Tropical Newton-Puiseux polynomials

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Abstract

We introduce tropical Newton-Puiseux polynomials admitting rational exponents. A resolution of a tropical hypersurface is defined by means of a tropical Newton-Puiseux polynomial. A polynomial complexity algorithm for resolubility of a tropical curve is designed. The complexity of resolubility of tropical prevarieties of arbitrary codimensions is studied. Tropical Newton-Puiseux rational functions are introduced, and we prove that any tropical polynomial has a resolution in tropical Newton-Puiseux rational functions (this can be treated as a tropical analog of the algebraic closedness of the field of Newton-Puiseux series).

Introduction

Recall (see e. g. [6]) that in the tropical semiring $\oplus$ denotes min and $\otimes$ denotes the (classical) addition $+$. As examples of tropical semirings one can take $\mathbb{Z}$, $\mathbb{R}$. A *tropical* (respectively, *tropical Laurent) monomial* has the form

$$a \otimes x^{\otimes I} := a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$$

where $a \in \mathbb{R}$ and $0 \leq i_1, \ldots, i_n \in \mathbb{Z}$ (respectively, $i_1, \ldots, i_n \in \mathbb{Z}$). Thus, classically $a \otimes x^{\otimes I}$ equals a linear function $a + \sum_{1 \leq j \leq n} i_j \cdot x_j$. A *tropical polynomial* f has the form $\bigoplus_I a_I \otimes x^{\otimes I}$, being classically a convex piece-wise linear function.

A vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is a *tropical root* of f if the minimum of $a_I \otimes \mathbf{x}^{\otimes I}$ is attained at least for two different tropical monomials of f. The set of all tropical roots of f constitute a *tropical hypersurface* $T(f) \subset \mathbb{R}^n$ being a finite union of polyhedra of dimensions $n - 1$.

We extend the concept of a tropical polynomial by allowing the exponents $i_1, \dots, i_n$ to be rational calling it a *tropical Newton-Puiseux polynomial.* Assume that

$$f = \bigoplus_{0 \le i \le d} f_i \otimes y^{\otimes i} \tag{1}$$

for some tropical polynomials $f_0, \dots, f_d$ in the variables $x_1, \dots, x_n$. We call a Newton-Puiseux polynomial y a *resolution* of f (or of the tropical hypersurface $T(f)$) if for any point $\mathbf{x} \in \mathbb{R}^n$ the point $(\mathbf{x}, y(\mathbf{x})) \in \mathbb{R}^{n+1}$ provides a tropical root of f (the formal definitions one can find below in Section 1).

This resembles Newton-Puiseux series from algebraic geometry with the difference that we consider finite supports since in the tropical semiring one takes min. Tropical Newton-Puiseux polynomials can be viewed as a tropical analog of algebraic functions.

In Section 1 we show that the set of all the resolutions of a tropical hypersurface is finite and closed under taking min. Thus, there exists a minimal resolution, and in case of a monic tropical polynomial

$$f = y^{\otimes d} \oplus \bigoplus_{0 \le i < d} f_i \otimes y^{\otimes i}$$

we provide a simple formula for the minimal resolution. In addition, a geometric description of resolutions is given. Also we show that the resolubility of a tropical hypersurface belongs to the complexity class NP.

In Section 2 a polynomial (bit-size) complexity algorithm is exhibited for resolving degree 1 tropical polynomials of the form $f_1 \otimes y \oplus f_0$, which is equivalent to the divisibility of f_0 by f_1.

In Section 3 we design a polynomial (bit-size) complexity algorithm for testing resolubility of a tropical curve in a real space of a fixed dimension, moreover the algorithm provides a succinct description of the set of all the resolutions.

In Section 4 we study the problem of resolubility of a system of tropical polynomials in a single variable x and in several indeterminates $y_1, \dots, y_s$ and establish its NP-hardness.

In Section 5 we study tropical Newton-Puiseux rational functions, being tropical quotients (or in other words, the classical subtraction) of pairs of tropical Newton-Puiseux polynomials. We prove (see Remark 1.8) a tropical analog of the algebraic closedness of the field of Newton-Puiseux series, namely, that any tropical polynomial f (see (2)) has a resolution in tropical Newton-Puiseux rational functions, and moreover, provide an explicit formula

$$\bigoplus_{0 \le i \le d} (f_{d-i} \oslash f_d)^{\otimes (1/i)}$$

for the minimal resolution of f. Also an algorithm is suggested which tests resolubility of a tropical curve by means of tropical Newton-Puiseux rational functions. The complexity of the algorithm is polynomial for a fixed dimension of the ambient space.

1 Resolution of a tropical hypersurface

Let an algebraic (classical) equation

$$F := \sum_{0\leq i\leq d} F_i \cdot Y^i = 0 \tag{2}$$

where the coefficients $F_i \in K[X_1, \dots, X_n]$ for the field $K = \mathbb{C}((t^{1/\infty}))$ of Newton-Puiseux series, have a Laurent polynomial solution

$$Y = \sum_I A_I \cdot X^I \tag{3}$$

with a finite sum over multiindices $I \in \mathbb{Z}^n$ and the coefficients $A_I \in K$.

Denote the *tropicalization*

$$Trop(Y) := \bigoplus_I Trop(A_I) \otimes X^{\otimes I} \tag{4}$$

where for a Newton-Puiseux series $A_I = \sum_{0\leq j<\infty} b_j \cdot t^{s_j/q}$ with $b_j \in \mathbb{C}$, $b_0 \neq 0$ and increasing integers $s_0 < s_1 < \dots$ its tropicalization $Trop(A_I) := s_0/q \in \mathbb{Q}$.

Remark 1.1 *$Trop(Y)$ is a solution of the tropical equation*

$$\bigoplus_{0\leq i\leq d} Trop(F_i) \otimes (Trop(Y))^{\otimes i} \tag{5}$$

This means that for any point $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ the minimal value of $Trop(F_i) \otimes (Trop(Y))^{\otimes i}$ at $\mathbf{x}$ for $0 \leq i \leq d$ is attained at least for two different $0 \leq i_1 < i_2 \leq d$.

Remark 1.2 *Observe that the validity of (5) would not change if one multiplies all the rational coefficients in $Trop(F_i)$, $0 \leq i \leq d$ by their common denominator m and simultaneously all $Trop(A_I)$ (see (4)) by m to make all the coefficients in $Trop(F_i)$, $0 \leq i \leq d$ integers.*

Remark 1.1 motivates the following definition.

Definition 1.3 *A tropical hypersurface $T(f) \subset \mathbb{R}^{n+1}$ defined by a tropical polynomial (1) where f_i are tropical polynomials in the variables $x_1, \dots, x_n$ with integer coefficients (cf. Remark 1.2) has a resolution being a tropical Newton-Puiseux polynomial*

$$y = \bigoplus_I a_I \otimes x^{\otimes I} \tag{6}$$

for a finite sum over multiindices $I \in \mathbb{Q}^n$ and $a_I \in \mathbb{Q}$, if for any point $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ the minimal value among $f_i \otimes y^{\otimes i}$, $0 \le i \le d$ (treated as piecewise linear functions) at $\mathbf{x}$ is attained at least for two different $0 \le i_1 < i_2 \le d$.

Denote by N the common denominator of all the rational coordinates of multiindices I from (6). Then $y^{\otimes N}$ is a tropical (Laurent) polynomial which equals classically to $N \cdot \min_I \{a_I + i_1 x_1 + \cdots + i_n x_n\}$.

Proposition 1.4 *Let y be a resolution of f (see (1), (6)), then $(\mathbf{x}, y(\mathbf{x})) \in T(f)$.*

Example 1.5 *The tropical polynomial $f = y \oplus x \oplus 0$ has a resolution $y = x \oplus 0$. Its graph $\{(\mathbf{x}, y(\mathbf{x})) : \mathbf{x} \in \mathbb{R}\} \subset T(f) \subset \mathbb{R}^2$ consists of two half-lines, while the tropical curve $T(f)$ consists of three half-lines.*

Proposition 1.6 *Let y (see (6)) and $\bigoplus_I b_I \otimes x^{\otimes I}$ be resolutions of (1). Then $\bigoplus_I (a_I \oplus b_I) \otimes x^{\otimes I}$ is also a resolution of (1).*

The proof follows from an observation that for any point $\mathbf{x} \in \mathbb{R}^n$ the minimum on the tropical monomials after opening the parenthesis in a power $y^{\otimes i}$ (see (6)) is attained on the powers of the kind $(a_I \otimes \mathbf{x}^{\otimes I})^{\otimes i}$. $\Box$

Below in Remark 1.10 we show that there is at most a finite number of resolutions of (1). Hence according to Proposition 1.6, there exists a minimal resolution.

Proposition 1.7 *If*

$$f = y^{\otimes d} \oplus \bigoplus_{0 \le i < d} f_i \otimes y^{\otimes i}$$

(see (1)) is monic then

$$y = \bigoplus_{1 \le i \le d} f_{d-i}^{\otimes (1/i)}$$

is the minimal resolution.

Proof. For any point $\mathbf{x} \in \mathbb{R}^n$ the minimal $y_0 \in \mathbb{R}$ such that $(\mathbf{x}, y_0)$ belongs to the tropical hypersurface $T(f) \subset \mathbb{R}^{n+1}$ satisfies a (classical) equation $d \cdot y_0 = f_i(\mathbf{x}) + i \cdot y_0$ for suitable $0 \le i < d$ (due to analyzing the Newton polygon). $\Box$

Note also that if $f_{d-i} = \bigoplus_J c_J \otimes x^{\otimes J}$ then $f_{d-i}^{\otimes (1/i)} = \bigoplus_J (c_J / i) \otimes x^{\otimes (J/i)}$.

Remark 1.8 *When f is not monic, a resolution does not necessary exist as in the example $f = (x \oplus 0) \otimes y \oplus 0$. On the other hand, one can write a similar formula*

$$y = \bigoplus_{1 \le i \le d} (f_{d-i} \oslash f_d)^{\otimes(1/i)}$$

where $\oslash$ stays for the tropical division, i. e. the classical subtraction. In this case y is not necessary a convex function, while being piece-wise linear (we call them tropical Newton-Puiseux rational functions, see Section 5), and y provides the minimal resolution of f. This can be treated as a tropical analog of the algebraic closedness of the field of Newton-Puiseux series.

Now we proceed to a geometric description of resolutions. Let (6) be a resolution of (1). Assume that for some I the (convex) polyhedron $M_I \subset \mathbb{R}^n$ of points at which the (tropical) monomials $\{a_J \otimes x^{\otimes J}\}_J$ of y attain the minimum for $a_I \otimes x^{\otimes I}$, has the full dimension n. Observe that if M_I has a dimension less than n one can discard the monomial $a_I \otimes x^{\otimes I}$ from y.

Assume that for some $0 \le i_1 < i_2 \le d$ and a pair of monomials $c_{i_1,I_1} \otimes x^{\otimes I_1}$, $c_{i_2,I_2} \otimes x^{\otimes I_2}$ from the polynomials f_{i_1}, f_{i_2}, respectively, it holds

$$I_1 + i_1 \cdot I = I_2 + i_2 \cdot I;\ c_{i_1,I_1} + i_1 \cdot a_I = c_{i_2,I_2} + i_2 \cdot a_I, \tag{7}$$

in other words, the monomials

$$(c_{i_1,I_1} \otimes x^{\otimes I_1}) \otimes (a_I \otimes x^{\otimes I})^{\otimes i_1} = (c_{i_2,I_2} \otimes x^{\otimes I_2}) \otimes (a_I \otimes x^{\otimes I})^{\otimes i_2}$$

coincide. Consider the convex polyhedron $M_{I,i_1,I_1,i_2,I_2} \subset M_I$ of the points from M_I at which the minimum of the monomials $(c_{i,I} \otimes x^{\otimes I}) \otimes (a_I \otimes x^{\otimes I})^{\otimes i}$ for the monomials $c_{i,I} \otimes x^{\otimes I}$ from f_i, $0 \le i \le d$ is attained for $(c_{i_1,I_1} \otimes x^{\otimes I_1}) \otimes (a_I \otimes x^{\otimes I})^{\otimes i_1}$. We get the following lemma.

Lemma 1.9 *Let (6) be a resolution of (1) and the polyhedron $M_I \subset \mathbb{R}^n$ have the full dimension n. Then the polyhedra M_{I,i_1,I_1,i_2,I_2} having the full dimension n constitute a partition of M_I, i. e. every two elements of the partition either coincide or intersect by a set (face) of dimension less than n.*

It would be interesting to clarify, how many resolutions a tropical hypersurface might have?

Let the tropical degrees $trdeg(f_i) \le D$, $0 \le i \le d$.

Remark 1.10 *The problem of resolving a tropical polynomial (1) belongs to the complexity class NP. This follows from the observation that each coefficient a_I satisfies (7) (or equals infinity), and therefore, there are at most $d^2 \cdot \binom{D+n}{n}$ possibilities for a_I, taking into account that $\binom{D+n}{n}$ bounds the number of monomials in each f_i.*

Note that when f_i, $0 \leq i \leq d$ are in sparse encoding, in the latter bound one can replace $\binom{D+n}{n}$ by the number of monomials in f_i, $0 \leq i \leq d$. Thus, the problem of resolubility of (1) belongs to NP for both dense and sparse encodings of (1).

It would be interesting to say more about the complexity of resolubility of (1).

Remark 1.11 *One can extend the results of this section to an input tropical Newton-Puiseux polynomials in place of (1).*

2 Polynomial complexity testing divisibility of tropical polynomials

If (1) has degree 1, i. e. $f = f_1 \otimes y \oplus f_0$ then according to (7) a resolution (6) is equivalent to the divisibility $f_1 \otimes y = f_0$ with y being a tropical Laurent polynomial. We agree that two tropical (Laurent) polynomials are equal if they are equal as (convex piece-wise linear) functions.

We expose an algorithm for testing divisibility within polynomial complexity. First the algorithm deletes from f_0 all the monomials of the form $b \otimes x_1^{b_1} \otimes \cdots \otimes x_n^{b_n}$ which do not change f_0 as a function. Geometrically, it means that the hyperplane defined as the graph

$$\{(x_1, \dots, x_n, \sum_{1 \leq j \leq n} b_j \cdot x_j + b) : (x_1, \dots, x_n) \in \mathbb{R}^n\}$$

of this monomial in $\mathbb{R}^{n+1}$ is higher (with respect to the last coordinate) than the polyhedron P defined by the other monomials of f_0 (observe that P is the graph of f_0 as a function). The latter is a problem of linear programming. Thus, one can suppose f_0 to be reduced, i. e. do not contain unnecessary monomials. Also we suppose that f_1 is reduced.

For every candidate $I = (i_1, \dots, i_n) \in \mathbb{Z}^n$, $\sum_{1 \leq j \leq n} |i_j| \leq D$ (see (7)) to be in the support of a resolution y the algorithm calculates (again involving linear programming) the minimal a_I such that for each monomial $c \otimes x^{\otimes C}$ of f_1 the hyperplane in $\mathbb{R}^{n+1}$ defined by the monomial $(c \otimes x^{\otimes C}) \otimes (a_I \otimes x^{\otimes I})$ is (non-strictly) higher than P.

Then $y = \bigoplus_I a_I \otimes x^{\otimes I}$ is a resolution of $f_1 \otimes y \oplus f_0$ iff for each monomial $b \otimes x^{\otimes B}$ of f_0 there exists I and a monomial $c \otimes x^{\otimes C}$ of f_1 such that $(a_I \otimes x^{\otimes I}) \otimes (c \otimes x^{\otimes C}) = b \otimes x^{\otimes B}$. Reducing further y as described above, we conclude that there is a unique reduced resolution y (provided that it does exist).

Summarizing, we obtain the following proposition.

Proposition 2.1 *One can test resolubility of degree 1 tropical polynomial $f_1 \otimes y \oplus f_0$ (or equivalently, the divisibility $f_1 \otimes y = f_0$) within polynomial complexity. In case of the divisibility the algorithm yields the unique reduced resolution y.*

3 Polynomial complexity algorithm for resolving tropical curves

Let a system of tropical polynomials

$$f_i,\ 1 \le i \le k \tag{8}$$

in n variables $x, y_1, \dots, y_{n-1}$ with integer coefficients determine a tropical prevariety $T := T(f_1, \dots, f_k) \subset \mathbb{R}^n$. Let the tropical degrees $trdeg(f_i) \le d$, $1 \le i \le k$ and the bit-sizes of the coefficients of f_i, $1 \le i \le k$ do not exceed L.

First, the algorithm constructs T as a union of polyhedra (see e. g. [6]). Each of these polyhedra (including faces of all the dimensions) is defined by specifying the monomials of f_i, $1 \le i \le k$ (treated as linear functions) on which the minima are attained (cf. e. g. [4]). The algorithm can find the partition of $\mathbb{R}^n$ into polyhedra defined by given feasible tuples of signs (i. e. either the positive, either the negative or zero) of all the differences of the monomials of f_i, $1 \le i \le k$ (in other words, by all the feasible orderings of the monomials of f_i, $1 \le i \le k$). Namely, the algorithm finds the partition by recursion on the number of the differences. If for a current subset of the differences the partition of $\mathbb{R}^n$ w.r.t. this subset is already constructed, the algorithm picks up the next difference and for each element (being a polyhedron) of the current partition verifies which signs of the picked up difference are feasible on this polyhedron (with the help of linear programming). Thereupon, the algorithm discards the unfeasible tuples of signs, which completes the recursive step.

The number of the elements of a current partition at every step of the recursion is bounded by

$$n^2 \cdot 2^n \cdot \binom{k \cdot \binom{d+n}{n}^2}{n} < k^n \cdot d^{2 \cdot n^2}$$

due to the Buck's formula on the number of faces in an arrangement of hyperplanes [3]. Hence the complexity of the recursion is bounded by a polynomial in L, k^n, d^{n^2} because the algorithm invokes linear programming the number of times being polynomial in $k^n \cdot d^{n^2}$.

Since the tropical prevariety T is a union of appropriate subset of the elements of the constructed partition of $\mathbb{R}^n$, we get the following proposition.

Proposition 3.1 *There is an algorithm which constructs the tropical prevariety $T(f_1, \dots, f_k) \subset \mathbb{R}^n$ determined by (8) within the complexity polynomial in L, k^n, d^{n^2}.*

Now we assume that $\dim T = 1$, thus T is a tropical curve. We design an algorithm which verifies the resolubility of T, i. e. whether there exist tropical Newton-Puiseux polynomials $y_1(x), \dots, y_{n-1}(x)$ assuring a resolution of (8). The latter is equivalent to that every piece-wise linear function y_j, $1 \leq j \leq n-1$ is convex.

The algorithm produces a directed graph G whose vertices being the edges of T (including the unbounded ones) not lying in a hyperplane of the form $x = c$. Two edges $e^{(-)}$, $e^{(+)}$ of T (being vertices of G) with the same endpoint of the kind

$$e^{(-)} = ((x^{(-)}, y_1^{(-)}, \dots, y_{n-1}^{(-)}),\ (x, y_1, \dots, y_{n-1})),$$

$$e^{(+)} = ((x, y_1, \dots, y_{n-1}),\ (x^{(+)}, y_1^{(+)}, \dots, y_{n-1}^{(+)}))$$

are linked by an edge directed from $e^{(-)}$ to $e^{(+)}$ in G if

$$x^{(-)} < x < x^{(+)}; \quad \frac{y_j - y_j^{(-)}}{x - x^{(-)}} \geq \frac{y_j - y_j^{(+)}}{x - x^{(+)}},\ 1 \leq j \leq n-1. \tag{9}$$

When $e^{(-)}$ (respectively, $e^{(+)}$) is unbounded with an endpoint $(x, y_1, \dots, y_{n-1})$ (so, is a half-line), which we call unbounded from the left, we take an arbitrary point of $e^{(-)}$ with $x^{(-)} < x$ (respectively, if $e^{(+)}$ is a half-line, we take a point of $e^{(+)}$ with $x^{(+)} > x$, and we call $e^{(+)}$ unbounded from the right). When an edge of T has no endpoints, so is a line, it provides a resolution of T.

After that the algorithm produces a subset S of the vertices of G. It starts with including into S all the edges of T (so, the vertices of G) unbounded from the left (denote this set by S_0). Thereupon, the algorithm includes into S all the vertices of G reachable from S_0. If a vertex of G corresponding to an edge of T unbounded from the right, belongs to S, a path in G leading to such a vertex from S_0 provides a resolution of T (i. e. each piece-wise linear function $y_j(x)$, $1 \leq j \leq n-1$ corresponding to the path, is convex due to (9)). Moreover, the paths in G from S_0 to the vertices corresponding to the edges of T unbounded from the right, are in a bijective correspondence with the resolutions of T.

Summarizing and taking into account Proposition 3.1, we obtain the following theorem.

Theorem 3.2 *There is an algorithm which tests resolubility of a tropical curve $T \subset \mathbb{R}^n$ determined by (8), and in case of the resolubility yields a resolution. The complexity of the algorithm is polynomial in L, k^n, d^{n^2}. In particular, the complexity is polynomial for a fixed ambient dimension n.*

Remark 3.3 *Let a system of tropical polynomials of the form (8) depend on the variables $x_1, \dots, x_m, y_1, \dots, y_n$ and the tropical prevariety $T \subset \mathbb{R}^{m+n}$*

have dimension m. Then one can try different subsets of all m-dimensional faces of T as candidates to constitute a graph of a resolution

$$(x_1, \dots, x_m) \to (x_1, \dots, x_m, y_1(x_1, \dots, x_m), \dots, y_n(x_1, \dots, x_m)) \in T$$

of T similar to Remark 1.10. The latter, in fact, means that firstly, the projections of the chosen m-dimensional faces on $\mathbb{R}^m$ with the coordinates $x_1, \dots, x_m$ form a partition of $\mathbb{R}^m$ and secondly, that each piece-wise linear function $y_j(x_1, \dots, x_m)$, $1 \le j \le n-1$ is convex.

4 Resolution of systems of tropical polynomials with several indeterminates

In this section we consider systems of tropical polynomials (instead of a single polynomial (1)) in one variable x and several indeterminates $y_1, \dots, y_s$. Thus, in a resolution (cf. (6)) each y_i is a tropical Newton-Puiseux polynomial. We show the following proposition.

Proposition 4.1 *The problem of resolubility of a system of tropical polynomials in a single variable and in several indeterminates is NP-hard.*

Proof. We reduce 3-SAT to the problem under consideration, so we construct a system R of tropical polynomials. For an instance of 3-SAT problem in n variables $u_1, \dots, u_n$ we introduce indeterminates $y_1, \dots, y_n$, $z_1, \dots, z_n$ and add to R tropical polynomials

$$y_i \otimes z_i \oplus x,\ 1 \le i \le n \tag{10}$$

Formula (10) means that the resolutions of y_i and of z_i are both monomials in x. Informally, $0 = x^{\otimes 0}$ encodes the truth and $x = x^{\otimes 1}$ encodes the falsity, y_i corresponds to u_i and z_i corresponds to $\neg u_i$.

For every j-th 3-clause of the 3-SAT formula, say, $u_m \vee \neg u_k \vee u_l$ we add to R the following tropical (linear) polynomials

$$y_m \oplus z_k \oplus y_l \oplus v_j; \tag{11}$$

$$v_j \oplus x^{\otimes 1} \oplus w_j; \tag{12}$$

$$w_j \oplus x^{\otimes 1} \oplus 0 \tag{13}$$

with indeterminates v_j, w_j. Note that (13) ensures that in a resolution the reduced $w_j = x^{\otimes 1} \oplus 0$, then (12) ensures that the reduced v_j contains the constant monomial 0 (and possibly, monomials of the form $c \otimes x^{\otimes b}$ with $0 < b \leq 1$, $c \geq 0$). Finally, (11) ensures that one of the resolutions of y_m, z_k, y_l equals 0.

Thus, existence of a resolution of the system R for all j implies the solvability of the initial 3-SAT formula.

The converse is obvious: for a Boolean vector $(u_1, \dots, u_n)$ providing a solution of the initial 3-SAT formula put $y_i = 0$, $z_i = x^{\otimes 1}$ when u_i is true and $y_i = x^{\otimes 1}$, $z_i = 0$ for u_i being false. Thereupon put $v_j = y_m \oplus z_k \oplus y_l$. $\Box$

We mention that the problem of solvability of a system of tropical polynomials is NP-complete [8].

It would be interesting to understand more about the complexity of the problem under consideration in this section.

5 Tropical Newton-Puiseux rational functions

Any tropical Newton-Puiseux rational function $f_1 \oslash f_2$ where f_1, f_2 are tropical Newton-Puiseux polynomials, is a piece-wise linear (continuous) function (cf. Remark 1.8). The converse is also true (see e. g. [1], [5]): any piece-wise linear continuous function is a difference of two piece-wise linear convex functions. In [7] an algorithm is suggested which represents a piece-wise linear function as a difference of piece-wise linear convex functions with the complexity bound being exponential. In case of one-variable functions a polynomial complexity algorithm for this problem is exhibited in [2].

Let a tropical curve $T \subset \mathbb{R}^n$ be determined by a system (8). As in Section 3 the algorithm finds T. Thereupon, similar to Section 3 constructs a graph which comprises all the paths consisting of the edges of T of the form

$$\{(x_l, y_1^{(l)}, \dots, y_{n-1}^{(l)}),\ (x_{l+1}, y_1^{(l+1)}, \dots, y_{n-1}^{(l+1)}) \,:\, 0 \leq l \leq s\}$$

where $x_0 := -\infty < x_1 < \cdots < x_s < x_{s+1} := \infty$, thus, this path contains $s+1$ edges. The difference with Section 3 is that now we do not impose a requirement on convexity.

The algorithm can pick up any such path (provided that it does exist), then this path yields $n-1$ piece-wise linear functions $y_i(x)$, $1 \leq i < n$. Making use of [2] the algorithm represents $y_i(x) = g_i(x) - h_i(x)$ with piece-wise linear convex functions g_i, h_i. This produces a tropical Newton-Puiseux rational function resolution of T.

Summarizing and invoking the complexity bounds from Section 3, we get the following proposition.

Proposition 5.1 *There is an algorithm which tests resolubility of a tropical curve determined by (8) by means of tropical Newton-Puiseux rational functions within the complexity polynomial in L, k^n, d^{n^2}. The algorithm yields a* *resolution, provided that it does exist. Therefore, the complexity is polynomial for a fixed dimension n of the ambient space.*

It would be interesting to estimate the complexity of resolubility of tropical prevarieties or arbitrary dimensions by means of tropical Newton-Puiseux rational functions.

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