times U_e)$ being the closure of the torus $\phi_{\mathcal{E}_e}(\mathbb{T}^n)$ in $\mathbb{A}^{n+N}$ and the sets $\mathcal{E}_e$ as introduced above.

Moreover, $X = \cup_{e\in\Gamma_{\overline{\mathcal{E}}}} X_e$ with $\Gamma_{\overline{\mathcal{E}}} \subset \overline{\mathcal{E}}$ as introduced above, since whenever $e_0 \in \overline{\mathcal{E}} \cap (\mathrm{Conv}\,(\Gamma_{\overline{\mathcal{E}}}) + \mathbb{R}^n_+)$ it follows that there is a nonempty subset $I_0 \subset \Gamma_{\overline{\mathcal{E}}}$ with $\{q_e\}_{e\in I_0} \subset \mathbb{Z}_+$ and $\omega \in \mathbb{Z}^n \cap \mathbb{Q}^n_+$ such that $\left(\prod_{e\in I_0} (w_e/w_{e_0})^{q_e} \cdot x^\omega\right)\big|_{X_{e_0}} = 1$. Consequently the chart $X_{e_0} \subset X_e$ for any $e \in I_0 \subset \Gamma_{\overline{\mathcal{E}}}$. Then X is nonsingular iff all X_e , $e \in \Gamma_{\overline{\mathcal{E}}}$, are nonsingular, and the nonsingularity Criterion 3.18 in terms of the sets $\mathcal{E}_e$, $e \in \Gamma_{\overline{\mathcal{E}}}$, applies. But the special case at hand provides a substantial simplification since among the exponents $\mathcal{E}_e$ for $e \in \Gamma_{\overline{\mathcal{E}}}$ exponents corresponding to the 'z-coordinates' (as in the definition of $\mathcal{E}''$ following (3.1)) do not occur and therefore a simpler Criterion 3.16 applies, i. e. that over $\mathbb{Z}_+$ the set $\mathcal{E}_e$ is generated by its n elements. Indeed, otherwise the set $(\mathcal{E}_e)'' \neq \emptyset$ implying that there is a nonempty $I_e \subset (\mathcal{E}_e)''$ with $\sum_{\vec{v}\in I_e} q_{\vec{v}} \cdot \vec{v} = \mathbf{0}$ and $\{q_{\vec{v}}\}_{\vec{v}\in I_e} \subset \mathbb{Z}_+$. Then $I_e \cap \{e' - e : \ e' \in \overline{\mathcal{E}}\} \neq \emptyset$ and so $e \in (\mathrm{Conv}\,(\Gamma_{\overline{\mathcal{E}}}) + \mathbb{R}^n_+) \setminus \Gamma_{\overline{\mathcal{E}}}$, contrary to our assumption. □

4. Reduction of the Nash algorithm to a combinatorial one.

We introduced Nash blowings up and the Euclidean multidimensional division algorithm in Sections 1.1 and 1.3 . For binomial varieties the stabilization of the algorithm of successive Nash blowings up reduces to the stabilization of the same algorithm for their respective essential affine toric subvarieties (due to Remark 1.2 and Theorem 3.7 , as summarized in Remark 3.15). In this section we establish the equivalence of the Nash algorithm for the latter varieties of dimension m to the Euclidean m-dimensional division algorithm. To that end we first establish in Constructions 4.4 and 4.5 a combinatorial 'bookkeeping' for the sequences of successive Nash blowings up of these varieties. We then in Claim 4.6 show that Nash blowings up of the essential affine varieties admit coverings by varieties from the same class. In

Claim 4.9 , we show that a criterion for the stabilization of the Nash algorithm corresponds to the termination criterion of the corresponding Euclidean division algorithm. We conclude this section by summarizing in Section 4.3 the well-known combinatorial translation of the effect of normalization for affine toric varieties. Finally, in this section $\mathbb{K}$ denotes an algebraically closed field of characteristic zero.

4.1. **Gauss map and Nash blow up of an essential subvariety.** Let $\{\vec{\delta}_i \times \mathbf{0}\}_{1\le i\le m} \subset \mathbb{Z}^N$, where $\vec{\delta}_i := (\delta_{1i}, \dots, \delta_{Li})$, generate the integral lattice of $\operatorname{Ker} \hat{E} \cap (\mathbb{Q}^L \times \mathbf{0}) \subset \mathbb{Q}^N$ over $\mathbb{Z}$ and denote $\mathcal{E} := \{\vec{\Delta}_j\}_{1\le j\le L} \subset \mathbb{Z}^m$, where each $\vec{\Delta}_j := (\delta_{j1}, \dots, \delta_{jm})$. Then

$$(4.1)\quad (\phi_{\mathcal{E}})_j(x) := \prod_{1\le i\le m} x_i^{\delta_{ji}},\ 1 \le j \le L; \quad (\phi_{\mathcal{E}})_s \equiv 1,\ L < s \le N\ ,$$

are components of the isomorphism $\phi_{\mathcal{E}} : (\mathbb{K}^*)^m \to Y^* := Y \cap (\mathbb{K}^*)^N \hookrightarrow \mathbb{K}^L$ of tori ($\phi := \phi_{\mathcal{E}}$ in this section). The closure $Y \hookrightarrow \mathbb{K}^L$ of Y^* contains $\mathbf{0} \in \mathbb{K}^L$ (Corollary 3.5) and one may choose $\delta_{ji} \in \mathbb{Z}_+$ (Claim 3.3).

Remark 4.1. The map $\phi|_{(\mathbb{R}_+ \setminus\{\mathbf{0}\})^m} : (\mathbb{R}_+ \setminus \{\mathbf{0}\})^m \to Y \cap (\mathbb{R}_+ \setminus \{\mathbf{0}\})^N$ is an isomorphism. Therefore, also its tangent (at $\mathbb{I}_m \in \mathbb{R}^m$) map

$$(\mathbb{R}^m)^{dual} \ni h \mapsto (h(\vec{\Delta}_1), ..., h(\vec{\Delta}_L)) \times \mathbf{0} \in \operatorname{Ker} \hat{E} \cap (\mathbb{R}^L \times \{\mathbf{0}\})$$

is an isomorphism. Then, due to the choice of the vector $\vec{\xi}^+$ from Corollary 3.5, there is a functional $h^+ \in (\mathbb{Q}^m)^{dual}$ such that each $h^+(\vec{\Delta}_j) = (\vec{\xi}^+)_j > 0$. Hence $\mathbf{0} \notin \operatorname{Conv}(\mathcal{E}) \subset \mathbb{R}^m$ for $\mathcal{E}$ introduced above (4.1).

The latter **essential** *property of $\mathcal{E} \subset \mathbb{Z}^m$ (hereditary due to Claim 4.6) enables 'bookkeeping' of the Nash (and/or normalized Nash) algorithm by following the changes in the successive sets of the exponents $\mathcal{E}$.*

To 'control' the closure of the torus Y^* we prove (in Section 5) the following lemma.

Lemma 4.2. *For every point $\mathcal{P} \in Y \setminus Y^*$ there exist $g \in Y^*$ and a vector $\vec{\xi} \in \operatorname{Ker} \hat{E} \cap ((\mathbb{Z}_+)^L \times \{\mathbf{0}\})$ such that $\{\mathcal{P}\} = g \cdot (X_{\mathcal{E}^+} \setminus X^*_{\mathcal{E}^+})$, where the set $\mathcal{E}^+ := \{(\vec{\xi})_j\}_j$ consists of coordinates $(\vec{\xi})_j$ of $\vec{\xi}$. (Of course $g \cdot X^*_{\mathcal{E}^+} \hookrightarrow Y^*$.) Moreover, every coordinate of $\vec{\xi}$ is positive or is equal to zero depending on the respective coordinate of $\mathcal{P}$ being equal to zero or not.*

When $\delta_{ji} < 0$, the map ϕ would not extend to all of $\mathbb{K}^m$ and even if all $\delta_{ji} > 0$, as in Claim 3.3 , the map $\phi : \mathbb{K}^m \to Y$ may not be surjective. Nevertheless, every $\mathcal{P} \in Y \setminus Y^*$ is in the closure of the ϕ-image of a translation of $\{(t^{h_1}, \dots, t^{h_m}\}_{t \in K^*} \hookrightarrow (\mathbb{K}^*)^m$, where $(h_1, \dots, h_m) \in \mathbb{Z}^m$, due to the following corollary.

Corollary 4.3. *For any $\mathcal{P} \in Y \setminus Y^*$ there is an $h \in (\mathbb{Q}^m)^{dual} \cap \mathbb{Z}^m$ such that $\{\mathcal{P}\} = g \cdot (X_{h(\mathcal{E})} \setminus X^*_{h(\mathcal{E})})$. (Of course for $h \in (\mathbb{Q}^m)^{dual} \cap \mathbb{Z}^m$, the set $X_{h(\mathcal{E})} \setminus X^*_{h(\mathcal{E})} \neq \emptyset$ iff either all $h(\vec{\Delta}_j) \geq 0$ or all $h(\vec{\Delta}_j) \leq 0$.)*

Proof. Apply Remark 4.1 to pick an $h \in (\mathbb{Q}^m)^{dual} \cap \mathbb{Z}^m$ such that $h(\vec{\Delta}_j) = q \cdot (\vec{\xi})_j \in \mathbb{Z}_+ \cup \{\mathbf{0}\}$, $q \in \mathbb{Z}_+$, with $\mathcal{E}^+ = \{(\vec{\xi})_j\}_j \subset \mathbb{Z}$ from Lemma 4.2 . Of course replacing the vector $\vec{\xi}$ and the set $\mathcal{E}^+$ by $q \cdot \vec{\xi}$ and $h(\mathcal{E}) = q \cdot \mathcal{E}^+$ does not change the outcome of Lemma 4.2. $\square$

In particular, by identifying $\mathbb{K}^L$ with $\mathbb{K}^L \times \mathbb{I}_{N-L} \hookrightarrow \mathbb{K}^N$ and by making use of Corollary 3.5 and Claim 3.2 , it follows that the origin of $\mathbb{K}^L$ is in Y . Equivalently, there is an $h^+ \in (\mathbb{Q}^m)^{dual}$ such that for $1 \leq j \leq L$ the values $h^+(\vec{\Delta}_j) = (\vec{\xi^+})_j > 0$, which is also equivalent to $\mathcal{E} := \{\vec{\Delta}_j\}_{1 \leq j \leq L} \subset \mathbb{Z}^m$ being essential, i. e. $\mathrm{Conv}\,(\mathcal{E}) \not\ni \mathbf{0}$.

Construction 4.4. *Explicit construction of the composition $G_Y \circ \phi$.*

Let $\mathcal{G}_m(\mathbb{K}^L) \hookrightarrow \mathbb{KP}^{\binom{L}{m}-1}$ be the embedding of the Grassmanian $\mathcal{G}_m(\mathbb{K}^L)$ of the m-dimensional subspaces of $\mathbb{K}^L$ by means of Plücker coordinates. Consider the composition of the Gauss map G_Y of Y on Y^* with a monomial parametrization (4.1) of Y^* . Also, identify $G_Y(\phi(x)) \in \mathcal{G}_m(\mathbb{K}^L)$ with $T_{\phi(x)}Y$, which coincides with the image of $T_x\mathbb{K}^m \simeq \mathbb{K}^m$ under the tangent map to ϕ at $x \in (\mathbb{K}^*)^m$. The homogeneous (Plücker) coordinates $\tilde{w} = [\ \dots : \tilde{w}_J : \dots\ ]$ of $G_Y(\phi(x)) = \mathrm{Im}\,\frac{\partial \phi}{\partial x}(x)$ are the subdeterminants $\det_J(\mathcal{J}_\phi)(x)$ of the $m \times m$ size submatrices of the jacobian matrix $\mathcal{J}_\phi(x)$ of the map $y = \phi(x)$ and are listed by the choices of $J = \{j_1, \dots, j_m\} \subset \{1, \dots, L\}$ of m distinct rows of the $L \times m$ matrix $\mathcal{J}_\phi$, i. e. $\tilde{w}_J = \det_J(\mathcal{J}_\phi(x)) = \det_J(\delta) \cdot x^{\sum_{j \in J} \vec{\Delta}_j}/(x_1 \cdot \dots \cdot x_m)$, where $\det_J(\delta)$ are the respective subdeterminants of the exponent matrix δ in (4.1) . Denote $\mathcal{S} := \mathcal{S}(\mathcal{E}) := \{J : \det_J(\delta) \neq 0\}$ and $L^* := \#\mathcal{S} - 1$ (the notation $\mathcal{S}(\mathcal{E})$ is justified since $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}\{\vec{\Delta}_j\}_{j \in J} = m$ iff $\det_J(\delta) \neq 0$). Let $\mathbb{KP}^{L^*} := \bigcap_{\{J : \det_J(\delta) = 0\}} \{\tilde{w}_J = 0\} \hookrightarrow \mathbb{KP}^{\binom{L}{m}-1}$. Then $G_Y \circ \phi(x) \in \mathbb{KP}^{L^*}$ for all $x \in (\mathbb{K}^*)^m$. Moreover, then $G_Y \circ \phi : (\mathbb{K}^*)^m \to \cap_{J \in \mathcal{S}} \{\tilde{w}_J \neq 0\} =: T$.

Of course each $\mathcal{W}_J := \{\tilde{w}_J \neq 0\} \simeq \mathbb{K}^{L^*}$ and via this isomorphism T is identified with $(\mathbb{K}^*)^{L^*} \subset \mathbb{K}^{L^*}$. In abuse of notation let then $\mathcal{W}^*_J$

denote $T \hookrightarrow \mathcal{W}_J$. Similarly, denote $\mathcal{U}_J := \mathbb{K}^L \times \mathcal{W}_J$, $\mathcal{U}_J^* := (\mathbb{K}^*)^L \times \mathcal{W}_J^*$ and, also, the affine charts $N(Y)_J := N(Y) \cap \mathcal{U}_J$, $N(Y)_J^* := N(Y) \cap \mathcal{U}_J^*$. Of course the tori $N(Y)_{J_0}^* = \cap_{J \in \mathcal{S}} N(Y)_J$ for any $J_0 \in \mathcal{S}$. For the sake of convenience we replace coordinates $\tilde{w}_J$ by $w_J := (\det_J(\delta))^{-1} \cdot \tilde{w}_J$.

Construction 4.5. *Charts $N(Y)_J$ and parametrizations of their tori:*

As a consequence of the preceeding construction, $\mathcal{U}_J^* \hookrightarrow \mathcal{U}_J$ is isomorphic to $(\mathbb{K}^*)^{L+L^*} \hookrightarrow \mathbb{K}^{L+L^*}$ and the affine toric variety $N(Y)_J$ is the closure of the image $N(Y)_J^*$ of the torus $(\mathbb{K}^*)^m \subset \mathbb{K}^m$, under an algebraic group monomorphism $x \mapsto \psi(x) := (\phi(x)\ ,\ G_Y \circ \phi(x))$. For $J \in \mathcal{S}$ let $\vec{\Delta}_J := \sum_{j \in J} \vec{\Delta}_j$. An explicit formula for $G_Y \circ \phi$ of Construction 4.4 (in the w_J-coordinates of $\mathbb{KP}^{L^*}$) and, consequently, for the map ψ (in the affine coordinates of chart $\mathcal{U}_{J_0}$, for $J_0 \in \mathcal{S}$) is the monomial map $\phi_{\mathcal{E}_{J_0}}$ whose exponent set is $\mathcal{E}_{J_0} := \{\vec{\Delta}_j\}_{1 \le j \le L} \cup \{\vec{\Delta}_J - \vec{\Delta}_{J_0}\}_{J \in \mathcal{S} \setminus \{J_0\}}$.

Corollary 4.3 implies that, for any $J_0 \in \mathcal{S}$ 'one may reach' all points $\tilde{\mathcal{P}} \in N(Y)_{J_0} \setminus N(Y)_{J_0}^*$ by means of $g \cdot X_{h(\mathcal{E}_{J_0})}^* \hookrightarrow Y^*$ with $g \in Y^*$, i. e. $\{\mathcal{P}\} = g \cdot (X_{h(\mathcal{E}_{J_0})} \setminus X_{h(\mathcal{E}_{J_0})}^*)$, where $h \in (\mathbb{Q}^m)^{dual} \cap \mathbb{Z}^m$. Also, $h(\mathcal{E}_{J_0}) \subset \mathbb{Z}_+ \cup \{\mathbf{0}\}$ since $X_{h(\mathcal{E}_{J_0})} \setminus X_{h(\mathcal{E}_{J_0})}^* \neq \emptyset$. Moreover, the affine chart $N(Y)_{J_0}$ contains the origin of $\mathcal{U}_{J_0} \simeq \mathbb{K}^{L+L^*}$, i. e. is essential, iff there is $\tilde{h} \in (\mathbb{Q}^m)^{dual}$ such that $\tilde{h}(\mathcal{E}_{J_0}) \subset \mathbb{Z}_+$. The latter is also equivalent (Lemma 3.4) to all coordinates on $\mathcal{U}_{J_0}$ being 'y-variables' for $N(Y)_{J_0}$. Equivalently (Corollary 4.3) $\mathrm{Conv}\,(\mathcal{E}_{J_0}) \not\ni \mathbf{0}$.

Claim 4.6. *Assuming* $\mathbf{0} \in Y = X_{\mathcal{E}} \hookrightarrow \mathbb{K}^L \simeq \mathbb{K}^L \times \mathbb{I}_{N-L}$ *it follows that* $N(Y) = \cup_{J \in \mathcal{S}'} N(Y)_J$ *, where* $\mathcal{S}'$ *is the subset of all* $J \in \mathcal{S}$ *such that the affine charts* $N(Y)_J$ *are essential.*

Proof. Due to Claim 3.2 and Corollary 4.3 our assumption is $\mathrm{Conv}\,(\mathcal{E}) \not\ni \mathbf{0}$. Let the cone $\tilde{\mathcal{C}} := \{h \in (\mathbb{R}^m)^{dual} : h_{|\mathcal{E}} \ge 0\}$ and, likewise, for every $J \in \mathcal{S}$ let $\tilde{\mathcal{C}}_J := \{h \in \tilde{\mathcal{C}} : h_{|\mathcal{E}_J} \ge 0\}$. Then h^+ from Corollary 4.3 is in the interior of the cone $\tilde{\mathcal{C}}$ (in particular $\dim_{\mathbb{R}} \tilde{\mathcal{C}} = m$). We refer to $h = (h_1, \ldots, h_m) \in (\mathbb{R}^m)^{dual}$ with $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}\{h_1, \ldots, h_m\} = m$ as an *irrational* point of $(\mathbb{R}^m)^{dual}$. For any irrational $h \in \tilde{\mathcal{C}}$ there is (a unique) $J \in \mathcal{S}$ such that h is in the interior of $\tilde{\mathcal{C}}_J$. Therefore $\dim_{\mathbb{R}} \tilde{\mathcal{C}}_J = m$ iff $\mathrm{Conv}\,(\mathcal{E}_J) \not\ni \mathbf{0}$. The latter is equivalent to $J \in \mathcal{S}'$ implying $\tilde{\mathcal{C}} = \cup_{J \in \mathcal{S}'} \tilde{\mathcal{C}}_J$.

Consider any $J_0 \in \mathcal{S}$. The torus $N(Y)_{J_0}^*$ coincides with the image $\psi((\mathbb{K}^*)^m) \subset \cap_{J \in \mathcal{S}'} N(Y)_J$. Let $\mathcal{P} \in N(Y)_{J_0} \setminus N(Y)_{J_0}^*$. Then, as in Corollary 4.3 , there are $g \in N(Y)_{J_0}^*$ and $h \in (\mathbb{Q}^m)^{dual} \cap \mathbb{Z}^m$ such that $\{\mathcal{P}\} = g \cdot (X_{h(\mathcal{E}_{J_0})} \setminus X_{h(\mathcal{E}_{J_0})}^*)$. Moreover, the values $h(\vec{\Delta}_j)$, $1 \le j \le L$, and all $h(\vec{\Delta}_J - \vec{\Delta}_{J_0})$, $J \in \mathcal{S}\setminus\{J_0\}$, are positive or vanish depending on the respective coordinates of $\mathcal{P}$ being equal to zero or not (Lemma 4.2). Thus $h \in \tilde{\mathcal{C}} = \cup_{J\in\mathcal{S}'}\tilde{\mathcal{C}}_J$ and, therefore, there exists $J_1 \in \mathcal{S}'$ such that $h \in \tilde{\mathcal{C}}_{J_1}$. As a consequence $h(\vec{\Delta}_{J_0}) = h(\vec{\Delta}_{J_1})$. It follows that the ratio w_{J_0}/w_{J_1} of the homogeneous coordinates is identically one on $X^*_{h(\mathcal{E}_{J_0})}$. Consequently it is constant on $g \cdot X^*_{h(\mathcal{E}_{J_0})}$ and coincides with the ratio $w_{J_0}(g)/w_{J_1}(g)$. Hence $\mathcal{P} \in N(Y)_{J_1} \setminus N(Y)^*_{J_1}$, as required. □

In the next two sections we summarize our 'translation' of Nash and normalized Nash blowings up into respective combinatorial versions. These combinatorial versions are in terms of the smallest subsets of generators for additive semigroups $\mathbb{Z}_+(\mathcal{E})$ generated by $\mathcal{E} \subset \mathbb{Z}^m$ and $\mathbb{Q}_+(\mathcal{E})_\mathbb{Z} := \mathrm{Span}_\mathbb{Z}(\mathcal{E}) \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) \setminus \{\mathbf{0}\}$, where sets $\mathcal{E}$ are finite and $\mathrm{Conv}\,(\mathcal{E}) \not\ni \mathbf{0}$.

For an additive semigroup without zero, say G_+ , we introduce a notion of the set $\mathcal{E}xtreme(G_+)$ of all *Z_+-extremal points* of G_+ , i. e. of all $g \in G_+$ such that $g \neq g_1 + g_2$ for any g_1 , $g_2 \in G_+$.

Let $\nabla(J) := \mathrm{Conv}\,(J \cup \{\mathbf{0}\})$ and $\mathrm{int}(\nabla(J)) :=$ the interior of $\nabla(J)$.

Claim 4.7. *Assume that the set $\mathcal{E} \subset \mathbb{Z}^m$ is finite and essential. Then*

(i) The set $\mathcal{E}xtreme(\mathbb{Q}_+(\mathcal{E})_\mathbb{Z})$ is finite and it generates the semigroup $\mathbb{Q}_+(\mathcal{E})_\mathbb{Z}$ (the set $\mathcal{E}xtreme(\mathbb{Z}_+(\mathcal{E})) \subset \mathcal{E}$ is obviously finite and it generates $\mathbb{Z}_+(\mathcal{E})$);

(ii) $\mathcal{E}' = \mathcal{E}xtreme(\mathbb{Q}_+(\mathcal{E}')_\mathbb{Z})$, where $\mathcal{E}' = \mathcal{E}xtreme(\mathbb{Q}_+(\mathcal{E})_\mathbb{Z})$;

(iii) Let $\mathcal{S}'$ be as in Claim 4.6 . If $\mathcal{E} = \mathcal{E}xtreme(\mathbb{Q}_+(\mathcal{E})_\mathbb{Z})$ and $J \in \mathcal{S}'$ then int$(\nabla(J)) \cap \mathbb{Q}_+(\mathcal{E})_\mathbb{Z} = \emptyset$.

Proof. (i) is a consequence of Gordan's lemma (Prop.1 in 1.2 [5]) since $\mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E})$ coincides with the dual cone $(\tilde{\mathcal{C}})^{dual}$ of its own dual cone $\tilde{\mathcal{C}}$ and $\mathbb{Q}_+(\mathcal{E})_\mathbb{Z}$ is the set of its integral points (meaning the points in $\mathrm{Span}_\mathbb{Z}(\mathcal{E})$).

To show (ii) note, by making use of (i), that $\mathbb{Z}_+(\mathcal{E}') = \mathbb{Q}_+(\mathcal{E})_\mathbb{Z} \subset \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) = \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}')$. The latter implies $\mathbb{Q}_+(\mathcal{E}')_\mathbb{Z} = \mathbb{Q}_+(\mathcal{E})_\mathbb{Z}$. Thus (ii) follows.

We prove (iii) by contradiction. Indeed, if $\vec{a} \in \mathrm{int}(\nabla(J)) \cap \mathbb{Q}_+(\mathcal{E})_\mathbb{Z}$ let us choose an irrational $h \in \tilde{\mathcal{C}}$, as in Claim 4.6, such that $h(\vec{\Delta}_J) = \min_{J'\in\mathcal{S}} h(\vec{\Delta}_{J'})$. Let $j_0 \in J$ be such that $h(\vec{\Delta}_{j_0}) = \max_{j\in J} h(\vec{\Delta}_j)$. Then $\vec{a} \notin \mathcal{E}$, since otherwise the collection $J_0 := (J \cup \{\vec{a}\}) \setminus \{\vec{\Delta}_{j_0}\}$ is in $\mathcal{S}$, but $h(\vec{\Delta}_{J_0}) < h(\vec{\Delta}_J)$. Consequently $\vec{a} \in \mathbb{Z}_+(\mathcal{E})$, due to (i) and the assumption on $\mathcal{E}$. Therefore there is a vector $\vec{b} \in \mathcal{E}$ such that $J_1 := (J \cup \{\vec{b}\}) \setminus \{\vec{\Delta}_{j_0}\}$ is in $\mathcal{S}$, but $h(\vec{\Delta}_{J_1}) < h(\vec{\Delta}_J)$ because, if $\vec{a} \in \vec{b} + \mathbb{Z}_+(\mathcal{E})$, then the inequalities $h(\vec{\Delta}_{j_0}) > h(\vec{a}) > h(\vec{b})$ hold, contrary to the choice of h . $\square$

4.2. **Multidimensional Euclidean division as a bookkeeping.** In this section we complete the translation of the process of Nash blowings up into a *combinatorial* tree-like branching algorithm on finite essential subsets of $\mathbb{Z}^m$. To that end we choose $\{(\delta_{1i}, \dots, \delta_{Li})\}_{1\le i\le m} \subset \mathbb{Z}^L$ as in (4.1). The input of this algorithm is the collection $\mathcal{E}xtreme(\mathbb{Z}_+(\mathcal{E}))$, where $\mathcal{E} = \{\vec{\Delta}_j = (\delta_{j1}, \dots, \delta_{jm})\}_{1\le j\le L}$ is the essential collection (see Corollary 4.3) of exponents of a monomial parametrization of the torus Y^* of an essential variety Y ; we may assume that $\mathcal{E} = \mathcal{E}xtreme(\mathbb{Z}_+(\mathcal{E}))$.

In the notations of Claim 4.6 the record of changes (derived in Section 4.1) in the collections of exponents parametrizing the tori of the essential charts of Nash blowings up starting with the variety Y is the

Multidimensional Euclidean algorithm on essential collections: *Let $\mathcal{S} = \mathcal{S}(\mathcal{E})$ be the set of all m-tuples of linearly independent vectors in a finite essential (***input***) collection $\mathcal{E} = \{\vec{\Delta}_j\}_j \subset \mathbb{Z}^m$. We extend $\mathcal{E}$ to a collection $\mathcal{E}_J$ by adjoining the set $\{\vec{\Delta}_{J'} - \vec{\Delta}_J\}_{J\ne J'\in\mathcal{S}}$ provided that $J \in \mathcal{S}'(\mathcal{E}) := \{J' \in \mathcal{S} : \mathcal{E}_{J'}$ is essential $\}$, i. e. J determines* **branching***. Then the finite set $N_J(\mathcal{E}) := \mathcal{E}xtreme(\mathbb{Z}_+(\mathcal{E}_J))$ is essential. It generates the semigroup $\mathbb{Z}_+(\mathcal{E}_J)$ and it is the* **output** *of the branching algorithm corresponding to the choices of $J \in \mathcal{S}'$.*

A branch of this algorithm terminates at a node with an associated collection $\mathcal{E} = \{\vec{a}_j\}_j \subset \mathbb{Z}^m$ whenever $\#(\mathcal{E}) = m$.

Remark 4.8. Note that the differences $\vec{\Delta}_{J'} - \vec{\Delta}_J$ with $\#(J' \setminus J) = 1$ generate over $\mathbb{Z}_+$ all other differences in the collections $\mathcal{E}_J$. That is, it suffices to include in $\mathcal{E}_J$ only these differences in $\mathcal{E}_J$. Indeed, the matrix $(a_{ji})_{j\in J'\ ,\ i\in J}$ transforming the basis J of $\mathbb{Q}^m$ into the basis J' is nondegenerate. This implies the existence of a bijection $J' \ni j \mapsto i = i(j) \in J$ with all $a_{j\ i(j)} \ne 0$ and $\vec{\Delta}_{J'} - \vec{\Delta}_J = \sum_{j\in J'}(\vec{\Delta}_j - \vec{\Delta}_{i(j)}) = \sum_{j\in J'}(\vec{\Delta}_{J\cup j\setminus i(j)} - \vec{\Delta}_J)$, as required.

The Nash desingularization of the essential affine toric subvariety Y of an affine binomial variety $\hat{V}$ leads to a Nash desingularization of $\hat{V}$ by making use of Property **A.**, Theorem 3.7 **C.** and of Remark 1.2 . The variety Y' resulting from a sequence of Nash blowings up of Y is a union of the essential affine charts $Y' \cap \mathcal{U}' \hookrightarrow \mathcal{U}' \simeq \mathbb{K}^{L'}$ due to Claim 4.6 . Every affine chart $Y' \cap \mathcal{U}'$ corresponds to a *node* of a branch of our combinatorial 'bookkeeping' algorithm. Let $\{\vec{a}_j\}_{1\le j\le L'} \subset \mathbb{Z}^m$ be the essential collection associated with the latter node. It follows that the essential affine toric variety $Y' \cap \mathcal{U}'$ corresponding to the node admits a monomial parametrization of its torus by $(\mathbb{K}^*)^m$ in coordinates y'_j , $1 \le j \le L'$, on $\mathcal{U}'$ as follows: $y'_j = (\Phi)_j(x) := x^{\vec{a}_j}$, $1 \le j \le L'$. Finally, we show below that the stabilization of the sequence of Nash blowings up of Y is equivalent to the termination of our combinatorial algorithm.

Claim 4.9. *A branch $\mathcal{B}$ of the multidimensional analogue of Euclidean division algorithm terminates at a node iff the essential affine chart $Y' \cap \mathcal{U}'$ corresponding to this node of $\mathcal{B}$ is nonsingular.*

Proof. Assume $\mathcal{E}' = \{\vec{a}_j\}_{1 \le j \le k}$ is the collection corresponding to a node of the branch $\mathcal{B}$. Let $Y' \cap \mathcal{U}' \hookrightarrow \mathcal{U}' \simeq \mathbb{K}^{L'}$ be the corresponding essential affine chart. Then the exponents of the monomial parametrization $y'_j = x^{\vec{a}_j}$, $1 \le j \le L'$, of the torus $(Y' \cap \mathcal{U}')^* = (Y' \cap \mathcal{U}') \cap (\mathbb{K}^*)^{L'}$ include the collection $\mathcal{E}'$ and, moreover, they are in $\mathbb{Z}_+(\mathcal{E}')$. That is, they can be expressed as nonnegative integral linear combinations $\vec{a}_j = \sum_{1 \le l \le k} n_{jl} \cdot \vec{a}_l$, $k+1 \le j \le L'$.

Therefore, if the branch terminates, i. e. the collection $\mathcal{E}'$ associated with its *terminal node* is of size m , then $Y' \cap \mathcal{U}'$ is nonsingular because it is the graph of the map $y'_j = (y'_1)^{n_{j1}} \cdot \ldots \cdot (y'_m)^{n_{jm}}$, $m+1 \le j \le L'$.

Conversely, as in Criterion 3.16, if $Y' \cap \mathcal{U}'$ is nonsingular at the origin of $\mathcal{U}'$, it follows that it is the graph of an étale map-germ $\mathcal{G}$ at the origin over a coordinate subspace $\mathcal{V} := \mathbb{K}^m \subset \mathbb{K}^{L'}$. Since the closure $Y' \cap \mathcal{U}'$ of the torus $(Y' \cap \mathcal{U}')^*$ contains the origin $\mathbf{0}$ of $\mathcal{U}' \simeq \mathbb{K}^{L'}$, Claim 3.3 implies that there is a monomial parametrization $y'_j = x^{\vec{\omega}_j}$, $1 \le j \le L'$, of $(Y' \cap \mathcal{U}')^*$ with $\{\vec{\omega}_j\}_{1 \le j \le L'} \subset \mathbb{Z}^m_+$. Then the map-germ $\mathcal{G}$ is monomial. This is so because of the uniqueness of the Taylor series expansion of the composition of $\mathcal{G}$ with the components of the parametrization $y'_{j_l} = x^{\vec{\omega}_{j_l}}$, $1 \le l \le m$, associated with $\mathcal{V}$. We may conclude now that the vectors $\vec{a}_j$, $1 \le j \le L'$, are generated over $\mathbb{Z}_+$ by a subset of $\{\vec{a}_j\}_{1 \le j \le L'}$ (of size m) corresponding to the coordinate subspace $\mathcal{V}$ of the previous sentence. $\square$

Remark 4.10. The proof of Claim 4.9 shows that an essential toric variety is nonsingular iff it is nonsingular at the origin.

4.3. **Effect of normalization.** The normalization $\mathcal{N}(Y)$ of an essential affine variety Y adjoins as regular functions on $\mathcal{N}(Y)$ all monomials $\mathcal{M}$ in coordinates y_j , $1 \le j \le L$, on $\mathbb{K}^L$ whenever $\mathcal{M}^d$ for some $d \in \mathbb{Z}_+$ coincides on Y with another monomial $\mathcal{M}'$ in y_j's with non negative integral exponents (see Section 2.1 in [5]). Since the torus Y^* is parametrized by monomials $y_j = x^{\vec{\Delta}_j}$, $1 \le j \le L$,

normalization translates into a combinatorial algorithm: *augment an essential input set* $\mathcal{E} = \{\vec{\Delta}_j\}_j \subset \mathbb{Z}^m$ *to a semigroup* $\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}}$ *generated by its finite essential subset* $\mathcal{N}(\mathcal{E}) := \mathcal{E}xtreme(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$ *(Remark 4.7 (i)) - the output of combinatorial normalization.*

Of course a sequence of compositions of normalized Nash blowings up followed by normalization coincides with normalization followed by the sequence of Nash blowings up composed with normalizations. For the convenience of exposition (and reflecting the latter observation) the essential collection $\mathcal{N}(\mathcal{E})$, with $\mathcal{E} = \{\vec{\Delta}_j\}_{1\le j\le L}$ from (4.1), is the initial input for the

Normalized multidimensional Euclidean division algorithm: *for each step of the algorithm* **the input** *is an essential collection* $\mathcal{E} = \mathcal{N}(\mathcal{E})$ *,* $J \in \mathcal{S}'(\mathcal{E}) := \{J' \in \mathcal{S} : \mathcal{E}_{J'}$ *is essential* $\}$ *determines the* **branching** *and* **the output** *is the essential collection* $\mathcal{N}(N_J(\mathcal{E}))$ *.*

The latter algorithm records a sequence of normalized Nash blow ups (followed by normalization) of an essential toric variety Y . By definition a branch of this tree-like algorithm terminates at a node with an essential collection $\mathcal{E}$ provided that the size of $\mathcal{E}$ is m .

The proof of Claim 4.9 applies to show that a branch $\tilde{\mathcal{B}}$ of the normalized multidimensional Euclidean division terminates iff the essential chart corresponding to the terminal node of $\tilde{\mathcal{B}}$ is nonsingular. Since normalization separates all local étale irreducible components (and due to Property **A.**, Theorem 3.7 **C.** and Remark 1.2) the lengths of the normalized Nash desingularization of the essential subvariety Y of an affine binomial variety $\hat{V}$ and that of $\hat{V}$ coincide.

5. Structure of binomial varieties, proofs.

In this section we prove several assertions from Section 3 . We again consider affine binomial varieties $\hat{V} := \overline{V^*(\hat{f})}$ in $\mathbb{A}^N$ determined by a set $\hat{f} := \{\hat{f}_j\}_{1\le j\le M}$ of binomials from (2.1). Also, V^* denotes the irreducible component of $V^*(\hat{f})$ that contains $\mathbb{I}_N$.

Construction 5.1. *Calculation of binomials* f *with* $V^*(f) = V^*$ *:*

Let $\hat{E}$ be the exponent matrix of $\hat{V}$, as in Section 3 . Set $r := \operatorname{rank} \hat{E}$, $n := N - r$. Denote by $E = \{E_{ji}\}$ a matrix of size $r \times N$ with rows being a basis over $\mathbb{Z}$ of $(\hat{E})^{tr}(\mathbb{Q}^M) \cap \mathbb{Z}^N$. Then the ideal generated in $\mathbb{Z}$ by all $r \times r$ minors of the matrix E is the unit ideal, i. e. $d(\operatorname{Ker} E) = 1$ (Remark 3.1), which is equivalent to

(ℤ) $\{\xi \in \mathbb{R}^N : E\xi \in \mathbb{Z}^r\} = \operatorname{Ker} E \cap \mathbb{R}^N + \mathbb{Z}^N \subset \mathbb{R}^N$.

Let $\alpha_{ji} := \max\{E_{ji}\ ,\ 0\}$, $\beta_{ji} := -\min\{E_{ji}\ ,\ 0\}$. Furthermore, let $V^*(f) := \{w \in \mathbb{T}^N : f_j(w) = 0\ ,\ 1 \le j \le r\}$, with binomials given by

$$f_j := w_1^{\alpha_{j1}} \cdots w_N^{\alpha_{jN}} - w_1^{\beta_{j1}} \cdots w_N^{\beta_{jN}} \ . \tag{5.1}$$

Both $V^*(f) \subset V^*(\hat{f})$ are subgroups of $\mathbb{T}^N$. Also, $V^*(\hat{f}) \subset \operatorname{Reg} \hat{V}$. Since $\operatorname{Ker} E = \operatorname{Ker} \hat{E}$, the sets of exponents parametrizing V^* and $V^*(f)$ coincide (Remark 3.1). Consequently $V^* = V^*(f)$ and $V = \overline{V^*(f)}$. (Recall that V is defined as $\overline{V}^*$ in Theorem 3.7 and is an irreducible component of $\hat{V}$ that contains $\mathbb{I}_N$ due to **Property A**.)

Let $G := \{w \in \hat{V}^* : |w| = \mathbb{I}_N\} \hookrightarrow \mathbb{T}^N$, where $|w| \in \mathbb{R}^N$ is the point with coordinates being the absolute values $|w_j|$ of coordinates w_j of $w \in \mathbb{C}^N$, and $G_0 := \{w = \exp(2\pi\sqrt{-1} \cdot h) : h \in \mathbb{R}^N\ ,\ Eh = \mathbf{0}\} \hookrightarrow G$, where $\exp((h_1, \ldots, h_N)) := (e^{h_1}, \ldots, e^{h_N})$. Note that G and G_0 are subgoups of $\hat{V}^* := V^*(\hat{f})$. Recall that Γ is defined as $\hat{V}^*/V^*$.

Remark 5.2. Γ *coincides with* G/G_0 *when* $\mathbb{K} = \mathbb{C}$.

To see this, first note that $G_0 = G \cap V^*(f)$ due to the property (ℤ) of the matrix E . Moreover, $\Gamma \simeq G/G_0$, since $|w| \in V^*$ and $g := w \cdot |w|^{-1} \in G$ for any $w \in \hat{V}^*$. The map $\xi \mapsto \exp(2\pi\sqrt{-1} \cdot \xi)$ provides a bijection onto Γ of an additive group $\Gamma_* := \{\xi \in \mathbb{R}^N : \hat{E}(\xi) \in \mathbb{Z}^M\}/(\mathbb{Z}^N + \operatorname{Ker} E)$. Also, Γ_* is finite, since for any choice of a basis $\{\vec{h_j}\}_{1\le j\le r}$ of $\hat{E}(\mathbb{R}^N) \cap \mathbb{Z}^M$ over $\mathbb{Z}$ there is a choice of $\{\vec{\xi_j}\}_j \subset \mathbb{Q}^N$ with each $\vec{h_j} = \hat{E}(\vec{\xi_j})$.

We will make use of the following

Claim 5.3. *Assume* $\mathcal{P} \in \hat{V} \setminus V^*(\hat{f})$ *and that upon splitting all variables* w_j , $1 \le j \le N$, *into* $w = (u, v)$ *the u-coordinates of* $\mathcal{P}$ *vanish. Let* $b := v(\mathcal{P}) \in \mathbb{T}^{N''}$. *Then there are* $a \in \mathbb{T}^{N'}$, *where* $N' := N - N''$, *and* $\vec{\xi} \in (\mathbb{Z}_+)^{N'} \times \{\mathbf{0}\}$ *such that* $g \cdot X^*_{\mathcal{E}^+} \hookrightarrow V^*(\hat{f})$, *where* $g := (a, b) \in \mathbb{T}^N$ *and* $\mathcal{E}^+ := \{(\vec{\xi})_j\}_{1\le j\le N} \subset \mathbb{Z}$.

In particular, the point $\{\mathcal{P}\} = (g \cdot X_{\mathcal{E}^+}) \setminus \big(g \cdot X^*_{\mathcal{E}^+}\big)$.

Proof. Let $X \hookrightarrow \hat{V}$ be an irreducible curve with $\mathcal{P} \in X$. Then the normalization $\mathcal{N}(X)$ of X is a nonsingular curve and the morphism $\mathcal{N}_X : \mathcal{N}(X) \to X$ is finite and surjective. Let $\mathcal{Q} \in (\mathcal{N}_X)^{-1}(\mathcal{P})$. Since $\mathcal{N}(X)$ is nonsingular (at $\mathcal{Q}$) it follows that the completion $\hat{\mathcal{O}}$ (in the Krull topology) of the local ring $\mathcal{O} \hookrightarrow \hat{\mathcal{O}}$ of $\mathcal{N}(X)$ at $\mathcal{Q}$ is the ring $\mathbb{F}[[t]]$ of the formal power series expansions in one variable, say t , with coefficients in the residue field $\mathbb{F}$ of $\mathcal{O}$ (hence $[\mathbb{F} : \mathbb{K}] < \infty$).

Denote by $\gamma_j(t) \in \mathbb{F}[[t]]$ the pull back $(\mathcal{N}_X)^*(w_j|_X) \in \mathcal{O} \hookrightarrow \hat{\mathcal{O}}$ of the restriction $w_j|_X$ of the w_j-coordinate to X . It follows that $\gamma(t)^{\hat{E}} = \mathbb{I}_M$ in $\mathbb{F}[[t]]^M$ and that $w(\mathcal{P}) = (\mathbf{0}, b) = \gamma(0)$. For each j , $1 \leq j \leq N'$, let the initial form of $\gamma_j(t)$ be $in(\gamma_j) = a_j \cdot t^{\xi_j}$, $a_j \in \mathbb{F}^*$ and $\xi_j \in \mathbb{Z}_+$. Then $a := (a_1, \ldots, a_{N'}) \in (\mathbb{F}^*)^{N'}$ and $\vec{\xi} := (\xi_1, \ldots, \xi_{N'}, 0, \ldots, 0) \in \mathbb{Z}^N$ satisfy $X^*_{\mathcal{E}+} \hookrightarrow V^*(\hat{f})$ and $(a,b)^{\hat{E}} = \mathbb{I}_M$, i. e. are as required. □

Corollary 5.4. *(a) the equality* $\hat{V} \cap (\mathbb{A}^L \times \mathbb{I}_{N-L}) = \overline{V^*(\hat{f}) \cap (\mathbb{A}^L \times \mathbb{I}_{N-L})}$ *of Theorem 3.7 B., (b) Lemma 4.2 and (c) Lemma 3.4 hold :*

Proof. To prove (a) we apply Claim 5.3 to $\mathcal{P} \in \hat{V} \cap (\mathbb{A}^L \times \mathbb{I}_{N-L})$ and obtain $g \in V^*(\hat{f})$ and $\vec{\xi} \in \mathbb{Z}^N$ such that
(i) each coordinate of $\vec{\xi}$ is either in $\mathbb{Z}_+$ or vanishes depending on whether the respective coordinate of $\mathcal{P}$ vanishes or not and, moreover,
(ii) g and $\vec{\xi}$ satisfy the conclusions of Claim 5.3 .
Therefore it follows that $\mathcal{P} \in \overline{V^*(\hat{f}) \cap (\mathbb{A}^L \times \mathbb{I}_{N-L})}$, as required.

Claims (b) and (c) follow by applying the proof of (a) with an appropriate choice of the point $\mathcal{P}$. □

Remark 5.5. The equality $\operatorname{Ker} \hat{E} = \operatorname{Ker} E$ and Lemma 3.4 imply that the splitting of variables w into y- and z-variables for the variety $\hat{V} \subset \mathbb{A}^N$ and for the irreducible component $V \ni \mathbb{I}_N$ of $\hat{V}$ coincide.

Define the matrices $(\hat{\Omega} \quad \hat{\Xi}) := \hat{E}$ and $(\Omega \quad \Xi) := E$ with the columns of $\hat{\Omega}$ and Ω corresponding to y- and the columns of $\hat{\Xi}$ and Ξ to z-variables. The following Claim implies that $\pi(V)$ is a closed binomial variety and **completes the proof of Theorem 3.7 B.** (using Property **A.** of Remark 3.1)

Claim 5.6. $\pi(V^*(\hat{f}))$ *is closed in* $\mathbb{A}^{N-L}$ *, equals* $\pi(\hat{V})$ *and is binomial.*

Proof. Let the matrix T of size $M' \times M$, $M' := M - \operatorname{rank}(\hat{\Omega})$, have as rows a basis over $\mathbb{Z}$ of $\operatorname{Ker}(\hat{\Omega})^{tr} \cap \mathbb{Z}^M$. Then $\operatorname{Ker} H = \pi(\operatorname{Ker} \hat{E})$ for $H := T \cdot \hat{\Xi}$. Moreover

Lemma 5.7. $\pi(V^*(\hat{f})) = \{z \in \mathbb{T}^{N-L} : z^H = \mathbb{I}_{M'}\}$.

We return to proving Claim 5.6 following the proof of this Lemma.

Proof. The matrix T admits (cf. Construction 5.1) a right inverse matrix $\mathcal{L}$ with entries in $\mathbb{Z}$, i. e. $T \cdot \mathcal{L} = \operatorname{Id}_{M'}$. Therefore $T \cdot (\operatorname{Id}_M - \mathcal{L} \cdot T) = 0$, $(\operatorname{Ker} T) \cap (\operatorname{Im} \mathcal{L} \cdot T) = \{0\}$, $\operatorname{Im} \mathcal{L} = \operatorname{Im} \mathcal{L} \cdot T$. Hence $\mathbb{Q}^M = \operatorname{Im}(\operatorname{Id}_M - \mathcal{L} \cdot T) \oplus \operatorname{Im} \mathcal{L} \cdot T$ which implies that

$\mathrm{Im}\ (\mathrm{Id}_M - \mathcal{L}\cdot T) = \mathrm{Ker}\ T = \mathrm{Im}\ \hat{\Omega}$. Of course there are square matrices Λ and λ with entries in $\mathbb{Z}$, $\det(\Lambda) = 1 = \det(\lambda)$ and such that the matrix $\tau := \Lambda\cdot\hat{\Omega}\cdot\lambda$ has a diagonal upper left corner of size $M'\times M'$ and zero entries otherwise. Then $\mathrm{Im}\ \tau = \mathrm{Im}\ \Lambda\cdot\hat{\Omega} = \mathrm{Im}\ \theta$, where $\theta := \Lambda\cdot(\mathrm{Id}_M - \mathcal{L}\cdot T)$. This implies for any $v\in\mathbb{T}^M$ the existence of $y_*\in\mathbb{T}^L$ with $y_*^\tau = v^\theta$, which for $v := z^{\hat{\Xi}}$ with $z^H = \mathbb{I}_{M'}$ and $y := y_*^{-\lambda}$ implies $y^{-\hat{\Omega}} = z^{\hat{\Xi}}$. Consequently, we proved that if $z^H = \mathbb{I}_{M'}$ then $z\in\pi(V^*(\hat{f}))$, while the converse is obvious. $\square$

In other words $\pi(V^*(\hat{f}))$ is the vanishing set of binomials and H is a matrix associated with the variety $\hat{W} = \overline{\pi(V^*(\hat{f}))}$ for which all variables are the 'z-variables' (follows using $\vec{\xi}^+$ of Corollary 3.5). Therefore $\pi(V^*(\hat{f}))$ is a closed binomial variety and coincides with $\pi(\hat{V})$. This completes the proof of Claim 5.6 and, also, the proof of Theorem 3.7 B. $\square$

Corollary 5.8. *It follows that* $\pi(V) = \pi(V^*(f)) = \pi(V^*) \hookrightarrow \mathbb{T}^{N-L}$ *is a torus (Remark 3.1) closed in* $\mathbb{A}^{N-L}$ *and, being nonsingular, is a connected component of* $\pi(\hat{V})$.

Next we prove Theorem 3.7 C.

Proof. We start by showing the claim of existence in part **C.**. Namely, following the arguments of Criterion 3.18 let $V^* = X^*_{\tilde{\mathcal{E}}}$ and split the exponents of the set $\tilde{\mathcal{E}}\subset\mathbb{Z}^n$ into subsets $\mathcal{E}'$ and $\mathcal{E}''$ according to the splitting of all coordinates w on $\mathbb{A}^N$ into y and z-coordinates. Let the matrix $\widetilde{\mathcal{M}}$ complete the matrix $\mathcal{M}$ of Criterion 3.18 to a square size matrix with $\det(\widetilde{\mathcal{M}}) = 1$ and entries in $\mathbb{Z}$ by attaching a matrix $\underline{\mathcal{M}}$ of size $n\times(n-m)$ as the last $n-m$ columns. Then, respectively, the columns of the matrices $\mathcal{E}_Y := \tilde{\mathcal{E}}\cdot\mathcal{M}$ and $\mathcal{E}_V := \tilde{\mathcal{E}}\cdot\widetilde{\mathcal{M}}$ form $\mathbb{Z}$-bases of $\mathrm{Ker}\,E\cap(\mathbb{Z}^L\times\{\mathbf{0}\})$ and $\mathrm{Ker}\,E\cap\mathbb{Z}^N$ implying that $Y^* = X^*_{\mathcal{E}_Y}\hookrightarrow\mathbb{T}^N$ and $X^*_{\tilde{\mathcal{E}}} = X^*_{\mathcal{E}_V}$. Moreover, letting $\mathcal{E}_Z := \tilde{\mathcal{E}}\cdot\underline{\mathcal{M}}$ it follows that $\mathcal{E}_Z$ is a $\mathbb{Z}$-basis and that as the set of exponents $\mathcal{E}_V = (\mathcal{E}_Y||\mathcal{E}_Z)$, as required.

We next prove that the torus $Z^* := X^*_{\mathcal{E}_Z}$ is closed in $\mathbb{A}^N$. Applying the projection π to the columns of matrices $\mathcal{E}_V$ and $\mathcal{E}_Z$ it follows that $\mathrm{Span}_{\mathbb{Z}}(\pi(\mathcal{E}_V^{tr})) = \mathrm{Span}_{\mathbb{Z}}(\pi(\mathcal{E}_Z^{tr}))$ implying $\dim_{\mathbb{Q}}(\mathrm{Span}_{\mathbb{Q}}(\pi(\mathcal{E}_Z^{tr}))) = \dim_{\mathbb{Q}}(\mathrm{Span}_{\mathbb{Q}}(\mathcal{E}_V^{tr})) - \dim_{\mathbb{Q}}(\mathrm{Span}_{\mathbb{Q}}(\mathcal{E}_Y^{tr})) = \dim_{\mathbb{Q}}(\mathrm{Span}_{\mathbb{Q}}(\mathcal{E}_Z^{tr}))$. (Note though that $\pi(\mathcal{E}_Z^{tr})$ is not necessarily a $\mathbb{Z}$-basis of $\mathrm{Span}_{\mathbb{Q}}(\mathcal{E}_Z^{tr})$ as Example 3.17 demonstrates.) The inclusion $\pi(\overline{Z})\subset\pi(V)\subset\mathbb{T}^{N-L}$ (Corollary 5.8) implies that all 'z-variables' for $\hat{V}$ are the 'z-variables' for Z and then the criterion of the iterative construction preceding (3.2) implies that all w_j variables, $1 \le j \le N$, are the 'z-variables' for $\overline{Z}$, i. e. $\mathbb{T}^N \supset \overline{Z} = Z$, as required.

The properties of the morphism $\pi|_Z : Z \to \pi(V)$ follow (Remark 3.12) from the analogous properties of $\phi_{(\pi(\mathcal{E}_Z^{tr}))^{tr}} : \mathbb{T}^{n-m} \to \pi(V)$. The surjectivity of the latter (when $\mathbb{K}$ is perfect) is a consequence of Corollary 5.8. Applying Remark 3.1 by replacing the matrix $\hat{E}$ by $(\pi(\mathcal{E}_Z^{tr}))^{tr}$ implies that the morphism $\pi|_Z$ is finite of degree $\underline{d} = d((\pi(\mathcal{E}_Z^{tr}))^{tr})$ with the size of all fibres equal to $[\underline{d}] = \#(\hat{Y}^*/Y^*)$ (cf. Claim 3.10) and that it is an étale isomorphism iff $\underline{d} \ne 0$ in $\mathbb{K}$.

Next we establish the properties of $\mu : Z \times \hat{Y} \to V$ and of $\mu|_{Z\times Y}$ listed in Theorem 3.7 **C.**. The surjectivity and the quasifiniteness of both with all fibres of μ being of the same size $[\underline{d}]$ as those of morphism $\pi|_Z$ are straightforward consequences of the surjectivity of $\pi_{|Z} : Z \to \pi(V)$ as a group homomorphism and of the definition of $\hat{Y} := V \cap (\mathbb{A}^L \times \mathbb{I}_{N-L})$.

Besides the morphism μ being an étale isomorphism whenever $[\underline{d}] = \underline{d}$ (which we prove at the very end) it remains to show that both μ and $\mu|_{Z\times Y}$ are finite morphisms of the same degree $\underline{d}$ as $\pi|_Z$. The proof is similar to the calculation in the special case of Example 3.9 . Thus, we carry it out only in the case of the morphism μ . Indeed, since $Z \hookrightarrow \mathbb{A}^N$ is isomorphic to a closed torus $\underline{\mathbb{T}}^{n-m} \hookrightarrow \mathbb{A}^{2\cdot(n-m)}$, the ring of regular functions on Z is $\mathbb{K}[Z] \simeq \mathbb{K}[s_1 , \dots , s_{n-m} , s_1^{-1} , \dots , s_{n-m}^{-1}]$, while $\mathbb{K}[Z \times \hat{Y}] \simeq (\mathbb{K}[Z])[y]/\mathcal{I}$, where $\mathcal{I}$ is the ideal in $(\mathbb{K}[Z])[y]$ generated by equations defining $\hat{Y}$ in $\mathbb{A}^L$. We split the exponents $e \in \mathcal{E}_Z$ of the parametrization $\mathbb{T}^{n-m} \ni s \to \phi_{\mathcal{E}_Z}(s) \in Z$ of Z according to the y and z-coordinates. Consequently we arrive at the formulas $s^{e'_j} = \phi^*_{\mathcal{E}_Z}(y_j)$, $1 \le j \le L$, and $s^{e''_i} = \phi^*_{\mathcal{E}_Z}(z_i)$, $1 \le i \le N - L$, where $s = (s_1 , \dots , s_{n-m})$. It follows that

$$\mathbb{K}[Z] \simeq \mathbb{K}[s^{e'_1} , \dots , s^{e'_L} , s^{e''_1} , \dots , s^{e''_{N-L}}] ,$$

$$\mathbb{K}[\pi(Z)] \simeq \mathbb{K}[s^{e''_1} , \dots , s^{e''_{N-L}}] \text{ and}$$

$$\mu^*(\mathbb{K}[V]) \simeq \mathbb{K}[\pi(Z)][y_1 \cdot s^{e'_1} , \dots , y_L \cdot s^{e'_L}]/\mathcal{I} \hookrightarrow (\mathbb{K}[Z])[y]/\mathcal{I} .$$

Recall that $\pi(Z) = \pi(V)$ and $(\pi|_V)^* : \mathbb{K}[\pi(Z)] \hookrightarrow \mathbb{K}[V]$. We conclude that $\mathbb{K}[Z \times \hat{Y}]$ is integral over $\mu^*(\mathbb{K}[V])$ since $\mathbb{K}[Z]$ is integral over $\mathbb{K}[\pi(Z)]$, because of the finiteness of $\pi|_Z$, and since each element $s^{-e'_j} \in \mathbb{K}[Z]$, $1 \le j \le L$.

Next, the degree of $\pi|_Z$ is $\underline{d}$. This means that $\dim_{\mathbb{F}} \mathbb{K}[Z] \cdot \tilde{S}^{-1} = \underline{d}$, where $\tilde{S} := \mathbb{K}(\pi(Z))\backslash\{\mathbf{0}\}$ and $\mathbb{F} := \mathbb{K}(\pi(Z))$. Note that $\mathbb{K}[Z] \cdot \tilde{S}^{-1} \simeq \mathbb{F}[s^{e'_1} , \dots , s^{e'_L}]$ and $\mu^*(\mathbb{K}[V]) \cdot \tilde{S}^{-1} \simeq \mathbb{F}[y_1 \cdot s^{e'_1} , \dots , y_L \cdot s^{e'_L}]/\mathcal{I}$.

Also $(y_j \cdot s^{e'_j}) \in \mu^*(\mathbb{K}[V])$, $y_j \in \mathbb{K}[Z \times \hat{Y}]$, $1 \le j \le L$ and the element $y_j \in (y_j \cdot s^{e'_j}) \cdot \mathbb{K}[Z] \subset (y_j \cdot s^{e'_j}) \cdot \mathbb{K}[Z] \cdot \tilde{S}^{-1} \subset \mathbb{K}[Z \times \hat{Y}] \cdot \tilde{S}^{-1}$. Then $\mathbb{K}[Z] \cdot \tilde{S}^{-1} \otimes_{\mathbb{K}(\pi(Z))} \mathbb{K}(V) \simeq \mathbb{K}[Z \times \hat{Y}] \cdot S^{-1}$. This implies that $\dim_{\mu^*(\mathbb{K}(V))} \mathbb{K}[Z \times \hat{Y}] \cdot S^{-1} = \dim_{\mathbb{K}(\pi(Z))} \mathbb{K}[Z] \cdot \tilde{S}^{-1} = \underline{d}$, cf. Remark 3.8 .

Finally, the property of the morphism μ to be an étale isomorphism is a consequence of the analogous property for $\pi|_Z : Z \to \pi(V)$ (proved above to be equivalent to $[\underline{d}] = \underline{d}$). In the special case of $\mathbb{K} = \mathbb{C}$ the étale inverse $(\pi_{Z,a})^{-1}$ is an analytic map from $\pi(Z)$ to Z (from a neighbourhood in the classical topology of $\pi(a)$ to that of a). Then the étale inverse $(\hat{\mu}_{(a,b)})^{-1}$ of μ as an analytic map germ (at the point (a, b)) is

$$V_{\mu(a,b)} \ni v \mapsto ((\pi_{Z,a})^{-1}(\pi(v)) \times [(\pi_{Z,a})^{-1}(\pi(v))]^{-1} \cdot v) \in (\mathbb{Z} \times \hat{Y})_{(a,b)} \ ,$$

where $[g]^{-1} : v \to [g]^{-1} \cdot v$ is the action of $g := (\pi_{Z,a})^{-1}(\pi(v)) \in Z$ on V and $V_{\mu(a,b)}$, $(\mathbb{Z} \times \hat{Y})_{(a,b)}$ are the germs as analytic sets at the respective points $\mu(a, b) \in V$, $a \in Z$ and $b \in \hat{Y}$. In the general case we exploit the calculations of the previous two paragraphs.

For any prime ideal $\mathfrak{p} \in \mathrm{Spec}\,(\mathbb{K}[Z \times \hat{Y}])$ and $\mathfrak{q} := \mathfrak{p} \cap \mu^*(\mathbb{K}[V]) \in \mathrm{Spec}\,(\mu^*(\mathbb{K}[V]))$ consider the respective localizations at $\mathfrak{p}$ and $\mathfrak{q}$ followed by the completions in the Krull topologies. We must show that the latter local rings are isomorphic. Note that, since $\pi|_Z$ is an étale isomorphism, the analogous procedure starting with prime ideals $\tilde{\mathfrak{p}} := \mathfrak{p} \cap \mathbb{K}[Z]$ and $\tilde{\mathfrak{q}} := \mathfrak{q} \cap \mathbb{K}[\pi(Z)]$ leads to the same ring, say $\hat{\mathcal{O}}$. It suffices to show that adjoining $(\mathbb{K}[Z] \setminus \tilde{\mathfrak{p}})^{-1}$ to $\mathbb{K}[Z \times \hat{Y}]$ and $(\mathbb{K}[\pi(Z)] \setminus \tilde{\mathfrak{q}})^{-1}$ to $\mu^*(\mathbb{K}[V])$, followed by the completions in the Krull topologies induced by the powers of the ideals generated by $\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{q}}$ in the respective rings, leads to isomorphic rings (even prior to the localizations at $\mathfrak{p}$ and $\mathfrak{q}$ followed by the respective completions). But the partial localizations followed by the respective completions of the previous sentence transform the rings $\mu^*(\mathbb{K}[V]) \hookrightarrow \mathbb{K}[Z \times \hat{Y}]$ into the pair of rings $\hat{\mathcal{O}}[y_1 \cdot s^{e'_1} , \ \dots \ , \ y_L \cdot s^{e'_L}]/\mathcal{I} \hookrightarrow \hat{\mathcal{O}}[y]/\mathcal{I}$, which are of course isomorphic since each element $s^{-e'_j} \in \mathbb{K}[Z] \hookrightarrow \hat{\mathcal{O}}$, $1 \le j \le L$. This completes the proof of Theorem 3.7 **C.** . ☐

We now prove (in the respective order) Claims 3.14 , 3.3 and 3.6 .

Proof. of **Claim 3.14 .** The binomial variety $\pi(\hat{V}) = \pi(\hat{V}^*) \subset \mathbb{T}^{N-L}$ and therefore is nonsingular. Consequently, its irreducible components are pairwise disjoint and smooth. To prove the first statement of Claim 3.14 it suffices (due to property **A.** of Remark 3.1 and Remark 5.5) to consider a nonsingular subvariety W of the component

$\pi(V)$ and a subvariety $\tilde{V}$ of V, obtained by restricting the original z-variables to a nonsingular subvariety W. Similarly, we define $\tilde{Z} \hookrightarrow Z$ by restricting the z-variables to W. Then $\tilde{Z}$ is nonsingular since μ is an étale isomorphism. Moreover, the morphism $\pi|_{\tilde{Z}} : \tilde{Z} \to W$ and the coordinatewise multiplication $\mu : \tilde{Z} \times \hat{Y}_V \to \tilde{V}$ are surjective étale isomorphisms and $\pi|_{\tilde{Z}}$ is finite (due to Theorem 3.7 **C.**), which proves the first half of Claim 3.14 .

Next we show that a quasi-binomial variety, say $\tilde{X}$, arises from a special case of the preceding construction. Without loss of generality we may assume that quasi-binomial equations defining $\tilde{X}$ are linear combinations of two monomials with the first coefficients being equal 1 . We start by replacing the 'second' coefficients of quasi-binomial equations (one per each) by minus a variable, say $-c_j$, introducing simultaneously another variable $\tilde{c}_j$ and a binomial equation $c_j \cdot \tilde{c}_j = 1$. We thus construct a binomial variety, say X , with all of the new variables among the 'z-variables' for X . Let $\pi(X)$ be the projection of the binomial variety X to the affine subspace of its z-variables. It suffices to show that the intersection W of the projection $\pi(X)$ with the specialization of variables c_j (according to their values in the quasi-binomial equations defining the variety $\tilde{X}$) is nonsingular. This will reduce the claim to a special case of the construction of the previous paragraph. Due to Theorem 3.7 **B.** $\pi(X) = \pi(X^*)$ and is a closed binomial variety (implying W is a quasi-binomial variety). Therefore $\pi(X) = \pi(X^*) \subset \mathbb{T}^{N-L}$. Consequently $W = W^* := W \cap \mathbb{T}^{N-L} \subset \operatorname{Reg} W$. The latter is due to the algebraic group structure of $\mathbb{T}^{N-L}$ (similarly to the 'Gauss elimination' argument of Remark 3.1 and the analogous claim $\hat{V}^* \subset \operatorname{Reg} \hat{V}$ of Remark 3.1). $\square$

Proof. of **Claim 3.3 .** The 'only if' implication is obvious. Assume that $\mathbf{0} \in X$. It follows that there are no z-coordinates. Then Corollary 3.5 implies the existence of $\vec{\xi}^{+} \in \operatorname{Ker} E \cap (\mathbb{Z}^N_+)$. Say $m := \dim X = N - \operatorname{rank} E$. To construct a monomial parametrization of the torus of X with positive integral exponents $\mathcal{E} = \{\vec{\Delta}_j\}_{1\le j\le N} \subset \mathbb{Z}^m$ it suffices to find a $\mathbb{Z}$-basis $\{\vec{\delta}_i\}_{1\le i\le m}$ of $\operatorname{Ker} E \cap \mathbb{Z}^N$ with positive coordinates, as in Remark 3.1. The construction of the latter is provided by the lemma below. $\square$

Lemma 5.9. *For any matrix E of size $M \times N$ with entries in $\mathbb{Q}$ and $m := N - \operatorname{rank} E$ the following properties are equivalent:*

(i) there is $\vec{v} \in \operatorname{Ker} E \cap (\mathbb{Z}^N_+)$;

(ii) there is a $\mathbb{Q}$-basis $\{\vec{\delta}_i\}_{1\le i\le m} \subset \mathbb{Z}^N_+$ of $\operatorname{Ker} E \cap \mathbb{Q}^N$;

(iii) there is a $\mathbb{Z}$*-basis* $\{\vec{\delta}_i\}_{1\le i\le m}$ *of* $\operatorname{Ker} E \cap \mathbb{Z}^N$ *with all positive coordinates (equivalently, there exists a* $\mathbb{Q}$*-basis* $\{\vec{\delta}_i\}_i \subset \mathbb{Z}_+^N$ *of* $\operatorname{Ker} E \cap \mathbb{Q}^N$ *such that* $I = \mathbb{Z}$ *, where* $I = I(\vec{\delta}_1 \wedge \cdots \wedge \vec{\delta}_m)$ *is the ideal generated in* $\mathbb{Z}$ *by all coordinates of* $\vec{\delta}_1 \wedge \cdots \wedge \vec{\delta}_m$ *in the standard basis* $\{(j_1) \wedge \cdots \wedge (j_m)\}_{1\le j_1<\cdots<j_m\le N}$ *).*

Remark 5.10. (i) of Lemma 5.9 is equivalent to $(\mathrm{Im}\ E^{tr}) \cap \mathbb{Q}_+^N = \{\mathbf{0}\}$.

Proof. (Also, cf. Gordan theorem from [2] , communicated to us by Dima Pasechnik.) Our proof is based on simple linear algebra. To prove that (i) implies (ii) it suffices to choose any basis $\{\vec{v}_i\}_i \subset \mathbb{Z}^N$ of $\operatorname{Ker} E \cap \mathbb{Q}^N$ with $\vec{v}_1 := \vec{v}$; then let $\vec{\delta}_1 := \vec{v}$ and $\vec{\delta}_i := t \cdot \vec{v} + \vec{v}_i$, $i > 1$. Then (ii) follows for a sufficiently large $t \in \mathbb{Z}_+$.

The proof of the remaining implication "(iii) follows from (ii)" is slightly harder. Let $\{\vec{\delta}_i\}_{1\le i\le m} \subset \mathbb{Z}_+^N$ be a $\mathbb{Q}$-basis of $\operatorname{Ker} E \cap \mathbb{Q}^N$. Also, let $s \in \mathbb{Z}_+$ be the generator of the ideal I , i. e. $(s \cdot \mathbb{Z}) = I$. If $s = 1$ we are done. Otherwise, we modify the basis $\{\vec{\delta}_i\}_{1\le i\le m}$ by reducing the size of s . Pick a prime factor p of s . Denote the field $\mathbb{Z}/(p \cdot \mathbb{Z})$ by $\mathbb{F}_p$. Now our collection of vectors $\{\vec{\delta}_i\}_{1\le i\le m}$, considered modulo the ideal $(p \cdot \mathbb{Z})$ in $(\mathbb{F}_p)^N$ is linearly dependent, i. e. $\sum_{1\le i\le m} \lambda_i \cdot \vec{\delta}_i = \mathbf{0}$ in $(\mathbb{F}_p)^N$ for a collection of coefficients $\{\lambda_i\}_{1\le i\le m} \subset (\mathbb{F}_p)^m \setminus \{\mathbf{0}\}$. Choose $\tilde{\lambda}_i \in \mathbb{Z}$ so that $\lambda_i = \tilde{\lambda}_i \ (mod\, p)$ and $0 \le \tilde{\lambda}_i < p$, $1 \le i \le m$. Then $\tilde{\lambda}_{i_0} \ne 0$ for some i_0 , $1 \le i_0 \le m$, and $\vec{\delta}_0 := (1/p) \cdot \sum_{1\le i\le m} \tilde{\lambda}_i \cdot \vec{\delta}_i \ \in\ \mathbb{Z}_+^N$. It follows that all coordinates of the modified $\mathbb{Q}$-basis of $\operatorname{Ker} E \cap \mathbb{Q}^N$ obtained by replacing the vector $\vec{\delta}_{i_0}$ of $\{\vec{\delta}_i\}_{1\le i\le m}$ by the vector $\vec{\delta}_0$ are positive integers and, also, that $I(\vec{\delta}_1 \wedge \cdots \wedge \vec{\delta}_{i_0-1} \wedge \vec{\delta}_0 \wedge \vec{\delta}_{i_0+1} \wedge \cdots \wedge \vec{\delta}_m) = \tilde{\lambda}_{i_0} \cdot (s/p) \cdot \mathbb{Z}$. Due to the choice of $\{\tilde{\lambda}_i\}_{1\le i\le m}$ in $\mathbb{Z}^m$ the size of $\tilde{\lambda}_{i_0} \cdot (s/p)$ is smaller than the size of s , which suffices. □

Remark 5.11. The complexity of the construction of a basis satisfying property (iii) of the algorithm arising in the proof of '(ii) implies (iii)' is polynomial in the maxima of the absolute values of the coordinates of $\vec{\delta}_1 \wedge \cdots \wedge \vec{\delta}_m$ in the standard basis for the initial $\mathbb{Q}$-basis $\{\vec{\delta}_i\}_{1\le i\le m}$. That is, it is exponential in the binary size of the input, unlike the construction of a basis $\{\vec{\delta}_j\}_{1\le j\le m}$ of (ii) which is a typical problem of linear programming and carries a polynomial cost in the binary size of the input. But we do not need the output with property (iii) for the algorithms of this article.

Proof. of **Claim 3.6 .** The 'if' implication is obvious. We first prove the 'only if' implication in the case that there are no y-coordinates,

i. e. we must show that in this case $(\hat{f})$ is a radical ideal when $\hat{V} = \hat{V} \cap \mathbb{T}^N = V^*(\hat{f})$. We have that $V^*(\hat{f}) \subset \operatorname{Reg} \hat{V}$ due to Remark 3.1. Therefore, assuming that the polynomial $P \in \mathbb{K}[w]$ vanishes on $\hat{V}$ it follows that the polynomial P belongs to the ideals $I_{\mathfrak{m}}$ generated by the ideal $(\hat{f})$ in the local rings $\mathcal{O}_{\mathfrak{m}}$ of the localizations of the polynomial ring $\mathbb{K}[w]$ at its maximal ideals $\mathfrak{m}$. The result follows by the standard 'partition of unity' argument of commutative algebra. Indeed, for every $\mathfrak{m}$ there is a polynomial $Q_{\mathfrak{m}} \in \mathbb{K}[w]$ with $Q_{\mathfrak{m}} \notin \mathfrak{m}$ such that $Q_{\mathfrak{m}} \cdot P \in (\hat{f})$. Since the ideal generated by all $Q_{\mathfrak{m}}$ in $\mathbb{K}[w]$ is not in any maximal ideal $\mathfrak{m}$ of $\mathbb{K}[w]$ it follows that it coincides with $\mathbb{K}[w]$. Therefore there is a finite linear combination $\sum_k h_k \cdot Q_{\mathfrak{m}_k} = 1$, for an appropriate choice of polynomials $h_k \in \mathbb{K}[w]$, commonly refered to as a partition of unity. Expressing the inclusions $Q_{\mathfrak{m}_k} \cdot P \in (\hat{f})$ as equalities $Q_{\mathfrak{m}_k} \cdot P = \sum_j G_{\mathfrak{m}_k,j} \cdot \hat{f}_j$ it follows that $P = \sum_k h_k \cdot Q_{\mathfrak{m}_k} \cdot P = \sum_j (\sum_k h_k \cdot G_{\mathfrak{m}_k,j}) \cdot \hat{f}_j$.

Finally, we reduce to the special case considered in the preceding paragraph. Let $v := (v_1, \ldots, v_L)$ and $g_i := y_i \cdot v_i - 1$ denote auxiliary variables and polynomials. We have that $\hat{V} \cap \{(y,z) \in \mathbb{A}^N : y_1 \cdot \ldots \cdot y_L \neq 0\} = V^*(\hat{f})$, by definition of the y-variables. Therefore the assumption that $P \in \mathbb{K}[w]$ vanishes on $\hat{V}$ (and equivalently on $V^*(\hat{f})$) implies that the polynomial $P \in \mathbb{K}[w] \subset \mathbb{K}[w,v]$ vanishes on $V^*(\hat{f}, g) \subset \mathbb{A}^{N+L}$. Obviously all (w,v) variables for the collection $\mathcal{F}$ of binomials $\{\hat{f}_j\}_j \cup \{g_i\}_i$ are, as we refer to them, the 'z-variables'. Therefore the case we considered first implies that the polynomial $P(w)$ is in the ideal generated by polynomials from $\mathcal{F}$ in the ring $\mathbb{K}[w,v]$. Substitution of $v_j = 1/y_j$, $1 \leq j \leq L$, in the equality expressing the inclusion of the previous sentence, followed by 'clearing' the denominators, i. e. (in our setting) by multiplying by a sufficiently high power of $y_1 \cdot \ldots \cdot y_L$, completes the proof. □

Part 2. Essential dimension $m = 2$.

The main results of Part 2. are Theorems 6.6 and 6.8 proved in Sections 6.2 and 7.1. In Theorem 6.6 we establish an explicit apriori bound for the termination of the normalized Euclidean 2-dimensional division algorithm and then improve it in Theorem 6.8 . The latter allows us to establish in Sections 7.2 and 7.3 a polynomial complexity of this algorithm.

6. Termination of normalized Euclidean division: $\dim = \mathbf{2}$.

Conjecture 6.1. *The tree* $\overline{\mathcal{T}}$ *associated with the multidimensional Euclidean algorithm is finite for any initial data.*

By König's lemma the latter is equivalent to the property that the algorithm terminates along every branch of the tree $\overline{\mathcal{T}}$. In dimension > 2 the 'normalized' version of 6.1 is the following

Conjecture 6.2. *The tree* $\mathcal{T}$ *associated with the normalized multidimensional Euclidean algorithm is finite for any initial data.*

We start with an example from the Introduction of a normal toric surface in $\mathbb{C}^3$ whose Nash blow up is not normal. It also illustrates the kind of calculations we deal with in Sections 6 and 7 .

Example 6.3. With $\phi : (x_1, x_2) \mapsto (x_1 \cdot x_2 \ , \ x_1 \cdot x_2^2 \ , \ x_1^3 \cdot x_2^2)$ let $S := \overline{\phi(\mathbb{T}^2)} \subset \mathbb{C}^3$. Exponents $\mathcal{E} := \{(1,1) \ , \ (1,2) \ , \ (3,2)\} \subset \mathbb{Z}^2$ generate over $\mathbb{Z}_+$ the integral points $\mathbb{Z}^2 \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E})$ of the cone $\mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) \subset \mathbb{Q}^2$ spanned by $\mathcal{E}$. Indeed, $\det((3,2) \ , \ (1,1)) = 1 = \det((1,1) \ , \ (1,2))$ implies that the cones $\mathrm{Span}_{\mathbb{Q}_+}(\{(3,2) \ , \ (1,1)\})$ and $\mathrm{Span}_{\mathbb{Q}_+}(\{(1,1) \ , \ (1,2)\})$ are, respectively, generated by the pairs of vectors $\{(3,2) \ , \ (1,1)\}$ and $\{(1,1) \ , \ (1,2)\}$. Since the union of these two cones is exactly the cone generated by $\mathcal{E}$, this implies the claim. Then, due to a criterion of Section 2.1 in [5], it follows that the surface S is normal. Next, with reference to Section 4.2 there are exactly two elements in the set $\mathcal{S}(\mathcal{E})'$, namely: $J_1 = \{(1,1) \ ; \ (1,2)\}$ and $J_2 = \{(1,1) \ ; \ (3,2)\}$, - and the Nash blow up $N(S)$ of S is covered by two respective affine charts $N(S)_{J_j}$, $j = 1, 2$, as explained in Claim 4.6 . (In the remainder of this example we follow the notations of Construction 4.5 .) It turns out that $N(S)_{J_1} \subset \mathbb{C}^5$ is not normal, i. e. the collection of exponents $\mathcal{E}_{J_1}$ of the monomial parametrization

$$\psi : (x_1, x_2) \mapsto (x_1 \cdot x_2 \ , \ x_1 \cdot x_2^2 \ , \ x_1^3 \cdot x_2^2 \ , \ x_1^2 \cdot x_2 \ , \ x_1^2)$$

of the torus $N(S)^*_{J_1}$ does not generate $\mathbb{Z}^2 \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}_{J_1})$ over $\mathbb{Z}_+$. Indeed, obviously the point $(1,0) \in \mathbb{Z}^2 \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}_{J_1}) \setminus \mathbb{Z}_+(\mathcal{E}_{J_1})$, but $(1,0) \notin \mathbb{Z}_+(\mathcal{E} \cup \{(2,1) \ , \ (2,0)\})$, which implies that $N(S)$ is not normal. Note, that $\psi_3(x) = \psi_1(x) \cdot \psi_4(x)$, i. e. the exponent $(3,2)$ is generated over $\mathbb{Z}_+$ by 'others', illustrating the passage from $\mathcal{E}_J$ to $\mathcal{E}xtreme(\mathbb{Z}_+(\mathcal{E}_J))$ in the combinatorial algorithm recording Nash blowing up.

Consider a node τ of a tree $\mathcal{T}$ associated with the normalized multidimensional Euclidean division for the initial essential collection $\mathcal{N}(\mathcal{E})$ with $\mathcal{E}$ from Remark 4.1 . Let $C_\tau \subset \mathbb{Z}^2$ denote the essential collection associated with the node τ . In an abuse of notation we will not indicate the dependence of $\mathcal{S}_\tau := \mathcal{S}(C_\tau)$ and $\mathcal{S}'_\tau := \mathcal{S}(C_\tau)'$ on τ . For the definitions of $\mathcal{S}(\mathcal{E})$ and $\mathcal{S}'$ see Construction 4.4 and Claim 4.6. Note that $\mathrm{int}(\nabla(J)) \cap \mathrm{Span}_{\mathbb{Z}}(C_\tau) = \mathrm{int}(\nabla(J)) \cap \mathbb{Q}_+(C_\tau)_{\mathbb{Z}}$ for $J \in \mathcal{S}_\tau$. Also, $J \in \mathcal{S}'_\tau$ implies that $\mathrm{int}(\nabla(J)) \cap \mathbb{Q}_+(C_\tau)_{\mathbb{Z}} = \emptyset$, see Remark 4.7 (ii) , (iii) . Clearly, $\mathrm{Span}_{\mathbb{Z}}(C_\tau) = \mathrm{Span}_{\mathbb{Z}}(\mathcal{E})$ for any node τ . We may assume that $\mathbb{Z}^m = \mathrm{Span}_{\mathbb{Z}}(\mathcal{E})$, otherwise we 'rescale' replacing the latter span by $\mathbb{Z}^m$. Finally, we refer to the initial node τ_0 of $\mathcal{T}$ as its *root* and to the collection of the 'immediate descendants' of τ in $\mathcal{T}$ as *child nodes* of τ - terms commonly used in the 'theory of trees'.

6.1. **An apriori bound in (essential) dimension $m = 2$ on the length of desingularization by normalized Nash blow ups.** Below we assume that $m = 2$, nodes τ_0 and τ are not terminal and associate with node τ an integer $\mathcal{V}(\tau) := 2 \times$ the area of $\mathrm{Conv}\,(C_\tau)$. Note that (ii) of Theorem 2.1 is a consequence of the decrease of $\mathcal{V}(\tau)$ at least by a factor of 2 after two steps of the normalized Euclidean division algorithm, see Theorem 6.8 below. The behavior of $\mathcal{V}(\tau)$ after one step of the normalized Euclidean division algorithm is recorded by Theorem 6.6 . A detailed plan of the cases to be considered for the outcome of the first step of the algorithm (and the proof of Theorem 6.6 in Section 6.2) can be found following Remark 6.11 . For the outcome of the second step of the normalized Euclidean division algorithm additional splittings into subcases have to be considered. The latter and the proof of Theorem 6.8 in Section 7 are placed following the outcome of the respective cases of the first step of the algorithm in Section 6.2 .

Consider vectors $\{\vec{\Delta}_{j_i}\}_{i=1,2} \subset \mathcal{E} := \{\vec{\Delta}_j\}_{1\le j\le L} \subset \mathbb{Z}^2$ in the intersection of $\mathcal{E}$ with two extremal rays of the cone generated by $\mathcal{E}$ over $\mathbb{R}_+$, having minimal length. We refer to such vectors as the *extremal vectors* of $\mathcal{E}$. The extremal vectors of the input $\mathcal{N}(\mathcal{E})$ for the normalized 2-dimensional Euclidean division are the same vectors. The integer D of Theorem 2.1 (ii) equals $|\det(\vec{\Delta}_{j_1}\ ,\ \vec{\Delta}_{j_2})|$. In an abuse of notation we will not distinguish in this section between the subsets $J \in \mathcal{S}_\tau$ of indices of vectors in collections C_τ and the sets of the corresponding vectors themselves. Let $b_1,\ ,\ b_2 \in C_\tau$ be the extremal vectors of C_τ . Denote $D(\tau) := |\det(b_1\ ,\ b_2)|$ and pick a 2-tuple $J := \{u_j\}_{j=1,2} \in \mathcal{S}'$. In other words, J corresponds to a child node $\overline{\tau}$ of τ and determines the branching of $\mathcal{T}$ at node τ . Then $C_\tau = \mathcal{E}xtreme(\mathbb{Q}_+(C_\tau)_{\mathbb{Z}})$.

Every $J \in \mathcal{S}'$ is a *frame*, i. e. is a collection of linearly independent vectors, and moreover is a *minimal frame of* C_τ . By minimal we mean that, for an irrational functional h positive on the convex hull of the collection $C_{\overline{\tau}} \subset \mathbb{Z}^2$ the value of $h(\vec{\Delta}_J)$, where $\vec{\Delta}_J := u_1 + u_2$, is smaller than the value of $h(\vec{\Delta}_{J'})$ for any other choice of $J' \in \mathcal{S}$. This property of frames $J \in \mathcal{S}'$ does not depend on the choice of irrational h , provided h is positive on the convex hulls of the collections $C_{\overline{\tau}} \subset \mathbb{Z}^2$ corresponding to $\overline{\tau}$, and provides a bijective correspondence between the minimal frames of C_τ and the child nodes $\overline{\tau}$ of τ , cf. Claim 4.6 . We identify in explicit geometric terms the sets involved in the proof of an apriori bound Theorem 2.1 (ii) (see Corollary 6.9 below) in the following

Claim 6.4. *The generators $\mathcal{E}xtreme(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$ of any subset $\mathcal{E} \subset \mathbb{Z}^2$ with* $\mathrm{Conv}\,(\mathcal{E}) \not\ni \mathbf{0}$ *and* $\mathrm{Span}_{\mathbb{Z}}(\mathcal{E}) = \mathbb{Z}^2$ *are the integral points of the bounded edges Γ of $K :=$* $\mathrm{Conv}\,(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$ *. For any node τ of the tree $\mathcal{T}$*

$$D(\tau) - \mathcal{V}(\tau) = \#(C_\tau) - 1 \tag{6.1}$$

Proof. The inclusion of the integral points of the bounded edges Γ of K in $\mathcal{E}xtreme(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$ is obvious. To show the opposite inclusion we pick any pair J of adjacent integral points $\{u_1 \ , \ u_2\}$ on any bounded edge Γ of K . Then the only integral points of the triangle $\nabla(u_1 \ , \ u_2)$ are its vertices. Therefore the only integral points in the parallelogram $P(J)$ spanned by the vectors u_1 , u_2 are its extremal points, which implies (by tiling of $\mathbb{R}^2$ by translations of $P(J)$) that $\mathrm{Span}_Z(J) = \mathbb{Z}^2$. Consequently, $\mathbb{Z}^2 \cap \mathrm{Span}_{\mathbb{Q}_+}(J) \setminus \{\mathbf{0}\} = \mathbb{Z}_+(J)$ and $\mathrm{Span}_{\mathbb{Q}_+}(J) \cap \mathcal{E} = J$, which is equivalent to $1 = |\det(u_1 \ , \ u_2)| = 2 \cdot \mathrm{area}(\nabla(u_1 \ , \ u_2))$ for any pair of adjacent integral points u_1 , u_2 of any bounded edge Γ of $\mathrm{Conv}\,(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$ implying (6.1) for any node τ . Also the remainder of the claim ("the opposite inclusion") follows from the decomposition $\mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) = \cup_J \mathrm{Span}_{\mathbb{Q}_+}(J)$, where the union is over pairs J of the adjacent integral points of the bounded edges of K . ☐

Remark 6.5. Any $J = \{u_1 \ , \ u_2\} \in \mathcal{S}(\mathcal{E})'$ must lie on a bounded edge Γ of $\mathrm{Conv}\,(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$. Moreover, the frame J is a minimal frame iff u_1 , u_2 $\in \Gamma$ are adjacent integral points of the edge Γ and at least one of them is a vertex of Γ , since $J \in \mathcal{S}(\mathcal{E})'$ iff $\dim \tilde{\mathcal{C}}_J = 2$ (see proof of Claim 4.6). The fact that $|\det(u_1 \ , \ u_2)| = 1$ for any pair $\{u_1 \ , \ u_2\}$ of adjacent integral points on a bounded edge of $\mathrm{Conv}\,(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$ is a byproduct of the proof of Claim 6.4 above. Moreover, the converse also holds. Namely, let $u_1, \ \ldots \ , \ u_k \in \mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}}$ be such that u_1 , u_k are extremal vectors of $\mathcal{E}$. Assume that

$|\det(u_i\ ,\ u_{i+1})| = 1\ ,\ 1 \le i < k$, and that $|\det(u_i\ ,\ u_j)| \ge 2$ whenever $i \ge j+2$. Then $\mathcal{E}xtreme(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}}) = \{u_1,\ \dots\ ,\ u_k\}$ and the points $u_i\ ,\ u_{i+1}\ ,\ 1 \le i < k$, are the adjacent integral points on a bounded edge of $\mathrm{Conv}\,(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$.

Clearly $\mathcal{V}(\tau) = 0$ for a terminal node τ . Also, if the node τ is not terminal but $\mathcal{V}(\tau) = 0$, then there are exactly two child nodes of node τ and both are terminal due to a simple argument of the case 2a of the proof in Section 6.2 of the following weak version of Theorem 2.1 (ii)

Theorem 6.6. *Assume $\overline{\tau}$ is not terminal. With every step of the normalized 2-dimensional Euclidean algorithm the integer $\mathcal{V}(\tau)$ decreases, i. e. $\mathcal{V}(\tau) > \mathcal{V}(\overline{\tau})$.*

Corollary 6.7. *The normalized 2-dimensional Euclidean algorithm terminates after at most $\mathcal{V}(\tau_0) + 1 \le D(\tau_0) - 1$ steps.*

We derive Theorem 2.1 (ii) as a consequence of the following

Theorem 6.8. *Assume $\overline{\tau}$ is not terminal. It follows that either $\mathcal{V}(\overline{\tau}) < \mathcal{V}(\tau)/2$ or $\mathcal{V}(\overline{\overline{\tau}}) \le \mathcal{V}(\overline{\tau})/2 < \mathcal{V}(\tau)/2$.*

Of course Theorem 2.1 (ii) follows, namely

Corollary 6.9. *The normalized 2-dimensional Euclidean algorithm terminates after at most $2 \cdot \log_2(\mathcal{V}(\tau_0) + 2) \le 2 \cdot \log_2 D(\tau_0)$ steps.*

Claim 6.10. *For any node $\tau \ne \tau_0$ the collection C_τ contains at most 6 vectors. Moreover, $Conv(\mathbb{Q}_+(C_\tau)_{\mathbb{Z}})$ contains at most 3 bounded edges. If there are at least 2 bounded edges then no edge can have more than 4 integral points. If there are just 3 bounded edges then the middle edge among them has exactly two integral points and no edge can have more than 3 integral points. Finally, at most 3 child nodes of τ can be nonterminal.*

We begin with a proof of the weaker bound of Theorem 6.6. The proofs of Theorem 6.8 and Claim 6.10 we placed in Section 7.

6.2. **Proof of Theorem 6.6 .**

Proof. Fix an irrational $h \in \tilde{\mathcal{C}}_J$ for a $J \in \mathcal{S}'_\tau$. By reindexing arrange that $h(b_1) < h(b_2)$. Let $b'_1\ ,\ b'_2 \in C_{\overline{\tau}}$ be the extremal vectors of $C_{\overline{\tau}}$. Also, let $\tilde{b}'_1\ ,\ \tilde{b}'_2 \in N_J(C_\tau)$ be the minimal vectors in the intersection of $N_J(C_\tau)$ with two extremal rays of the cone generated by $N_J(C_\tau)$ over $\mathbb{R}_+$. Clearly, the latter cone does not change under 'normalization', i. e. coincides with the cone generated by $C_{\overline{\tau}}$ over $\mathbb{R}_+$, see Section 4.3 . In particular, it follows that after an appropriate choice of indices, the extremal vectors $\tilde{b}'_1$, $\tilde{b}'_2$ preceding normalization are proportional to the extremal vectors b'_1 , b'_2 with coefficients from $\mathbb{Z}_+$.

Remark 6.11. The node τ is terminal iff $|\det(b_1 , b_2)| = 1$, iff $\#(C_\tau) = 2$, iff $\{b_1 , b_2\}$ is a minimal frame in C_τ . To establish the only nonobvious implication (i. e. that the last property implies the first) it suffices to apply Claim 6.4 . The latter reference and the node τ not being terminal also imply that if $J \not\subset \mathrm{int}\nabla(b_1 , b_2)$, then $\#(\{b_1 , b_2\} \cap J) = 1$ and $b_2 \notin J$; otherwise $h(b_1) < \min h|_J < h(b_2)$ contrary to the choice of the irrational functional $h \in \tilde{\mathcal{C}}_J$.

Plan : Our proof of the decrease of $\mathcal{V}(\tau)$ splits into several cases identified below. First we consider the case that $J \subset \mathrm{int}\nabla(b_1 , b_2)$. Otherwise we may assume that $b_1 \in J$, $b_2 \notin J$ (due to Remark 6.11) and, also, $b_1 \in \{b'_1 , b'_2\}$ due to the equality $\mathrm{Span}_{\mathbb{Q}_+}(J) \cap C_\tau = J$ established in Claim 6.4 , cf Figures 1 , 2 and 3 . Say $b'_1 = b_1$ and $u_1 = b_1$. The remaining cases are split according to whether $u_2 \notin \mathrm{int}\nabla(b_1 , b_2)$ (and then $\overline{\tau}$ is terminal contrary to our assumption) or otherwise; then according to $\#(C_\tau) = 3$ (when $\#(C_\tau) = 2$ the node is terminal) or $\#(C_\tau) \geq 4$. We show that in the last case $\#(\mathbb{Z}^2 \cap \Gamma) > 2$ for the bounded edge $\Gamma \supset J$ of Conv $(\mathbb{Q}_+(C_\tau)_\mathbb{Z})$. Then the node $\overline{\tau}$ must be terminal, which is contrary to our assumption. In the previous case of $u_2 \in \mathrm{int}\nabla(b_1 , b_2)$ and $\#(C_\tau) = 3$ the arguments of our proof differ depending on $D(\tau)$ being even or odd : if $D(\tau) = 2k - 1$ is odd then it turns out that $C_{\overline{\tau}} = \{b_1 , u_2 , b_2 - (k-1) \cdot u_2 , b_2 - b_1\}$ and $\mathcal{V}(\tau) - \mathcal{V}(\overline{\tau}) = 1$, on the other hand if $D(\tau) = 2k$ is even then $C_{\overline{\tau}} = \{b_1 , u_2 , (b_2 - b_1)/2\}$ and $\mathcal{V}(\tau) - \mathcal{V}(\overline{\tau}) = \mathcal{V}(\tau)/2 + 1$. In each of the cases (with the nodes τ and $\overline{\tau}$ not being terminal) we establish that (after 'normalization') the integer $\mathcal{V}(\tau)$ decreases. We now start with

1. Points u_1 , u_2 in the interior of $\nabla(b_1 , b_2)$.
Then after one step of 2-dimensional Euclidean division (and prior to normalization) each extremal vector $\tilde{b}'_l = a_{(l)} - u_{j_l}$ for appropriate points $a_{(l)} \in C_\tau \cap (\mathrm{int}(\nabla(b_1 , b_2)) \cup \{b_1 , b_2\})$, $l = 1 , 2$, $j_l \in \{1 , 2\}$. Also, after one step of the normalized 2-dimensional Euclidean algorithm the extremal vectors b'_1 , b'_2 are proportional to their respective counterparts $\tilde{b}'_1$, $\tilde{b}'_2$ with positive coefficients majorated by 1 , so that $D(\overline{\tau}) \leq |\det(\tilde{b}'_1 , \tilde{b}'_2)|$. Denote by H and $\mathcal{A}_H$ the convex hull of $\{a_{(1)} , a_{(2)} , u_{j_1} , u_{j_2}\}$ and its area. Clearly, the areas of triangles $\nabla(b_1 , b_2)$ and $\nabla(b'_1 , b'_2)$ are $D(\tau)/2$ and, respectively, $D(\overline{\tau})/2$.

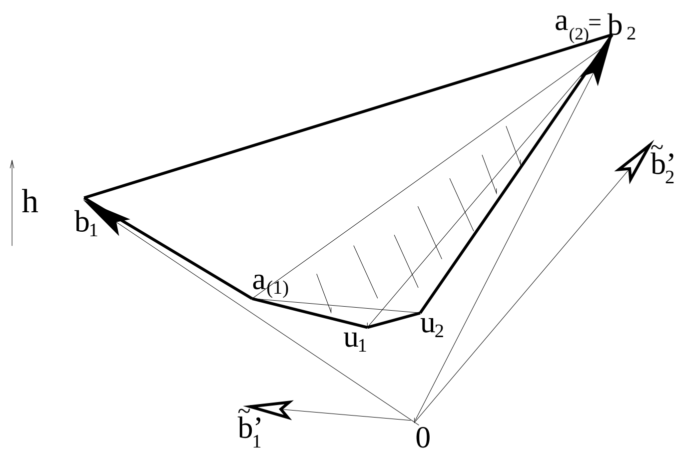

FIGURE 1. $C_\tau = \{ \; b_1 \; , \; a_{(1)} \; , \; u_1 \; , \; u_2 \; , \; b_2 \; \}$.

Then the claimed inequality follows from

$$\mathcal{V}(\overline{\tau}) < D(\overline{\tau}) \leq |\det(\tilde{b}_1' \; , \; \tilde{b}_2')| = 2 \cdot \mathcal{A}_H \leq \mathcal{V}(\tau) \; .$$

Remark 6.12. In the proofs of Theorem 6.8 and Claim 6.10 we will distinguish between the following subcases of case **1**.

1a The minimal frame $\{u_1 \; , \; u_2\} \subset \Gamma$ is not the set of all integral points of a bounded edge Γ of $Conv(\mathbb{Q}_+(C_\tau)_\mathbb{Z})$.

Then, due to Remark 6.5 , we may assume that u_2 is an endpoint of Γ and that the points $u_1 \; , \; u_2$ are adjacent integral points of Γ . Then there is also an integral point $a_{(1)}$ in Γ adjacent to u_1 and of course $a_{(1)} - u_1 = u_1 - u_2$. Also, there is a bounded edge $\Gamma' \ni u_2$ of $Conv(\mathbb{Q}_+(C_\tau)_\mathbb{Z})$ and an integral point, say $a_{(2)} \in \Gamma'$, adjacent to u_2 . Then $u_1 + a_{(2)} = l \cdot u_2$ for an integer $l \geq 3$ since due to Remark 6.5 $\det(u_1 + a_{(2)} \; , \; u_2) = 0$ and $\det(u_1 \; , \; u_1 + a_{(2)}) \geq 3$. We will refer to the subcases of **1a** with integer l being even or odd as **1a+** and, respectively, **1a–** .

1b $\{u_1 \; , \; u_2\} = \mathbb{Z}^2 \cap \Gamma$ for a bounded edge Γ of $Conv(\mathbb{Q}_+(C_\tau)_\mathbb{Z})$.

Then there are bounded edges $\Gamma_i \ni u_i$, $i = 1 \; , \; 2$, of $Conv(\mathbb{Q}_+(C_\tau)_\mathbb{Z})$ distinct from the edge Γ . Say $a_{(i)} \in \Gamma_i$ are the integral points adjacent to u_i , $i = 1 \; , \; 2$. Once again due to Remark 6.5 there are integers $l_1 \; , \; l_2 \; \geq 3$ such that $u_2 + a_{(1)} = l_1 \cdot u_1$, $u_1 + a_{(2)} = l_2 \cdot u_2$. We refer to the subcases of **1b** with $l_1 \; , \; l_2$ being even or both odd as **1b++** and, respectively, **1b– –** . Otherwise it is subcase **1b+ –** .

If case **1** does not hold then

2. The extremal vector $b_1 \in \{u_1\ ,\ u_2\}$.

Since τ is not terminal $b_2 \notin J = \{u_1\ ,\ u_2\}$ and $b_1 \in \{b_1'\ ,\ b_2'\}$ (see 'Plan'). Set both $b_1' = b_1$ and $u_1 = b_1$, i. e. $b_1' = \tilde{b}_1' = b_1 = u_1$ for the remainder of the proof. We split **Case 2.** into several subcases starting with

2a. Assume $u_2 \notin$ int$\nabla(b_1\ ,\ b_2)$.

Then, with reference to Claim 6.4 , u_2 is on the open edge $(b_1\ ,\ b_2)$ (i. .e. excluding the endpoints $b_1\ ,\ b_2$) of the triangle $\nabla(b_1\ ,\ b_2)$. Therefore $C_\tau \subset [b_1\ ,\ b_2] := (b_1\ ,\ b_2) \cup \{b_1\ ,\ b_2\}$. Then $\tilde{b}_2' = a - u_2 \neq 0$ for the point $a \in C_\tau \cap [u_2\ ,\ b_2]$ adjacent to u_2 which implies that $b_2' = \tilde{b}_2' = u_2 - u_1$. Hence, with reference to Claim 6.4 , $|\det(b_1'\ ,\ b_2')| = |\det(u_1\ ,\ u_2)| = 1$ and $\overline{\tau}$ is terminal (Remark 6.11).

In the remaining subcases of case **2**, $u_2 \in \mathrm{int}\nabla(b_1\ ,\ b_2)$ and the assumptions of the subcase **2b** below imply that $\overline{\tau}$ is terminal.

2b. Assume $u_2 \in$ int$\nabla(b_1\ ,\ b_2)$, $\#(C_\tau) \geq 4$ and $\#(\mathbb{Z}^2 \cap \Gamma) > 2$ for the bounded edge Γ of Conv$(\mathbb{Q}_+(C_\tau)_{\mathbb{Z}})$ such that $\Gamma \supset J$.

Then, with reference to Claim 6.4 , $\tilde{b}_2' = a - u_2 \neq 0$ for the point $a \in C_\tau \cap \Gamma \setminus \{u_1\}$ adjacent to u_2 which implies (as in the previous case) that $b_2' = \tilde{b}_2' = u_2 - u_1$, that $|\det(b_1'\ ,\ b_2')| = |\det(u_1\ ,\ u_2)| = 1$ and, finally, that $\overline{\tau}$ is a terminal node, contrary to initial assumption. Note that the proof remains valid without our assumption $\#(C_\tau) \geq 4$.

2c. Assume $u_2 \in$ int$\nabla(b_1\ ,\ b_2)$, $\#(C_\tau) \geq 4$ and $\#(\mathbb{Z}^2 \cap \Gamma) = 2$ for the bounded edge $\Gamma \supset J$ of Conv$(\mathbb{Q}_+(C_\tau)_{\mathbb{Z}})$.

Then $\mathbb{Z}^2 \cap \Gamma = J$, $\#(C_\tau \setminus J) \geq 2$ and, with reference to Remark 6.5 , there is a bounded edge $\Gamma' \ni u_2$ of Conv$(\mathbb{Q}_+(C_\tau)_{\mathbb{Z}})$ distinct from Γ and an integral point $a \in \Gamma'$ adjacent to u_2 with $\tilde{b}_2' = a - u_1$. Therefore the integer $\mathcal{V}(\tau) - 2 \cdot \mathrm{area}(u_1 + \nabla(u_2 - u_1\ ,\ a - u_1)) > 0$ implying $|\det(\tilde{b}_1'\ ,\ \tilde{b}_2')| = 2 + 2 \cdot \mathrm{area}(u_1 + \nabla(u_2 - u_1\ ,\ a - u_1)) \leq 2 + (\mathcal{V}(\tau) - 1)$. Combining with (6.1) and Remark 6.11 proves the inequality $\mathcal{V}(\overline{\tau}) < \mathcal{V}(\tau)$, as required:

$$2 + \mathcal{V}(\overline{\tau}) \leq D(\overline{\tau}) \leq |\det(\tilde{b}_1'\ ,\ \tilde{b}_2')| \leq 1 + \mathcal{V}(\tau) \quad .$$

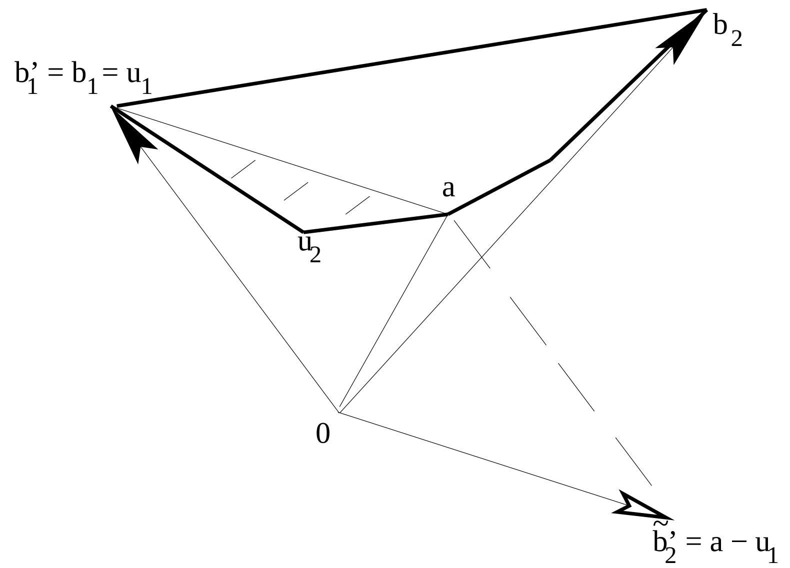

FIGURE 2. The area of $\operatorname{Conv}(C_\tau \setminus \{u_2\}) \geq 1$.

Remark 6.13. With a from case **2c** above and again due to Remark 6.5 (as in the argument in Remark 6.12 **1a**) there is an integer $l \geq 3$ with $u_1 + a = l \cdot u_2$. In the proofs of Theorem 6.8 and Claim 6.10 we will refer to the subcases of case **2c** with integer l being even or odd as **2c+** and, respectively, as **2c−** .

2d. Assume $u_2 \in \mathbf{int}\nabla(b_1\ ,\ b_2)$ **and** $\#(C_\tau) = 3$ **.**

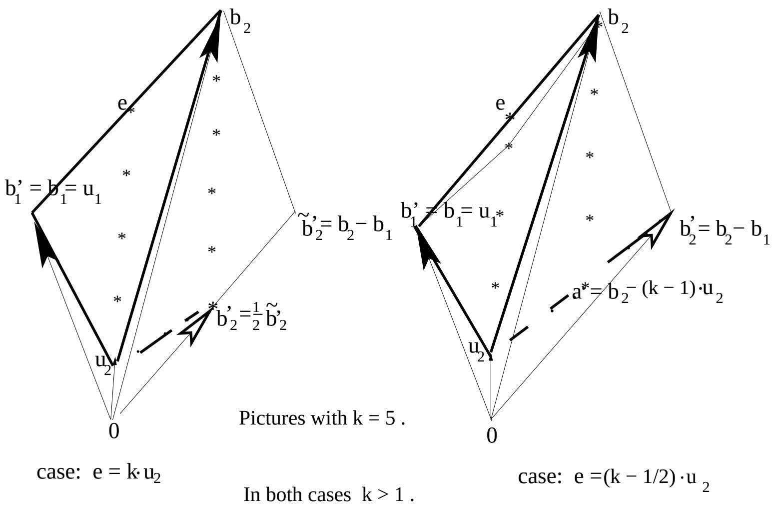

FIGURE 3. $D(\tau) = 2k$ or $2k-1 \Rightarrow \#C_{\overline{\tau}} = 3$ or 4 respectively.

Let e be the point of intersection of the edge $(b_1\ ,\ b_2)$ with the ray $\mathbb{R}_+ \cdot u_2$, say $\lambda \cdot u_2 = e$, $\lambda > 0$. Due to Claim 6.4 $\nabla(b_1\ ,\ b_2) \cap \mathbb{Z}^2 \setminus \{\mathbf{0}\ ,\ b_1\ ,\ b_2\} \subset \mathbb{Z}_+ \cdot u_2$ and $|\det(b_2\ ,\ u_2)| = 1 = |\det(u_2\ ,\ b_1)|$ implying that $\tilde{b}'_2 = b_2 - b_1$ and that the areas of the triangles $\nabla(b_2\ ,\ e)$ and $\nabla(b_1\ ,\ e)$ coincide. Hence $e = (b_1 + b_2)/2$ and, also, $\lambda = |\det(e\ ,\ b_1)| = D(\tau)/2$. The arguments in the remainder depend on $D(\tau)$ being even or odd and accordingly we split case **2d** into the following two subcases.

2d+ Assume $D(\tau)$ is even and let $k := D(\tau)/2$.

Then $b'_2 = \tilde{b}'_2/2$ since $\{(b_2 - b_1)/2\} = \mathbb{Z}^2 \cap (\mathbf{0}\ ,\ \tilde{b}'_2)$. Therefore $|\det((b_2 - b_1)/2\ ,\ u_2)| = |(\det(b_2\ ,\ u_2) + \det(u_2\ ,\ b_1))/2| = 1$, which implies that $C_{\overline{\tau}} = \{b_1\ ,\ u_2\ ,\ (b_2 - b_1)/2\}$ (Remark 6.5).

Remark 6.14. Claim 6.10 in case 2d+ is a consequence.

Finally, with reference to (6.1), it follows that

$$\mathcal{V}(\overline{\tau}) + 2 = D(\overline{\tau}) = |\det(b_1\ ,\ (b_2 - b_1)/2)| = D(\tau)/2 = (\mathcal{V}(\tau) + 2)/2$$

which implies that $\mathcal{V}(\tau) - \mathcal{V}(\overline{\tau}) = \mathcal{V}(\tau)/2 + 1$, as required.

Remark 6.15. Of course Theorem 6.8 in case 2d+ follows.

2d– Assume $D(\tau)$ is odd and let $k := (D(\tau) + 1)/2$.

Then there are no integral points on the edge $(b_1\ ,\ b_2)$ (as well as on the 'interval' $(\mathbf{0}\ ,\ \tilde{b}'_2)$) implying that $b'_2 = \tilde{b}'_2 = b_2 - b_1$. Denote the point $a := b_2 - (k-1) \cdot u_2 = (u_2 + b'_2)/2$. Then, since $|\det(b'_2\ ,\ u_2)| = 2$, it follows that $|\det(b'_2\ ,\ a)| = |\det(a\ ,\ u_2)| = 1$. Now, with reference to Remark 6.5 it follows that $C_{\overline{\tau}} = \{b_1\ ,\ u_2\ ,\ b_2 - (k-1) \cdot u_2\ ,\ b_2 - b_1\}$.

Remark 6.16. Note that Claim 6.10 in case 2d– follows.

The latter formula for $C_{\overline{\tau}}$ and (6.1) imply that

$$\mathcal{V}(\overline{\tau}) + 3 = D(\overline{\tau}) = |\det(b'_1\ ,\ b'_2)| = D(\tau) = \mathcal{V}(\tau) + 2\ .$$

Therefore $\mathcal{V}(\tau) - \mathcal{V}(\overline{\tau}) = 1$, which completes the proof of Theorem 6.6. $\square$

7. Sharp apriori bound and polynomial complexity. Proofs.

7.1. **Proofs of Theorem 6.8 and Claim 6.10.** We establish both results separately for all the cases introduced in the course of the proof of Theorem 6.6 (exluding the cases already covered by Remarks 6.14 , 6.15 , 6.16 and cases **2a** and **2b**, when $\overline{\tau}$ is terminal). We start with the case **1a+**.

Under the assumptions of case **1a+** the integer l is even. Let $k := l/2$. Then, due to Remark 6.5

$$C_{\overline{\tau}} = \{ \ u_1 - u_2 \ , \ u_2 \ , \ a_{(2)} - ku_2 = (a_{(2)} - u_1)/2 \ \} \ ,$$

unless $k - 1 = |\det(u_1 - u_2 \ , \ (a_{(2)} - u_1)/2)| = 1$ which implies that

$$C_{\overline{\tau}} = \{ \ u_1 - u_2 \ , \ (a_{(2)} - u_1)/2 \ \}$$

and then, due to Remark 6.11 , that $\overline{\tau}$ is terminal. The latter proves Claim 6.10 in case **1a+** . Moreover, then also

$$\mathcal{V}(\overline{\tau}) = |\det(u_1 - u_2 \ , \ (a_{(2)} - u_1)/2)| - 2 =$$
$$k - 3 < k - 1 = |\det(u_1 - u_2 \ , \ a_{(2)} - u_2)|/2 < \mathcal{V}(\tau)/2$$

(unless $k = 2$ and $\overline{\tau}$ is terminal, as we showed above), which establishes Theorem 6.8 in case **1a+** .

Under the assumptions of case **1a–** the integer l is odd. Let $k := (l+1)/2$. Then, due to Remark 6.5, it follows that

$$C_{\overline{\tau}} = \{ \ u_1 - u_2 \ , \ u_2 \ , \ a_{(2)} - (k-1)u_2 \ , \ a_{(2)} - u_1 \ \}$$

with the points u_2 , $a_{(2)} - (k-1)u_2$, $a_{(2)} - u_1$ lying on a bounded edge of $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$ and $a_{(2)} - (k-1)u_2 = (u_2 + (a_{(2)} - u_1))/2$ (unless $2k - 3 = |\det(u_1 - u_2 \ , \ a_{(2)} - u_1)| = 1$, which implies that

$$C_{\overline{\tau}} = \{ \ u_1 - u_2 \ , \ a_{(2)} - u_1 \ \}$$

and then, due to Remark 6.11 , that $\overline{\tau}$ is terminal). This proves Claim 6.10 in case **1a–** . Then

$$\mathcal{V}(\overline{\tau}) = |\det(u_1 - u_2 \ , \ a_{(2)} - u_1)| - 3 = 2k - 6 < 2k - 3 =$$
$$|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_2)|/2 \leq \mathcal{V}(\tau)/2$$

(unless $k = 2$ and $\overline{\tau}$ is terminal, as proved above), which establishes Theorem 6.8 in case **1a–** .

Under the assumptions of case **1b++** both of the integers l_1 and l_2 are even . Let $k_i := l_i/2$, $i = 1, 2$. Then, due to Remark 6.5,

$$C_{\overline{\tau}} = \{ \ a_{(1)} - k_1 \cdot u_1 = (a_{(1)} - u_2)/2 \ , \ u_1 \ , \ u_2 \ , \ a_{(2)} - k_2 \cdot u_2 = (a_{(2)} - u_1)/2 \ \}$$

(unless $|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)| = 4$ in which case

$$C_{\overline{\tau}} = \{ \ (a_{(1)} - u_2)/2 \ , \ (a_{(2)} - u_1)/2 \ \}$$

and then, due to Remark 6.11 , that $\overline{\tau}$ is terminal). This proves Claim 6.10 in case **1b++** . Then

$$\mathcal{V}(\overline{\tau}) = |\det((a_{(1)} - u_2)/2 \ , \ (a_{(2)} - u_1)/2)| - 3 <$$
$$|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)|/4 \leq \mathcal{V}(\tau)/4$$

(unless $|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)| = 4$ and $\overline{\tau}$ is terminal, as we proved), which establishes Theorem 6.8 in case **1b++** .

Under the assumptions of case **1b+−** the integers l_1 , l_2 are respectively odd and even (or vice versa, which is a similar case). Let $k_1 := (l_1+1)/2$ and $k_2 := 1 + l_2/2$. Then, due to Remark 6.5,

$$C_{\overline{\tau}} = \{ a_{(1)}-u_2 \ , \ a_{(1)}-(k_1-1)\cdot u_1 \ , \ u_1 \ , \ u_2 \ , \ a_{(2)}-(k_2-1)\cdot u_2 = \frac{a_{(2)}-u_1}{2} \}$$

with the first three points $a_{(1)} - u_2$, $a_{(1)} - (k_1 - 1) \cdot u_1$, u_1 lying on a bounded edge of $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$ and $a_{(1)} - (k_1 - 1) \cdot u_1 = (a_{(1)} - u_2 + u_1)/2$ (unless $|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)| = 2$, in which case

$$C_{\overline{\tau}} = \{ \ a_{(1)} - u_2 \ , \ (a_{(2)} - u_1)/2\}$$

and then, due to Remark 6.11 , that $\overline{\tau}$ is terminal). This proves Claim 6.10 in case **1b+−** . Then

$$\mathcal{V}(\overline{\tau}) = |\det(a_{(1)} - u_2 \ , \ (a_{(2)} - u_1)/2)| - 4 <$$

$$|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)|/2 \le \mathcal{V}(\tau)/2$$

(once again unless $|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)| = 2$ and, consequently, $\overline{\tau}$ is terminal), which establishes Theorem 6.8 in case **1b+−** .

Under the assumptions of case **1b− −** both of the integers l_1 and l_2 are odd. Let $k_i := (l_i+1)/2$, $i = 1, 2$. Then, due to Remark 6.5,

$$C_{\overline{\tau}} = \{ a_{(1)}-u_2 \ , \ a_{(1)}-(k_1-1)\cdot u_1 \ , \ u_1 \ , \ u_2 \ , \ a_{(2)}-(k_2-1)\cdot u_2 \ , \ a_{(2)}-u_1 \}$$

with the first three points $a_{(1)}-u_2$, $a := a_{(1)}-(k_1-1)\cdot u_1$, u_1 lying on a bounded edge of $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$ and $a = (a_{(1)}-u_2+u_1)/2$. Also, all of the last three points u_2 , $b := a_{(2)}-(k_2-1)\cdot u_2$ and $a_{(2)}-u_1$ lie on one bounded edge of $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$ and $b = (u_2 + a_{(2)} - u_1)/2$ (unless $|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)| = 1$, in which case

$$C_{\overline{\tau}} = \{ \ a_{(1)} - u_2 \ , \ a_{(2)} - u_1\}$$

and then, due to Remark 6.11, that $\overline{\tau}$ is terminal). Therefore

$$\mathcal{V}(\overline{\tau}) = |\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)| - 5 \le \mathcal{V}(\tau) - 5$$

(unless $|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)| = 1$ and $\overline{\tau}$ is terminal, as we proved) and Claim 6.10 is proved in case **1b− −** . It remains to prove Theorem 6.8 (passing from node $\overline{\tau}$ to $\overline{\overline{\tau}}$), but we will need to examine several options for choosing the minimal frames $J' = \{u'_1 \ , \ u'_2\}$ of $C_{\overline{\tau}}$ associated with the child node $\overline{\overline{\tau}}$ of $\overline{\tau}$ (unlike in the previously considered cases).

To begin with we assume that u_1 , u_2 are the endpoints of a bounded edge Γ of $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$.

The choice of $J' = \{u_1 \ , \ u_2\}$ is an option (see Remark 6.5). Then

$$\mathcal{V}(\overline{\overline{\tau}}) < |\det((a_{(1)} - u_2 + u_1)/2 - u_2 \ , \ (a_{(2)} - u_1 + u_2)/2 - u_1)| <$$

$$|\det(a_{(1)} - 2 \cdot u_2\ ,\ a_{(2)} - 2 \cdot u_1)|/2 \leq \mathcal{V}(\overline{\tau})/2\ ,$$

which establishes Theorem 6.8 in this subcase of case **1b– –**.

With the same assumption on $\{u_1\ ,\ u_2\}$ another possibility for the choice of a minimal frame J' of $C_{\overline{\tau}}$ is $u_1' = (a_{(1)} - u_2 + u_1)/2$, $u_2' = u_1$. Then, with reference to **1a** (passing from the node $\overline{\tau}$ to the node $\overline{\overline{\tau}}$),

$$\mathcal{V}(\overline{\overline{\tau}}) < |\det((a_{(1)} - u_2 + u_1)/2 - u_1\ ,\ (a_{(1)} - u_2 + u_1)/2 - u_2)| =$$

$$|\det(a_{(1)} - 2 \cdot u_2\ ,\ u_2 - u_1)|/2 < \mathcal{V}(\overline{\tau})/2\ ,$$

which implies Theorem 6.8 in this subcase.

Once again with the same assumption on $\{u_1\ ,\ u_2\}$, we choose $J' := \{u_1' = a_{(1)} - u_2\ ,\ u_2' = (a_{(1)} - u_2 + u_1)/2\}$. It follows with reference to case **2b** (passing from the node $\overline{\tau}$ to $\overline{\overline{\tau}}$) that the node $\overline{\overline{\tau}}$ is terminal. With the same assumption on $\{u_1\ ,\ u_2\}$ the remaining options for the choice of a minimal frame J' and, consequently, of a child node $\overline{\overline{\tau}}$ are either $J' := \{(u_2 + a_{(2)} - u_1)/2\ ,\ a_{(2)} - u_1\}$, which is similar to the case just considered, or $J' := \{u_2\ ,\ (u_2 + a_{(2)} - u_1)/2\}$, which is similar to the case considered in the previous paragraph. Consequently, in these cases Theorem 6.8 follows by means of analogous arguments.

To complete the proof of Theorem 6.8 in the case **1b– –** it remains to consider the case when u_1 , u_2 are not the endpoints of one bounded edge of $K := Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$. Then, following the constraints established in the first paragraph of case **1b– –**, there are at most 2 bounded edges of K . In the case that there is exactly one bounded edge Γ of K there are exactly two possible choices of minimal frames $\{u_1'\ ,\ u_2'\}$ of $C_{\overline{\tau}}$, cf. Remark 6.5 . Both choices lead to case **2a** (passing from the node $\overline{\tau}$ to the node $\overline{\overline{\tau}}$) and therefore $\mathcal{V}(\overline{\tau}) = 0$ establishing Theorem 6.8 in this case. In the case that there are exactly two bounded edges of K , it follows by making use of Remark 6.5 that there are exactly 4 possible choices of minimal frames $J' := \{u_1'\ ,\ u_2'\}$ of $C_{\overline{\tau}}$. We distinguish these choices only by the property of the intersection of the two edges being in J' or not. The latter case is the case **2b** (for passing from the node $\overline{\tau}$ to the node $\overline{\overline{\tau}}$). Consequently, it implies that the node $\overline{\overline{\tau}}$ is terminal, establishing Theorem 6.8 in this case. In the former case we are in the setting of case **1a** (but for passing from the node $\overline{\tau}$ to the node $\overline{\overline{\tau}}$). The inequalities on the values of $\mathcal{V}(\cdot)$ proved in both subcases of **1a** applied in our setting imply the second alternative of Theorem 6.8 in this last subcase of **1b– –**, as required.

The remaining cases to consider are **2c+** , **2c–** and **2d–**.

Under the assumptions of case **2c+** the integer l is even. Let $k := l/2$. Then, due to Remark 6.5,

$$C_{\overline{\tau}} = \{\ u_1\ ,\ u_2\ ,\ a - k \cdot u_2 = (a - u_1)/2\ \}\ ,$$

which proves Claim 6.10 in case **2c+**. Then

$$(7.1)\ \mathcal{V}(\overline{\tau}) = |\det(u_1\ ,\ (a - u_1)/2)| - 2 = k - 2 < (l - 1)/2 =$$
$$(|\det(u_1 - u_2, a - u_2)| + 1)/2 \le \mathcal{V}(\tau)/2\ ,$$

which establishes Theorem 6.8 in case **2c+**.

Under the assumptions of case **2c–** the integer l is odd. Let $k := (l + 1)/2$. Then, due to Remark 6.5,

$$C_{\overline{\tau}} = \{\ u_1\ ,\ u_2\ ,\ a - (k - 1) \cdot u_2\ ,\ a - u_1\ \}$$

with the points u_2 , $a - (k - 1) \cdot u_2$, $a - u_1$ lying on a bounded edge of $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$ and $a - (k - 1) \cdot u_2 = (u_2 + a - u_1)/2$. This proves Claim 6.10 in case **2c–**.

Once again, to establish Theorem 6.8 in the latter case we will examine the options for choosing the minimal frames $J' := \{\ u'_1\ ,\ u'_2\ \}$ in $C_{\overline{\tau}}$ and, consequently, corresponding child nodes $\overline{\overline{\tau}}$ of the node $\overline{\tau}$. There is an exception for $k = 2$ when $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$ has a single bounded edge with four integral points from $C_{\overline{\tau}}$, which implies that we are in the case **2a** for the node $\overline{\tau}$ and therefore the node $\overline{\overline{\tau}}$ is terminal.

There are three options for the choice of a minimal frame J' .

The first choice is $u'_1 = u_1$, $u'_2 = u_2$. For an integer $l_1 \ge 3$ the vector $u_1 + (a-(k-1) \cdot u_2) = l_1 \cdot u_2$. If l_1 is even then $\mathcal{V}(\overline{\overline{\tau}}) < \mathcal{V}(\overline{\tau})/2$ by inequality (7.1) with τ and $\overline{\tau}$ in (7.1) respectively replaced by $\overline{\tau}$ and $\overline{\overline{\tau}}$. If $l_1 = 2 \cdot k_1 - 1$ is odd then the passage from $\overline{\tau}$ to $\overline{\overline{\tau}}$ is similar to the passage from τ to $\overline{\tau}$ in case **2c–** considered above. Hence, with reference to Remark 6.5, and assuming $k \neq 2$,

$$C_{\overline{\overline{\tau}}} = \{\ u_1\ ,\ u_2\ ,\ a - (k - 1) \cdot u_2 - (k_1 - 1) \cdot u_2\ ,\ a - (k - 1) \cdot u_2 - u_1\ \}$$

with the points u_2 , $a-(k-1) \cdot u_2-(k_1-1) \cdot u_2$, $a-(k-1) \cdot u_2-u_1$ lying on a bounded edge of $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$ and $a-(k-1) \cdot u_2-(k_1-1) \cdot u_2 = (u_2 + a - (k - 1) \cdot u_2 - u_1)/2$. Consequently

$$\mathcal{V}(\overline{\overline{\tau}}) + 1 = |\det(u_1\ ,\ a - (k - 1) \cdot u_2 - u_1)| - 2 = l_1 - 2 =$$

$$|\det(u_1 - u_2\ ,\ a - (k-1) \cdot u_2 - u_1)| = \frac{|\det(u_1 - u_2\ , a - 2 \cdot u_1)|}{2} = \frac{\mathcal{V}(\overline{\tau})}{2}$$

and Theorem 6.8 follows in this subcase of **2c–** .

Another option for the choice of J' is $u'_1 = a - (k - 1) \cdot u_2$, $u'_2 = a - u_1$ which leads to case **2b** for the node $\overline{\tau}$. It follows that $\mathcal{V}(\overline{\overline{\tau}}) = 0$, which suffices. The last option for the choice of J' is $u_1' = u_2$, $u_2' = a - (k-1) \cdot u_2$. This choice leads to case **1a** for the node $\overline{\tau}$. In both of the latter subcases we derived the inequality $\mathcal{V}(\overline{\overline{\tau}}) < \mathcal{V}(\overline{\tau})/2$, as required in Theorem 6.8 .

This completes the proof of Claim 6.10 . But to complete the proof of Theorem 6.8 it remains to consider case **2d–**. (Remark 6.16 takes care of Claim 6.10 in this case.) Under the assumptions of case **2d–**

$$C_{\overline{\tau}} = \{b_1 \ , \ u_2 \ , \ b_2 - (k-1) \cdot u_2 \ , \ b_2 - b_1\}$$

with the points u_2 , $b_2 - (k-1) \cdot u_2$, $b_2 - b_1$ lying on a bounded edge of $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$ and $b_2 - (k-1) \cdot u_2 = (u_2 + b_2 - b_1)/2$. This is the setting of case **2c–** and, therefore, implies its conclusion $\mathcal{V}(\overline{\overline{\tau}}) \leq \mathcal{V}(\overline{\tau})/2$. This fully completes the proofs of both Claim 6.10 and Theorem 6.8 . □

Example 7.1. The example below demonstrates that the bound of Theorem 2.1 (ii), and of Corollary 6.9, is sharp. In the notations of case **2d+**, consider u_1 , u_2 , $b_2 \in \mathbb{Z}^2$ with $|\det(u_1 \ , \ u_2)| = |\det(u_2 \ , \ b_2)| = 1$ and $u_1 + b_2 = 2^l \cdot u_2$ for an integer $l > 0$, e. g. say $u_1 = (-1, 1)$, $u_2 = (0, 1)$, $b_2^{(l)} := (1 \ , \ 2^l - 1)$. Then $\mathcal{V}(\tau_0) = 2^l - 2$. Let us choose $\{u_1 \ , \ u_2\}$ as a minimal frame and follow the arguments of case **2d+**, i.e. $C_{\overline{\tau_0}} = \{u_1 \ , \ u_2 \ , \ b_2^{(l-1)}\}$ with $b_2^{(l-1)} = b_2^{(l)} - 2^{l-1} \cdot u_2 = (b_2^{(l)} - u_1)/2$. Then $\mathcal{V}(\overline{\tau_0}) = 2^{l-1} - 2$. Therefore in this example the normalized 2-dimensional Euclidean algorithm terminates after $l = \log_2 D(\tau_0)$ steps.

7.2. **Complexity issues.** We have constructed an algorithm by means of Lemma 3.4 (via linear programming). We then apply the algorithm of the first three lines of Section 4.1. Its input is the exponent matrix $\hat{E}$ (from (2.1)) and the output is an essential collection $\mathcal{E} = \{\vec{\Delta}_j\}_{1 \leq j \leq L}$ of the exponent vectors of a monomial parametrization of (4.1). The fact that the complexity of the designed algorithm is polynomial in the binary size of the input relies on the following two subroutines, namely:

(i) The first one by means of linear programming [14] separates variables w_j on $\mathbb{K}^N$ into two groups of z-variables and y-variables.

(ii) The second ([4]) yields a $\mathbb{Z}$-basis $\{(\vec{\delta}_{1i} \ , \ \ldots \ , \ \vec{\delta}_{Li}) \times \mathbf{0}\}_{1 \leq i \leq m}$ of the integral lattice in $\operatorname{Ker} \hat{E} \cap (\mathbb{Q}^L \times \{\mathbf{0}\}) \subset \mathbb{Q}^N$ and vectors from the collection $\mathcal{E}$ are derived by means of the formulae for $\vec{\Delta}_j = (\delta_{j1} \ , \ \ldots \ , \ \delta_{jm})$ for each j .

Combination of the latter two subroutines results in an algorithm whose input is an exponent matrix of an affine binomial variety $\hat{V} \subset \mathbb{K}^N$. The output of the algorithm is the collection of exponents

$\{\vec{\Delta}_j\}_{1\leq j\leq\ L} \subset \mathbb{Z}^m$ of a monomial parametrization $\mathbb{T}^m \to Y \cap \mathbb{T}^N \hookrightarrow \hat{V} \cap (\mathbb{T}^L \times \mathbb{I}_{N-L})$ of the torus of the essential toric subvariety $Y \hookrightarrow \hat{V}$, defined by the formulae $y_j = x^{\vec{\Delta}_j}$, $1 \leq j \leq\ L$. As explained in Remark 3.15, normalized Nash desingularization of the variety Y implies normalized Nash desingularization of the same length of the variety $\hat{V}$. We also observe that Criterion 3.18 invokes only subroutines (i),(ii) and thereby one can verify the nonsingularity of an affine binomial variety within polynomial complexity.

When $m = 2$ the sequence of normalizations followed by Nash blowings up stabilizes, as is proved in this section, and provides normalized Nash desingularization of Y . This process is recorded by means of a combinatorial algorithm on the exponents of monomial parametrizations of the dense tori of the successive compositions of the normalized Nash blowings up. We start with the normalization of the essential toric variety Y and follow by the normalized 2-dimensional Euclidean algorithm (described in Section 4.3 and in great detail here).

Below we estimate the complexity of both procedures in terms of the number D from Section 1.2 (see Remark 7.2 for the normalized Euclidean algorithm and Corollary 7.5 for the normalizing algorithm). Consequently, the complexity of the normalized Nash desingularizatiom of Y is polynomial in the binary size of the input, i. e. the exponents of the binomial equations defining an affine binomial variety whose essential toric subvariety is of dimension $m \leq 2$.

Remark 7.2. Let the set $\mathcal{E} = \mathcal{N}(\mathcal{E}) \subset \mathbb{Z}^2$ be the input of the algorithm from Section 4.3 . After each step of the normalized 2-dimensional Euclidean algorithm the maximal binary size of the points of the input increases at most by an additive constant. Since the length of any branch of the algorithm is bounded by $2 \cdot \log_2 D$ (Theorem 2.1 (ii)) and $\log_2 D$ is polynomial in the binary size of the initial input (combining the bounds for the subroutines considered above), it follows that the complexity of a single step of the algorithm as well as the complexity along a single branch are polynomial in the binary size of the initial input.

7.3. **Polynomial complexity of normalization.** Finally we establish a polynomial complexity bound for constructing the normalization $\mathcal{N}(\mathcal{E})$ starting with an initial essential collection $\mathcal{E} \subset \mathbb{Z}^2$. Let $K := Conv(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$ have k bounded edges with l_1 , ... , l_k integral points, respectively. We denote these points by

$$v_{1,1}\ ,\ v_{1,2}\ ,\ \dots\ ,\ v_{1,l_1} := v_{2,1}\ ,\ \ v_{2,2}\ ,\ \dots\ ,\ v_{2,l_2} := v_{3,1}\ ,\ v_{3,2}\ ,\ \dots\ ,\ v_{k,l_k}\ ,$$

where each pair of consecutive points consists of adjacent integral points, say A , B , on the boundary of K with $det(A\ ,\ B) = -1$ (cf. Remark 6.5) and the points $v_{i,1}$, $v_{i,2}$, $\ldots$, v_{i,l_i} lie on the i-th bounded edge with $v_{i,1}$, v_{i,l_i} being its endpoints.

Denote $v_i := v_{i,2} - v_{i,1} = \cdots = v_{i,l_i} - v_{i,l_i-1}$. Then

Remark 7.3. The point v_{i,l_i} is a common vertex of two bounded edges of K whenever $v_{i,l_i} + v_i \notin K$. Moreover, $v_{i+1,2} = B(s) := v_{i+1,1} + v_i + s \cdot v_{i+1,1}$ for $s = \lambda := det(v_i\ ,\ v_{i+1}) \in \mathbb{Z}_+$, which implies that λ is the smallest integer with $B(\lambda) \in \mathbb{Q}_+(\mathcal{E})$. This is because if $B(s) = v_{i+1,1} + t \cdot v_{i+1}$ for some t , $s \in \mathbb{R}$, then $t = det(B\ ,\ v_{i+1,1}) = det(v_i\ ,\ v_{i+1,1}) = 1$ and $1 + s = det(v_i\ ,\ B) = 1 + det(v_i\ ,\ v_{i+1})$. Finally $D = -\det(v_{1,1}\ ,\ v_{k,l_k})$.

Proposition 7.4. $l_1\ \cdots\ l_i \leq |\det(v_{1,1}\ ,\ v_{i,l_i})|$, $1 \leq i \leq k$.

Proof. By induction on i . The base of induction $l_1 = |\det(v_{1,1}\ ,\ v_{1,l_1})|$ is a consequence of Remark 6.5. For $v \in \mathbb{R}^2$ let $h(v)$ be the distance from v to the line $Span_{\mathbb{R}}(v_{1,1})$. Then the inductive hypothesis is

$$2 \cdot ||v_{1,1}|| \cdot h(v_{i,l_i}) \ = \ |\det(v_{1,1}\ ,\ v_{i,l_i})| \ \geq \ l_1\ \ldots\ l_i$$

Note that $h(v_{i+1,2} - v_{i,l_i}) = h(v_{i+1,2}) - h(v_{i,l_i})$. With $\lambda \geq 1$ it follows that

$$h((v_{i,l_i} + v_i) + \lambda \cdot v_{i,l_i}) \ > \ h((\lambda + 1) \cdot v_{i,l_i}) \ \geq 2 \cdot h(v_{i,l_i}) \ ,$$

which implies $|\det(v_{1,1}\ ,\ v_{i+1,2})| \ \geq 2 \cdot |\det(v_{1,1}\ ,\ v_{i,l_i})|$. Similarly, for $j \geq 2$, $h(v_{i+1,j}) = h(v_{i,l_i} + (j-1) \cdot v_{i+1}) = h(v_{i,l_i}) + (j-1) \cdot h(v_{i+1,2} - v_{i,l_i}) > j \cdot h(v_{i,l_i})$. Therefore $|\det(v_{1,1}\ ,\ v_{i+1,j})| \ > \ j \cdot |\det(v_{1,1}\ ,\ v_{i,l_i}| > j \cdot l_1 \cdots l_i$. Consequently, setting $j = l_{i+1}$ completes the inductive step of the proof. □

Corollary 7.5. *The number k of edges of K does not exceed* $\log_2 D$.

We describe, in dimension $m = 2$, in greater detail the normalization algorithm of Section 4.3 . Its input is $\mathcal{E} \subset \mathbb{Z}^m$ with $\mathrm{Conv}\,(\mathcal{E}) \not\ni \mathbf{0}$ and the output is $\mathcal{N}(\mathcal{E}) := \mathcal{E}xtreme(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}}) \subset \mathbb{Z}^m$. To carry out the normalized Euclidean algorithm with the initial input $\mathcal{N}(\mathcal{E})$ with $\mathcal{E}$ of the preceding sentence it suffices to indicate on the i -th bounded edge of K the endpoints $v_{i,1}$, v_{i,l_i} and, also, the point $v_{i,2}$, which then yields $v_i = v_{i,2} - v_{i,1}$ and $v_{i,l_i-1} = v_{i,l_i} - v_i$. The normalized Euclidean algorithm then starts by choosing a minimal frame $J \in \mathcal{S}(\mathcal{N}(\mathcal{E}))'$, which are (Remark 6.5) of the form $J = \{v_{i,1}\ ,\ v_{i,2}\}$ or $J = \{v_{i,l_i-1}\ ,\ v_{i,l_i}\}$ with $1 \leq i \leq k$, and the output of its first step for the choice of J is $\mathcal{N}(N_J(\mathcal{N}(\mathcal{E})))$.

First the normalization algorithm finds, by means of linear programming $v'_{1,1}$, $v' \in \mathcal{E}$ such that $\mathbb{Q}_+(\mathcal{E}) = \mathbb{Q}_+(v'_{1,1}\ ,\ v')$. Then, by dividing the coordinates of the points by their greatest common divisors, it finds the minimal integral non-zero points $v_{1,1}$, v on the corresponding rays $\mathbb{Q}_+(v'_{1,1})$, $\mathbb{Q}_+(v')$, i. e. the outcome is $v = v_{k,l_k}$ of the first paragraph of this subsection.

We execute the normalizing algorithm by recursion on i starting with the points $v_{1,1}$, v . For the base of the recursion of the algorithm we first find, by means of integer programming on the plane, an integral point $v'_{1,2} \in \mathbb{Q}_+(\mathcal{E})$ such that $|\det(v_{1,1}\ ,\ v'_{1,2})| = 1$. We then set $v_{1,2} := v'_{1,2} + \lambda \cdot v_{1,1}$ for the minimal integer λ such that $v'_{1,2} + \lambda \cdot v_{1,1} \in \mathbb{Q}_+(\mathcal{E})$, cf. Remark 7.3. Next, once again by means of integer programming, we construct $v_{1,l_1} := v_{1,1} + (l_1 - 1) \cdot (v_{1,2} - v_{1,1})$ for the largest integer l_1 such that $v_{1,l_1} \in \mathbb{Q}_+(\mathcal{E})$. Clearly, the integral points of the edge of K passing through $v_{1,1}$, $v_{1,2}$ are the points $v_{1,j} = v_{1,1} + (j-1) \cdot (v_{1,2} - v_{1,1}) \in \mathbb{Q}_+(\mathcal{E})$, $1 \le j \le l_1$.

Assuming that we have constructed the point v_{i,l_i} , and the vector v_i for an $i \ge 1$, we set (by applying integer programming) $v_{i+1,2} := (\lambda+1) \cdot v_{i,l_i} + v_i$ for the smallest integer λ such that $(\lambda+1) \cdot v_{i,l_i} + v_i \in \mathbb{Q}_+(\mathcal{E})$ (then $\lambda \ge 1$), cf Remark 7.3 . Therefore $v_{i+1} = v_{i+1,2} - v_{i,l_i}$. Then, by applying again integer programming and by following our algorithm, we set $v_{i+1,l_{i+1}} := v_{i,l_i} + (l_{i+1} - 1) \cdot v_{i+1}$ for the largest l_{i+1} such that $v_{i+1,l_{i+1}} \in \mathbb{Q}_+(\mathcal{E})$. Once again the integral points of the edge of K passing through $v_{i+1,1}$, $v_{i+1,2}$ are the points $v_{i+1,j} = v_{i+1,1} + (j-1) \cdot v_{i+1} \in \mathbb{Q}_+(\mathcal{E})$, $1 \le j \le l_{i+1}$. This completes the recursive step and the description of the normalizing algorithm.

The points $v_{i,1}$, $v_{i,2}$, $v_{i,l_i - 1}$, v_{i,l_i} provided by the algorithm lie in the triangle $\nabla(v_{1,1}\ ,\ v_{k,l_k})$. This implies that the binary sizes of these points are polynomial in the binary sizes of the input data. Now Corollary 7.5 combined with Remark 7.2 implies that the complexity of the algorithm of normalization is polynomial, as well as that of the normalized 2-dimensional Euclidean division algorithm.

Corollary 7.6. *The complexity of the normalized 2-dimensional Euclidean division algorithm along a single branch (or equivalently of the normalized Nash desingularization of affine binomial varieties of essential dimension* $m \le 2$ *) is polynomial in the binary size of the input.*

Finally, Corollary 7.5 combined with Claim 6.10 implies

Corollary 7.7. *The tree* $\mathcal{T}$ *(of Section 6) associated with the normalized 2-dimensional Euclidean algorithm applied to the normalization*

$\mathcal{N}(\mathcal{E}) \subset \mathbb{Z}^2$ *of* $\mathcal{E} \subset \mathbb{Z}^2$ *with* $\mathrm{Conv}\,(\mathcal{E}) \not\ni \mathbf{0}$ *contains less than* $O(D^{2\cdot\log_2 3} \cdot \log D) < O(D^{3.2})$ *nodes.*

We conclude this Section with two examples. The first one shows that the bounded edges of K can contain more than $D/2$ integral points, while the normalization algorithm of this section should not (and does not as we have described it) produce too many integral points on the edges, in order to proceed within the polynomial complexity (in fact it would construct at most four points on each edge).

Example 7.8. Let $v_{1,1} := (1\ ,\ 2)$, $v_{2,l_2} := (l_2\ ,\ 1)$. Obviously $D = 2 \cdot l_2 - 1$. Then K has just two bounded edges, the first of which contains two integral points $(1\ ,\ 2)$, $(1\ ,\ 1)$, while the second of which contains l_2 integral points $(i\ ,\ 1)$, $1 \le i \le l_2$.

The second example demonstrates the sharpness of the bound in Corollary 7.5.

Example 7.9. Denote $\Phi_1 := \Phi_2 := 1$ and by Φ_i the i-th Fibonacci number. Set $v_{1,1} := (\Phi_2\ ,\ \Phi_1)$, $v_{k,2} := (\Phi_{2k+2}\ ,\ \Phi_{2k+1})$. Then K has k bounded edges and the i-th among them contains just two integral points (being its endpoints) $(\Phi_{2i}\ ,\ \Phi_{2i-1})$, $(\Phi_{2i+2}\ ,\ \Phi_{2i+1})$.

8. Invariance of termination bounds.

This section is entirely devoted to the issue of the invariance of the integer D introduced in Sections 1.2 , 2 and 6 in terms of which the termination and complexity bounds are expressed. It has no evident bearing on the problem of termination of either normalized multidimensional Euclidean division or of its geometric counterpart for $m > 2$. In all three sections we considered the case of dimension $m = 2$. We associated a number D with a monomial parametrization $\mathbb{T}^m \ni x \mapsto y = \phi_{\mathcal{E}}(x) \in Y^*$, with components $y_j = (\phi_{\mathcal{E}})_j(x) := x^{\vec{\Delta}_j}$, of the torus Y^* of an essential toric subvariety Y of a binomial variety $\hat{V} \subset \mathbb{A}^N$. We expressed D in terms of the exponents $\mathcal{E} = \{\vec{\Delta}_j\}_{1 \le j \le L} \subset \mathbb{Z}^m$ of the map $\phi_{\mathcal{E}}$ as the area of a parallelogram generated by the extremal vectors. The latter are the smallest points of $\mathrm{Span}_{\mathbb{Z}}(\mathcal{E})$ on the (two) extremal rays of the cone spanned over $\mathbb{R}_+$ by the exponents in $\mathcal{E}$, see Section 6 .

Due to Theorem 3.7 , Corollary 3.5 and Claim 3.3 we may, as well, assume all exponents to be strictly positive, i. e. that $\mathcal{E} \subset \mathbb{Z}^m_+$. Also, we may assume without loss of generality that $\mathrm{Span}_{\mathbb{Z}}(\mathcal{E}) = \mathbb{Z}^m$. Recall that Y is 'essential', which means that $Y \ni \mathbf{0}$ and is equivalent to $\mathrm{Conv}\,(\mathbb{Z}_+(\mathcal{E})) \not\ni \mathbf{0}$, see Sections 3 and 4 . By *extremal vectors* for any m we mean a subset $\mathcal{E}xtremal(\mathcal{E}) \subset \mathcal{E}xtreme(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$, where $\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}} = \mathrm{Span}_{\mathbb{Z}}(\mathcal{E}) \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) \setminus \{\mathbf{0}\}$. $\mathcal{E}xtremal(\mathcal{E})$ consists of all points of $\mathcal{E}xtreme(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$, which are minimal in size on the extremal rays of the cone $\mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E})$. In terms of the exponents $\mathcal{E}$ the 'normality' property of Y is equivalent to the equality $\mathbb{Z}_+(\mathcal{E}) = \mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}}$. By construction this property is valid for both the input and the output of the normalized algorithms (Nash and/or 2-dimensional Euclidean) of Section 6 for which termination is proved. We may also assume, without loss of generality, that $\mathcal{E} = \mathcal{E}xtreme(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$ since the 'left out' exponents and corresponding affine coordinates are in $\mathbb{Z}_+(\mathcal{E}xtreme(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}}))$ and, respectively, coincide on Y with monomials in the coordinates corresponding to elements in $\mathcal{E}xtreme(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$. The definition of the number D admits a natural extension for an arbitrary m in terms of the set $\mathcal{E}$ as the smallest $D = D(\mathcal{E}) \in \mathbb{Z}_+$ such that $D \cdot \vec{\Delta}_j \in \mathbb{Z}_+(\mathcal{E}xtremal(\mathcal{E}))$ for all $\vec{\Delta}_j \in \mathcal{E}$.

Next we restate the definition of the *denominator* $D(\mathcal{E})$ as a local invariant of Y (as well as of any irreducible component V of $\hat{V}$) at any point $o \in Y$. The invariance we consider is with respect to the germs at o of local étale isomorphisms preserving coordinate hyperplanes that contain o . We restrict the variety $X := Y$ (or respectively $X := V$) to affine charts $\mathcal{U}_o$ obtained by exclusion of all coordinate hyperplanes off o , which we refer to as the *origin*. Recall that the 'y-variables' of the varieties Y , V and even of $\hat{V}$ coincide, see Section 3 and Remark 5.5 . To be precise charts $\mathcal{U}_o$ are constructed by introducing a 'double' $\tilde{z}_j$ of every affine coordinate $z_j := w_j$ with $w_j(o) \neq 0$, say $j = 1, ..., \tilde{L}$, and setting

$$\mathcal{U}_o := \{(z, \tilde{z}) \in \mathbb{A}^{2\tilde{L}} : z_j \cdot \tilde{z}_j = 1 \ , \ 1 \leq j \leq \tilde{L}\} \times \mathbb{A}^{L_o} \hookrightarrow \mathbb{A}^{L_o + 2\tilde{L}} \ ,$$

with the y-variables of the variety X being the remaining L_o variables induced by the original y-coordinates with $y_j(o) = 0$.

Then, according to Theorem 3.7 and Remark 3.12 , the germ X_o of the variety X at o is isomorphic to the product of a germ Z_a of a nonsingular subvariety Z at $a \in Z$ with a germ at $b \in (\pi|_X)^{-1}(\mathbb{I}_{2\tilde{L}}) =: \hat{Y}$ of the union of $\hat{Y}$ and possibly several, mutually isomorphic subvarieties (including the germ Y_b at b of the essential toric subvariety of X) and $o = \mu(a, b)$. Moreover, the germ Z_a is 'étale identified' with $\pi(Z_a) = \pi(X_o) \hookrightarrow \mathbb{A}^{2\tilde{L}}$ for projections $\pi : \mathbb{A}^{L_o + 2\tilde{L}} \to \mathbb{A}^{2\tilde{L}}$, whose components are the z-coordinates (Theorem 3.7 **C**).

Therefore (using the Krull completion) the morphisms $\mathcal{O}_{\pi(X_o)} \hookrightarrow \hat{\mathcal{O}}_{\pi(X_o)} \tilde{\to} \hat{\mathcal{O}}_{Z_a}$ and $(\pi|_{X_o})^* : \mathcal{O}_{\pi(X_o)} \hookrightarrow \mathcal{O}_{X_o}$ allow to consider the base change $\mathcal{R}_o := (\mathcal{O}_{X_o} \otimes_{\mathcal{O}_{\pi(X_o)}} \hat{\mathcal{O}}_{Z_a}) \otimes_{\hat{\mathcal{O}}_{Z_a}} \mathbb{F}$, where $\mathbb{F}$ is the field of fractions of $\hat{\mathcal{O}}_{Z_a}$. The morphism $\mu : Z_a \times \hat{Y}_b \to X_o$ is an étale isomorphism and $\pi|_{X_o} \circ \mu$ coincides with $\pi|_{Z_a} : Z_a \times \hat{Y}_b \ni (u \times v) \mapsto \pi|_{Z_a}(u) \in \pi|_{Z_a}(Z_a)$, while $Z_a \times Y_b$ is an irreducible component of $Z_a \times \hat{Y}_b$ and is a product of germs at a , b of the torus Z and, respectively, of the essential toric subvariety of X . Consequently, the base change above corresponds (via the étale isomorphism μ) to a base change of $Z_a \times \hat{Y}_b$ and is isomorphic to a very simple base change $\tilde{X}_b$ of $\hat{Y}_b$ via $- \otimes_{\mathbb{K}} \mathbb{F}$. Thus $\mathcal{R}_o$ is the local ring of a germ at b $(= \mathbf{0} \in \mathbb{A}^{L_o})$ of the variety $\tilde{X}$ obtained from $\hat{Y}$ by means of the base change via $- \otimes_{\mathbb{K}} \mathbb{F}$. Also, the germ at b of the base change $\tilde{Y}$ of Y via $- \otimes_{\mathbb{K}} \mathbb{F}$ is an essential toric variety and is a component of $\tilde{X}$, cf. Ch.1 [9]. We use these constructions below.

By attaching the subscript o we indicate the dependence on the new origin $o \in X$. Below we assume that all notations and assumptions of the second paragraph of this section are associated with the toric variety $X \hookrightarrow \mathcal{U}_o$; that includes the sets of the exponents $\mathcal{E}_o$ associated with the essential subvariety Y of $X \hookrightarrow \mathcal{U}_o$ and the extremal vectors $\mathcal{E}xtremal(\mathcal{E}_o) \subset \mathcal{E}_o$, as well as the numbers $m_o := \dim Y_o$ and $D_o := D(\mathcal{E}_o)$. By reindexing y_j's we may assume that $\mathcal{E}xtremal(\mathcal{E}_o) = \{y_j\}_{1 \le j \le L'_o}$. In abuse of notation we will write below $j \in \mathcal{E}xtremal(\mathcal{E}_o)$ instead of $y_j \in \mathcal{E}xtremal(\mathcal{E}_o)$.

For the sake of invariance we must consider notions which allow us to define the denominator $D(\mathcal{E}_o)$ in the respective local ring $\mathcal{O}_{X,o}$ (i.e. with X being the 'original' variety Y and/or V from the first paragraph of this section), while in $\mathcal{O}_{X,o}$ its 'defining equations' are no longer binomial. That is binomials do not generate the ideal of relations between local parameters. This is so even though we include among the latter all affine coordinates y_j with $y_j(o) = 0$, which we do since we examine the invariance with respect to the germs of local isomorphisms preserving all germs of sets $\{y_j = 0\}$. To overcome this problem we consider a base change (as above) passing to the germ $\tilde{X}_b$ of a binomial variety $\tilde{X}$ (defined over the field $\mathbb{F}$) and its local ring $\mathcal{R}_o$, whose maximal ideal $\mathfrak{m}_o$ is generated by the classes $\overline{y}_j$ in $\mathcal{R}_o$ of all affine coordinates y_j with $y_j(o) = 0$. Of course, the collection (of 'parameters') $\mathcal{P}ar(\mathcal{R}_o) := \{\overline{y}_j\}_{1 \le j \le L_o} \subset \mathfrak{m}_o$ induces a set that spans $\mathfrak{m}_o/\mathfrak{m}_o^2$ over the field $\mathbb{F}$.

Remark 8.1. $\mathcal{E}xtremal(\mathcal{P}ar(\mathcal{R}_o)) \subset \mathcal{P}ar(\mathcal{R}_o)$ can be defined in terms of $\mathcal{P}ar(\mathcal{R}_o) \subset \mathcal{R}_o$ as follows: $j \in \mathcal{E}xtremal(\mathcal{P}ar(\mathcal{R}_o))$ **iff**
(i) $\overline{y}_i^p = \overline{y}_j^q$, $(p, q) \in \mathbb{Z}_+^2$, $i \neq j$, implies $p < q$, and

(ii) $\overline{y}_j$ is not in the integral closure in $\mathcal{R}_o$ of the subring of $\mathcal{R}_o$ generated by $\overline{y}_i$'s such that $\overline{y}_i^p \neq \overline{y}_j^q$ for any $(p,q) \in \mathbb{Z}_+^2$.

Note that the ring $\mathcal{R}_o$ is the integral closure of its subring $\mathcal{R} \hookrightarrow \mathcal{R}_o$ generated by $\overline{y}_j$'s with $j \in \mathcal{E}xtremal(\mathcal{P}ar(\mathcal{R}_o))$ (using Section 2.1 of [5]). We may therefore introduce in terms of the collection $\mathcal{P}ar(\mathcal{R}_o)$ the smallest positive integer $D = D(\mathcal{P}ar(\mathcal{R}_o))$ such that for all j , $\overline{y}_j^D \in \mathcal{R}$. Obviously, the value of the *denominator* D of $\mathcal{P}ar(\mathcal{R}_o)$ coincides with $D_o = D(\mathcal{E}_o)$, where $\mathcal{E}_o$ is the collection of the exponents $\{\vec{\Delta}_j\}_j$ of any monomial map $\phi_{\mathcal{E}_o}$ (including the nonpositive exponents if there are any) parametrizing the torus Y^* of the essential subvariety Y of X . Consequently, $D(\mathcal{E}_o)$ is a local invariant due to the definition of $D = D(\mathcal{E}_o)$ being stated entirely in terms of the collection $\mathcal{P}ar(\mathcal{R}_o)$.

Remark 8.2. With reference to Section 4.3 the normalization $\mathcal{N}(Y)$ of $Y \subset \mathbb{A}^L$ is a toric variety in $\mathbb{A}^{L'}$ whose torus $\mathcal{N}(Y)^* := \mathcal{N}(Y) \cap \mathbb{T}^{L'}$ is parametrized by a map $\phi_{\mathcal{E}'} : \mathbb{T}^m \ni x \mapsto y = \phi_{\mathcal{E}'}(x) \in \mathcal{N}(Y)^*$ with components $y_j = (\phi_{\mathcal{E}'})_j(x) := x^{\vec{\Delta}_j}$. The collection of exponents, say $\mathcal{E}' := \{\vec{\Delta}_j\}_{1 \le j \le L'} \subset \mathrm{Span}_{\mathbb{Z}}(\mathcal{E}) \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) \subset \mathbb{Z}_+^m$, extends the set $\mathcal{E} = \{\vec{\Delta}_j\}_{1 \le j \le L}$ so that $\mathbb{Z}_+(\mathcal{E}') = \mathrm{Span}_{\mathbb{Z}}(\mathcal{E}) \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) \setminus \{\mathbf{0}\}$. It follows that $\mathbb{Z}_+(\mathcal{E}') = \mathrm{Span}_{\mathbb{Z}}(\mathcal{E}') \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}') \setminus \{\mathbf{0}\}$. In short, all assumptions of the following lemma (except on the size of $\mathcal{E}xtremal(\mathcal{E})$ when $m > 2$) are satisfied for Y being replaced by its normalization $\mathcal{N}(Y)$. Clearly, the elements of $\mathcal{E}xtremal(\mathcal{E}')$ and of $\mathcal{E}xtremal(\mathcal{E})$ span the same extremal rays with the extremal vectors of $\mathcal{E}xtremal(\mathcal{E}')$ being (equal or) shorter than their counterparts in $\mathcal{E}xtremal(\mathcal{E})$.

For a matrix $\mathcal{M}$ of size $m \times m$ with entries in $\mathbb{Z}$ let $\mathrm{den}\,(\mathcal{M}) \in \mathbb{Z}_+$ denote the least $d \in \mathbb{Z}_+$ such that the entries of $d \cdot \mathcal{M}^{-1}$ are integers. Obviously, the entries of the matrix $d \cdot \mathcal{M}^{-1}$ generate the unit ideal in $\mathbb{Z}$. Also, if $m = 2$ and the entries of $\mathcal{M}$ have no common divisor then $\mathrm{den}\,(\mathcal{M}) = |\det(\mathcal{M})|$. Recall that a matrix whose columns are elements of the collection $\mathcal{E} \subset \mathbb{Z}^m$ are denoted by the same letter $\mathcal{E}$.

Lemma 8.3. *If* $\mathrm{Span}_{\mathbb{Z}}(\mathcal{E}) = \mathbb{Z}^m$ *,* $\mathbb{Z}^m \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) \setminus \{\mathbf{0}\} = \mathbb{Z}_+(\mathcal{E})$ *and* $\#(\mathcal{E}xtremal(\mathcal{E})) = m$ *it follows that* $D(\mathcal{E}) = \mathrm{den}\,(\mathcal{E}xtremal(\mathcal{E}))$ *.*

Remark 8.4. Of course, if $\#(\mathcal{E}xtremal(\mathcal{E})) = m$ and $D(\mathcal{E}) = 1$, then the affine variety Y being of dimension m must be nonsingular and, if $m = 2$, then $\#(\mathcal{E}xtremal(\mathcal{E})) = m$ and $D(\mathcal{E}) = |\det(\mathcal{E}xtremal(\mathcal{E}))|$.

Proof. The inclusion $\mathrm{den}\,(\mathcal{E}xtremal(\mathcal{E})) \in D(\mathcal{E}) \cdot \mathbb{Z}$ is a simple consequence of the definitions. Therefore it suffices to show that for any prime number p and $s \in \mathbb{Z}_+$ if $\mathrm{den}\,(\mathcal{E}xtremal(\mathcal{E})) \in p^s \cdot \mathbb{Z}$ then $D(\mathcal{E}) \in p^s \cdot \mathbb{Z}$. Let $\mathcal{M} := \mathrm{den}\,(\mathcal{E}xtremal(\mathcal{E})) \cdot (\mathcal{E}xtremal(\mathcal{E}))^{-1}$. Then the entries of $\mathcal{M}$ generate the unit ideal in $\mathbb{Z}$ and, therefore, there is a column $\vec{\lambda}$ of the matrix $\mathcal{M}$ with a nonvanishing mod p entry. We modify the latter column to $\vec{\lambda}' := \vec{\lambda} + p^s \cdot t \cdot \mathbb{I}_m$ with a sufficiently large positive $t \in \mathbb{Z}_+$ so as to make all entries of $\vec{\lambda}'$ positive. It follows that $\vec{\lambda}' \neq \mathbf{0}$ $(mod\ p)$. Therefore the vector $\mathcal{E}xtremal(\mathcal{E}) \cdot \vec{\lambda}' \in (p^s \cdot \mathbb{Z}^m) \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) \setminus \{\mathbf{0}\}$. It follows that $D(\mathcal{E}) \in p^s \cdot \mathbb{Z}$, as required. □

Corollary 8.5. *The denominator $D(\mathcal{E})$ of the essential subvariety of a binomial variety $\hat{V}$ is the bound D appearing in Section 1.2 for $m = 2$ (and is a local integral invariant of $\hat{V}$ at $\mathbf{0}$).*

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9. **Appendix:** Length bound $1 + \log_2(\#\Gamma)$ on normalized Nash resolution with Γ the dual graph of the minimal resolution of a minimal surface singularity - **by M. Spivakovsky.**

Let (S, ξ) be a normal surface singularity and $\pi : X \to S$ its minimal desingularization.

Definition 9.1. The set $\pi^{-1}(\xi) \subset X$ is called **the exceptional divisor** of the resolution of singularities π.

The exceptional divisor is a curve on X, which may, in general, be reducible. Let $\pi^{-1}(\xi) = \bigcup\limits_{i=1}^{n} E_i$ be its decomposition into irreducible components. Two basic combinatorial invariants are usually associated to the singularity (S, ξ): the dual graph and the intersection matrix. The **dual graph** has vertices $\{x_i\}_{1 \leq i \leq n}$, one for each irreducible exceptional curve E_i; two vertices x_i and x_j are connected by an arc if and only if $E_i \cap E_j \neq \emptyset$. The **intersection matrix** is the $n \times n$ matrix $(E_i.E_j)$. Since (S, ξ) is normal, Zariski's main theorem implies that the exceptional divisor, and hence also the dual graph, are connected.

By a well-known theorem of Mumford and Grauert, the intersection matrix $(E_i.E_j)$ is negative definite.

Remark 9.1. We note the following consequences of the Mumford–Grauert theorem:

(1) We have $E_i^2 < 0$ for all $i \in \{1, \dots, n\}$.
(2) Take an index $i \in \{1, \dots, n\}$ and assume that $E_i \cong \mathbb{P}^1$. Then $E_i^2 \leq -2$. Indeed, if we had $E_i^2 = -1$ then such an exceptional curve could be contracted to a non-singular point by Castelnuovo's criterion, which would contradict the minimality of the desingularization π.
(3) There exists a cycle of the form

$$Z = \sum_{i=1}^{n} m_i E_i, \tag{9.1}$$

such that all the m_i are strictly positive integers and $Z.E_i \leq 0$ for all $i \in \{1, \dots, n\}$.

Among all the cycles Z satisfying (9.1), we can choose one which is *componentwise* minimal. Such a cycle is uniquely determined by the intersection matrix; it is called **the fundamental cycle** of the singularity (S, ξ).

Definition 9.2. The singularity (S, ξ) is called **minimal** if $E_i \cong \mathbb{P}^1$ for all $i \in \{1, \dots, n\}$, the intersections $E_i \cap E_j$ are transverse (whenever $E_i \cap E_j \neq \emptyset$), the dual graph of (S, ξ) is simply connected and the fundamental cycle Z is reduced (that is, $m_i = 1$ for all $i \in \{1, \dots, n\}$).

For more information on minimal singularities, we refer the reader to the article [11] by Janos Kollar where they were originally defined.

Definition 9.3. The singularity (S, ξ) is **a cyclic quotient** if each exceptional curve E_i intersects at most two other exceptional curves.

It follows easily from the definitions and Remark 9.1 (2) that every cyclic quotient singularity is minimal. The cyclic quotient singularities are precisely the toric ones among normal surface singularities (that is, they are precisely those normal surface singularities which can be defined by a binomial ideal in the ambient space). As the name suggests, they are also characterized by the fact that they can be obtained as quotients of the germ of a variety at a non-singular point by the action of a finite cyclic group.

Let (S, ξ) be a minimal singularity. For a graph Γ, the notation $\#\Gamma$ will stand for the number of vertices of Γ. For example, if Γ is the dual graph of ξ, we have $\#\Gamma = n$.

Theorem 9.2. ([15], Lemma 2.5, p. 442) Let $\sigma : S' \to S$ denote the normalized Nash blowing up of S, let ξ' be a singular point of S' and Γ' its dual graph. Then (S', ξ') is also a minimal singularity and

$$\#\Gamma' \leq \frac{n}{2}\,. \tag{9.2}$$

This bound is sharp in the sense that there are many examples for which equality holds in (9.2).

The simplest example of equality in (9.2) is the following. Let (S, ξ) be the A_n singularity with n even. This is the singularity defined in the three dimensional space by the equation $xy - z^{n+1}$. It can be obtained as the quotient of the two-dimensional space with coordinates (u, v) by the cyclic group action $(u, v) \to (\zeta u, \zeta^{-1} v)$, where ζ is the n-th root of unity. The dual graph of this singularity consists of n vertices, arranged in a straight line. The intersection matrix is given by

$$E_i^2 = -2, \quad i \in \{1, \ldots, n\}; \tag{9.3}$$

$$E_i.E_{i+1} = 1 \quad \text{for } i \in \{1, \ldots, n-1\} \tag{9.4}$$

$$E_i.E_j = 0 \quad \text{for all the other choices of } i, j \in \{1, \ldots, n\}. \tag{9.5}$$

As is shown in [6], the normalized Nash blowing up S' of (S, ξ) has two singular points ξ_1, ξ_2 of multipliciy three, and the dual graph of each of the singularities (S', ξ_1), (S', ξ_2) has $\frac{n}{2}$ vertices.

Corollary 9.3. The singularity (S, ξ) is resolved after at most $[\log_2 n]+1$ normalized Nash blowings up.

Proof of the Corollary: Let $l = [\log_2 n]+1$. Consider the sequence

$$S_l \overset{\sigma_l}{\to} S_{l-1} \overset{\sigma_{l-1}}{\to} \ldots \overset{\sigma_2}{\to} S_1 \overset{\sigma_1}{\to} S$$

of normalized Nash blowings up. We claim that S_l is non-singular. To see this, we will assume that S_l contains a singular point ξ_l and deduce a contradiction. Let ξ_i denote the image of ξ_l in S_i, $0 \leq i \leq l$ (we adopt the convention that $S_0 = S$ and $\xi_0 = \xi$). Let n_i denote the number of vertices in the dual graph of ξ_i. Since ξ_l is assumed to be singular, we have $n_l \geq 1$. By Theorem 9.2 and descending induction on i, we obtain $n_i \geq 2^{l-i}$ so, in particular, $n \geq 2^l$, that is, $l \leq \log_2 n$. This contradicts the definition of l. □

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