TIME COMPLEXITY OF MULTIDIMENSIONAL TURING MACHINES

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It is proved that the work of an indeterminate m-dimensional Turing machine with time complexity t can be simulated on an indeterminate k-dimensional ($k \le m$) Turing machine with time complexity $t^{1-(1/m)+(1/k)+\varepsilon}$ (for any $\varepsilon > 0$). Moreover, the following generalization to the multidimensional case of the familiar theorem of Hopcroft, Paul, and Valiant is proved: the work of an m-dimensional Turing machine with time complexity $t \log^{1/m} t$ [$t(n) \ge n$] can be simulated on an address machine working with time complexity t.

In the present paper it is proved that the work of an indeterminate m-dimensional Turing machine with time complexity t can be simulated on an indeterminate k-dimensional ($k \le m$) Turing machine with time complexity $t^{1+(1/k)-(1/m)+\varepsilon}$ (for any $\varepsilon > 0$).

In addition, it is remarked that the familiar result [1] on the time gain in passing from Turing machines to machines with arbitrary access to the memory (in other words, random access machines, RAM, cf. [2]) can be generalized to the multidimensional case, more precisely, to simulate an m-dimensional Turing machine working with time complexity $t \log^{1/m} t$ [$t(n) \ge n$ for any n], on a RAM with time complexity t. Moreover, the last simulation can be effected on the apparatus introduced by Slisenko and called in [3] an address machine (AM). It is a specification of a RAM and is characterized by the fact that in the course of the entire work to its conclusion, the length of the registers used does not exceed $\log_2 t + c$, where t is the time of work (the number c is fixed for a given AM).

By DTM (ITM) we shall denote a determinate (indeterminate) multidimensional Turing machine (for the precise definition, cf. [4]). In the case when some assertion is true both for DTM and for ITM, we use the notation TM, and here it is understood that either all apparatuses considered in the given assertion are determinate or they are all indeterminate.

1. In the first point of Theorem 1, which is proved below, there is given an estimate of the amount of time for simulating ITM of higher dimension on a machine of lower dimension. The method used is not simulation on-line, in contrast with the method applied in [5], with which there was obtained an estimate of the amount of time in lowering the dimension on DTM. We note that the estimate obtained in Sec. 1 for ITM is better than the corresponding estimate from [5] for DTM (which means also the estimate following from [5] for ITM). The upper bound given in Sec. 1 is slightly worse than the lower bound obtained in [4] for on-line simulation of TM on TM of lower dimension. Namely (we use the notation of [5] and the correction of the result of [4] made in [5]), it follows from [4] that for any $\varepsilon > 0$

$$MT^{m}(t) \not\subseteq MT^{k}(t^{1+\frac{1}{k}-\frac{1}{m}-\varepsilon}).$$

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The method used below allows one to do even more. In studying TM the question arises of the condensation of the trajectories of the heads. Trajectories can be "spread" over a multidimensional lattice. The method makes it possible to simulate the original ITM in such a way that the heads simulating the ITM do not leave the limits of a cube with small edge. We note that the method of [5] also allows one to get a similar result – to simulate the work of a TM with capacity complexity L and time complexity t, by a k-dimensional TM, working in a cube with edge $L^{1/k-1}$, but the estimate of time here is worse than $-tL^{1/k-1}$. Upon simulating ITM on ITM of the same dimension one can achieve condensation close to optimal for power (with arbitrary exponent larger than one) loss of time.

In proving the second point of Theorem 1 using the same method it is shown that upon lowering the dimension by one, one can get condensation close to optimal, with almost no loss in time.

THEOREM 1. Let $k \le m$ be natural numbers and $\varepsilon > 0$. Then for any m-dimensional ITM M, working with time complexity t and capacity complexity L, one can construct

1. a k-dimensional ITM M_1, working with time complexity $tL^{\frac{1}{k}-\frac{1}{m}+\varepsilon}$ in a cube with edge $L^{\frac{1}{k}+\varepsilon}$;

2. an $(m + 1)$-dimensional ITM M_2, working with time complexity $t\lg^2 L$ in a cube with edge $L^{\frac{1}{m+1}+\varepsilon}$,

where M_1 and M_2 both have the same output as M.

(For the case $k = 1$, point 1 of the theorem overlaps with the basic result of [6], extended to ITM.)

We give two auxiliary lemmas. The first of them is a multidimensional generalization of Lemma 2 of [7] and was used in proving a multidimensional generalization (whose formulation is given in [8]) of the basic result of [7].

LEMMA 1. Let the heads of the m-dimensional ITM M on the piece Δ of a zone (not necessarily connected), containing $S > 2m + 1$ cells, occur T times. Then one can find a hyperplane σ, orthogonal to one of the directions of the lattice, such that

1) on each of its sides there are situated no more than $\left(\frac{2m}{2m+1}\right)S$ cells of the piece Δ;

2) the number of passages of heads of the ITM M (in handling the piece Δ) through σ does not exceed $c_1T/S^{1/m}$, where c_1 depends only on m and the number of heads of the ITM M.

Proof. For each of the m axes of the lattice by convention we call one direction on the axis right, the other left. We single out the right (left) hyperplane passing through nodes of the lattice, orthogonal to the direction considered and such that on the left (right) side of it there are situated no more than $\left(\frac{1}{2m+1}\right)S$ cells of Δ.

The 2m hyperplanes singled out as a result (for all m directions) bound a parallelepiped Π, in which, by virtue of the choice of hyperplanes, are situated not less than $\left(\frac{1}{2m+1}\right)S$ cells of Δ. Hence one of the sides of Π has length not less than $\left(\frac{1}{2m+1}S\right)^{1/m}$. Consequently, one can find a hyperplane σ, orthogonal to this side and intersecting Π, through which heads of the ITM M pass not more than $c_1t/S^{1/m}$ times.

LEMMA 2. Let $P_1 = \{1\}, \ldots, P_{i+1}$ be obtained from P_i by replacing its maximal element a by some two a_1 and a_2 such that $a_j \ge ca$ ($j = 1, 2,\ 1/2 \ge c > 0$), where $a_1 + a_2 = a$. Then any element of P_N does not exceed $1/cN$.

By induction on N one can prove that if $a_1 \ge \ldots \ge a_N$ are all elements of P_N, then $a_N \ge ca_1$. Hence $1 = \sum_{1\le i\le N} a_i \ge a_1 \cdot cN$.

We proceed to the proof of Theorem 1 (both points will be proved in parallel).

We choose r sufficiently large that one has $\frac{1}{m} < \alpha = \frac{1}{k - \log_2(2m+1)} < \frac{1}{k} + \varepsilon$ in the case of point 1 and $\beta = \frac{1}{(m+1) - \log_2(2m+1)} < \frac{1}{m+1} + \varepsilon < \frac{1}{m}$ in the case of point 2.

The simulation of the work of M will consist of the following. We choose a (indeterminate) hyperplane with the property indicated in Lemma 1, then we apply Lemma 1 to the larger piece of the zone and thus r^k times (in the case of point 2 r^{m+1} times) we apply Lemma 1 (in both cases if there remains a piece of the zone containing no more than 2m + 1 cells, then we no longer subdivide it). Each time upon application of Lemma 1 we subdivide indeterminately the largest in number of cells of the pieces of the zone. Let us agree that the letter c with indices will denote constants, independent of t, L, s.

It will be proved by induction that the entire zone of the ITM M can be simulated in the memory of the ITM M_1 (or M_2), accommodating it in a cube with side $c_3L^{\alpha}\log_2^{1/k}L$ (respectively, $c_3L^{\beta}\log_2^{1/m+1}L$), while to each cell of the active zone of the ITM M corresponds its image, a cell of the memory of M_1 (or M_2), to which there is attached a cube of side $\log_2^{1/k}L$ (respectively, $\log_2^{1/m+1}L$) in which there is written the address of the original cell of the memory of the ITM M.

Let a piece of the active zone of the ITM M, consisting of s cells, be divided in the way described above in $N = r^k$ (respectively, $N = r^{m+1}$) pieces, containing $s_1 \geq \ldots \geq s_N$, respectively, active cells. We apply Lemma 2 to the collection of numbers $\{s_1/s, \ldots, s_N/s\}$ and we get that $s_1 \leq s/cN$, here and later $c = 1/(2m+1)$. By the inductive assumption, the pieces of the zone of the ITM M, corresponding to s_i, are already packed in cubes with sides $c_3s_i^{\alpha}\log_2^{1/k}L$ (respectively, $c_3s_i^{\beta}\log_2^{1/(m+1)}L$), so that the time required by M_1 (or M_2) for simulating the work of the ITM M on these pieces does not exceed $c_2t_is_i^{\alpha-1/m}\log_2L$ (respectively, $c_2t_i\log_2s_i\cdot\log_2L$); c_2 will be chosen at the end. For pieces of the zone containing no more than 2m + 1 cells, the inequalities indicated for the lengths of the sides of the cubes can be satisfied at the expense of a suitable choice of c_3.

Since the pieces corresponding to s_i can be disconnected, one estimates the sum of the times necessary for some head of the ITM M_1 (or M_2), over all intervals in which the head of the ITM M modeled by it are found in a piece of the zone corresponding to s_i. Moreover, one estimates that at the start of each such interval the corresponding head of the ITM M_1 (or M_2) is found in the image of the cell in which at the start of this interval the head of the ITM M modeled by it is situated.

The work of the ITM M_1 (or M_2) consists of steps of two types. Firstly, there is the direct simulation of the work of the ITM M for steps at which the heads of the ITM M do not pass through the cuts made by the hyperplanes (steps of the first type include consideration of the contents of cells, the entry of the new content, change of state). Secondly is the search for images of cells into which heads of the ITM M pass after intersecting cuts. The latter will be effected indeterminately by shortest paths, at the end of the search it is only necessary to verify that the address of the cell [it is entered in a cube with side $\log_2^{1/k}L$ (respectively, $\log_2^{1/(m+1)}$] is required.

Cubes of the memory of the ITM M_1 (or M_2) with sides $c_3s_i^{\alpha}\log_2^{1/k}L$ (respectively, $c_3s_i^{\beta}\log_2^{1/(m+1)}L$), where $1 \leq i \leq N$, can be packed into a cube with side $c_3rs_1^{\alpha}\log_2^{1/k}L$ (respectively, $c_3rs_1^{\beta}\log_2^{1/(m+1)}L$). Then by Lemma 2

$$r\cdot s_1^{\alpha}\log_2^{1/k}L \leq r\cdot\log_2^{1/k}L\cdot\frac{s^{\alpha}}{c^{\alpha}\cdot r^{k\alpha}} = s^{\alpha}\log_2^{1/k}L$$

and, respectively,

$$2s_1^\beta \log_2^{\frac{1}{m+1}} L \le 2\cdot \log_2^{\frac{1}{m+1}} L \frac{s^\beta}{c^\beta 2^{(m+1)\beta}} = s^\beta \log_2^{\frac{1}{m+1}} L$$

by virtue of the choice of α, β, which proves the inductive step on the length of a side of the cube of the memory of the ITM M_1 (or M_2).

It remains to estimate the time. In handling a piece of the zone corresponding to s, at a step of the first type the ITM M_1 (or M_2) by the inductive assumption spends time not greater than

$$T_1' = c_2\Big(\sum_i t_i s_i^{\alpha-\frac{1}{m}}\Big) \log_2 L,$$

respectively,

$$T_2' = c_2\Big(\sum_i t_i \log_2 s_i\Big) \log_2 L.$$

At a step of the second type the ITM M_1 (or M_2) spends time not exceeding

$$T_1 = c_3 s^\alpha (\log_2 L) \sum_j c_1 \frac{t_j'}{(s_j')^{1/m}},$$

respectively,

$$T_2 = c_3 s^\beta (\log_2 L) \sum_j c_1 \frac{t_j'}{(s_j')^{1/m}},$$

where s_j', t_j' $(1 \le j \le N)$ are the number of cells and the times of handling them on the ITM M in pieces of the zone which are cut out by the hyperplanes at the j-th step of the process described above.

The sum $\sum_j c_1 t_j'/(s_j')^{1/m}$ bounds (by Lemma 1) the number of steps in whose time cuts happen, and $c_3 s^\alpha \log_2 L$ (respectively, $c_3 s^\beta \log_2 L$) bounds the number of steps of the ITM M_1 (or M_2) in the search for the image of the necessary cell after passing through a cut. Since $s_j' \ge s\cdot c^N$, one has

$$T_1 \le c_4 t s^{\alpha-\frac{1}{m}} \log_2 L,$$

respectively,

$$T_2 \le c_4 t s^{\beta-\frac{1}{m}} \log_2 L.$$

Hence for the ITM M_1 one has $T_1 + T_1' \le c_4 t s^{\alpha-\frac{1}{m}} \log_2 L + c_2 t s^{\alpha-\frac{1}{m}} (1-c)^{\alpha-\frac{1}{m}} \log_2 L \le c_2 t s^{\alpha-\frac{1}{m}} \log_2 L$, the last inequality is achieved by a suitable choice of c_2 [we note that the choice of c_3, c_4 for the ITM M_1 (or M_2) did not depend on the choice of c_2). Analogously for the ITM M_2 one has

$$T_2' + T_2 \le c_2 t(\log_2 s + \log_2 (1-c)) \log_2 L + c_4 t s^{\beta-\frac{1}{m}} \log_2 L \le c_2 t \log_2 s \log_2 L$$

at the expense of a suitable choice of c_2. The inductive step on bounding the time is verified and Theorem 1 is proved.

2. In the second section we generalize to the multidimensional case one of the results of [1]. The proof uses the method proposed in [1] and the method of Schönhage [9] for simulating in real time a TM by the Kolmogorov – Uspenskii algorithm [10]. In connection with the fact that the proof has a compilational character, it is not recounted in great detail.

THEOREM 2. Let the k-dimensional TM M work with time complexity not exceeding t $(t(n) \ge n \log^{1/k} n)$. Then there exists an AM R, working with time complexity $t/\log^{1/k} t$ and having the same output as M.

In the first stage of the proof, just as in [1], we transform (with linear delay) the TM M into a TM M' so that M' becomes block-respected, cf. [1], with the block $c_1 \log^{1/k} t$, where the constant c_1 will be chosen later. We divide the memory of M' into cubes with side $c_1 \log^{1/k} t$. The requirement of being block-respected is that all the time of work of the TM M' is divided into intervals of length $c_1 \log^{1/k} t$, and in the course of one interval no head intersects boundaries of cubes.

To satisfy the requirement of block-respect we replace each head of the TM M by 3^k heads of the TM M' and we add further for each of these heads a head-indicator, which in some chosen cube with marked faces will simulate the position of the head in the cube and signal the time when it goes past the boundary.

All the time intervals of the work of M' are divided into basic and auxiliary. In the time of basic intervals there occurs simulation of the work of M, in the time of auxiliary intervals heads assume initial positions. The initial position of the 3^k heads corresponding to a head of the TM M, at the beginning time of a basic interval is the following. One head, we shall call it central, is found in a cell of the TM M' corresponding to that cell of the TM M in which its head being modeled is found. The remaining $(3^k - 1)$ heads which we call peripheral are found in neighboring cubes on the boundaries in cells close to that cell in which the central head is found (here and later we describe the work of the 3^k heads of the TM M', corresponding to one head of the TM M, the work of the remaining heads is simulated analogously). In the time of motion of the simulated head of the TM M inside a cube this condition is preserved.

Suppose at some time a head of the TM M passes through the boundary into one of the neighboring cubes. In this case the corresponding central head remains on the boundary, its role starts to be played by the corresponding peripheral head, and all the remaining $(3^k - 1)$ heads are found at each following moment in closest cells to the new central head. Then there can again occur a change of central head, etc. The constructions indicated allow the block-respect condition to be satisfied.

By the configuration of the TM M' for the start of a time interval we mean the content of all cubes in which at this time there is found at least one head of the TM M' (these cubes will be called active for this interval), and neighborhood relations between active cubes. By the choice of the constant c_1 one can achieve that the number of all possible configurations does not exceed $c_2 t^{\alpha}$ for some $\alpha < 1$.

The preliminary stage of the work of the AM R consists of the following. All possible configurations of the TM M' are entered in the memory of AM (for each configuration one needs a fixed number of registers). Next a fixed number of registers of the AM R are available for the entry of the content of the active cubes after the work of M' in the course of a time interval of length $c_1 \log^{1/k} t$, the new state, the situation of heads, and the indication of neighborhoods of new active cubes in relation to the old.

The proper simulation of the work of M' uses what was done by the AM R in the preliminary stage and the construction of Schönhage [9] (the author considered it inappropriate to reproduce in detail the construction of [9]). The memory of the TM M' can be described in the form of a tree analogous to Schönhage's tree, but to its leaves are "attached" cubes with side $c_1 \log^{1/k} t$ (the handling of this tree is easily simulated in real time on an AM). The content of the active cubes is changed in accord with the preliminary stage, the Schönhage tree is used for forming the configuration at the start of the next time interval – the tree with "attached" cubes allows one to show the content and neighborhood relations between active cubes.

The time of work at the preliminary stage is estimated as $c_3 t^{\alpha} \log^{1/k} t$, the time of proper simulation is estimated as $c_4 t/\log^{1/k} t$. Since the length of the entry of one active cube does not exceed $c_1^k \log_2 t$, by lowering c_1 it is easy to satisfy the condition on the length of registers formulated in defining AM (cf. [3]).

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