Formula Complexity Minicourse: Superpolynomial Lower Bound for Monotone Formulas

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Remind that a **formula over** $U_2$ (also called DeMorgan formula) is a circuit over $\{\lor, \land, \neg\}$ whose underlying graph is a tree. Such formulas can be defined inductively as follows:

1. every Boolean variable $x_i$ and its negation $\overline{x_i}$ is a formula of size 1 (these formulas also called leaves);
2. if $F_1$ and $F_2$ are formulas of size $l_1$ and $l_2$, then both $F_1 \lor F_2$ and $F_1 \land F_2$ are formulas of size $l_1 + l_2$.

For a function $f \in B_n$, by $L_{U_2}(f)$ we denote the size of the minimal formula over $U_2$ computing $f$. 
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For a function $f \in B_n$, by $L_{U_2}(f)$ we denote the size of the minimal formula over $U_2$ computing $f$. 
Let $A$ and $B$ be two disjoint subsets of $\{0, 1\}^n$. A Boolean formula $F$ separates $A$ and $B$ if $F(a) = 1$ for all $a \in A$ and $F(b) = 0$ for all $b \in B$.

A monochromatic rectangle is a subset $R' = A' \times B'$ of $A \times B$ such that $A'$ and $B'$ are separated by a variable $x_i$ or its negation $\overline{x_i}$, that is, there must be a coordinate $i \in \{1, \ldots, n\}$ such that $a_i \neq b_i$ for all vectors $a \in A'$ and $b \in B'$.

If we have a stronger condition that $a_i = 1$ and $b_i = 0$ for all $a \in A'$ and $b \in B'$ (i.e., we do not allow negations $\overline{x_i}$), then the rectangle is monotone.
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Covering by Rectangles

Lemma

If $A$ and $B$ can be separated by a (monotone) DeMorgan formula of size $t$ then the set $A \times B$ can be covered by $t$ mutually disjoint (monotone) rectangles.

Remark

- It was Khrapchenko (1971), who first used (implicitly) this lemma.
- Then it was explicitly stated by Ryckkov (1985).
- Then it was proved by Karchmer and Wigderson and independently by Razborov (1988) by establishing a connection between a formula size and a communication complexity of a certain communication game (and then using a well-known fact that communication complexity can be lowerbounded by a number of monochromatic rectangles needed to cover a matrix).
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Example: Khrapchenko’s Bound

Theorem (Khrapchenko, 1971)

Let $F$ be a formula separating $A$ and $B$. Then $F$ has size at least

$$\frac{|N(A, B)|^2}{|A| \cdot |B|},$$

where $N(A, B)$ is the set of all pairs $(a, b) \in A \times B$ such that $d(a, b) = 1$ (in this case we also say that $a$ and $b$ are neighbors).

Remark

- Intuitively, if $N(A, B)$ is large, then every formula separating $A$ and $B$ must be large, since the formula must distinguish many pairs of “very similar” inputs.
- $L_{U_2}(x_1 \oplus \cdots \oplus x_n) = \Omega(n^2)$.
- Khrapchenko’s theorem is not able to provide stronger than $\Omega(n^2)$ bounds.
Example: Khrpachenko’s Bound

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Canonical Covering

- Every set $\mathcal{R}$ of rectangles, whose union gives the whole set $A \times B$, is its cover.
- The canonical monotone cover $\mathcal{R}_{\text{mon}}(A, B)$ consists of $n$ rectangles $R_1, \ldots, R_n$, where

$$R_i = \{a \in A | a_i = 1\} \times \{b \in B | b_i = 0\}.$$

- By a matrix over $A, B$ we mean a matrix $M$ over a field $\mathbb{F}$ whose rows are indexed by elements of the set $A$ and columns by elements of the set $B$.
- Given a rectangle $R \subseteq A \times B$, we denote by $M_R$ the corresponding submatrix of $M$.
- By $\hat{M}_R$ we denote the matrix (over $A, B$) which is obtained from the matrix $M$ by changing all its entries $m_{u,v}$ with $(u, v) \not\in R$, to 0.
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Lower Bound for Monotone Formula Size

Theorem (Razborov, 1990)

For any $A, B \subseteq \{0, 1\}^n$ and a monotone Boolean function $f$ such that $f(A) = 1$, $f(B) = 0$ and any non-zero matrix $M$ over $A, B$ (over an arbitrary field $\mathbb{F}$), we have

$$L_M(f) \geq \frac{\text{rk}(M)}{\max_R \text{rk}(M_r)},$$

where the maximum is over all rectangles $R \in \mathcal{R}_{\text{mon}}(A, B)$.
Proof

- let \( t = L_M(f) \)
- by Rychkov’s lemma we know that there exists a set \( \mathcal{R} \) of \( |\mathcal{R}| \leq t \) mutually disjoint monotone monochromatic rectangles which cover the set \( A \times B \)
- then \( M = \sum_{R \in \mathcal{R}} \hat{M}_R \) and hence

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\text{rk}(M) = \text{rk} \left( \sum_{R \in \mathcal{R}} \hat{M}_R \right) \leq \sum_{R \in \mathcal{R}} \text{rk} \left( \hat{M}_R \right)
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- on the other hand, for every \( R \in \mathcal{R} \) there is a rectangle \( R' \in \mathcal{R}_{\text{mon}}(A, B) \) such that \( R \subseteq R' \)
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General (Non-monotone) Case

Remark

- The same lower bound holds for non-monotone case, if we extend the set $R_{mon}(A, B)$ by adding $n$ “dual” rectangles $R'_1, \ldots, R'_n$, where

$$R_i = \{a \in A | a_i = 0\} \times \{b \in B | b_i = 1\}.$$ 

- However Razborov has proved that in this case the result is useless: for any Boolean function in $n$ variables the fraction on the rhs does not exceed $O(n)$.
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Common Neighbors

Let $G = (V_1, V_2, E)$ with $|V_1| = |V_2| = n$ be a bipartite graph.

Define $f_{G,k} \in B_{2n}$ as follows. The function has $2n$ variables, one for each node of $G$, and accepts a set of nodes $X \subseteq V_1 \cup V_2$ iff $X$ contains some subset $S \subseteq V_1$ of size at most $k$, together with the set of its common neighbors

$$\Gamma(S) = \{ j \in V_2 : (i, j) \in E \text{ for all } i \in S \}.$$ 

That is, $f_{G,k}$ is an Or of all $\sum_{i=0}^{k} \binom{n}{i}$ monomials $\bigwedge_{i \in S \cup \Gamma(S)} x_i$ where $S \subseteq V_1$ and $|S| \leq k$.

By $\hat{\Gamma}(S)$ we will denote the set of all common non-neighbors of $S$, that is,

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Isolated Neighbor Condition

A bipartite graph $G = (V_1, V_2, E)$ satisfies the isolated neighbor condition for $k$ if for any two disjoint subsets $S, T \subseteq V_1$ such that $|S| + |T| = k$, there is a node $v \in V_2$ which is a common neighbor of all the nodes in $S$ and is isolated from all the nodes in $T$, i.e., if $\Gamma(S) \cap \widehat{\Gamma}(T) \neq \emptyset$.

It appears that if $G$ satisfies the isolated point condition, then a straightforward formula for $f_{G,k}$ (an Or of $\sum_{i=0}^{k} \binom{n}{i}$ And’s, each of length at most $2n$) is almost optimal.
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Main Lower Bound

Theorem (Gál 1998)

If $G$ satisfies the isolated neighbor condition for $2k$, then the function $f_{G,k}$ does not have a monotone DeMorgan formula of size smaller than $\sum_{i=0}^{k} \binom{n}{i}$. 
Proof

consider the following 0/1-matrix \( M \); its rows and columns are labeled by subsets of \( V_1 \) of size at most \( k \); the entries are defined by

\[
M_{S,T} = 1 \text{ iff } S \cap T = \emptyset
\]

it is known that this disjointness matrix has full rank over \( \mathbb{F}_2 \):

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Proof (Cont’d)

- In order to apply Razborov’s theorem, we will now label the rows and columns of this matrix by vectors from special subsets of vectors $A$ and $B$ in $\{0, 1\}^{2n}$ so that:
  - $f_{G, k}(A) = 1$ and $f_{G, k}(B) = 0$
  - For every $R \in R_{\text{mon}}(A, B)$, the submatrix $M_R$ has rank 1

- If a row of $M$ is labeled by a set $S$, then relabel this row by the incidence vector $v_S$ of $S \cup \Gamma(S)$ ($v_S(i) = 1$ iff $i \in S \cup \Gamma(S)$)

- If a column of $M$ is labeled by a set $T$, then relabel this row by the incidence vector $u_T$ of $V_1 \cup V_2 \setminus (T \cup \hat{\Gamma}(T))$ ($u_T(i) = 0$ iff $i \in T \cup \hat{\Gamma}(T)$)

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- let us verify that $f_{G,k}(A) = 1$ and $f_{G,k}(B) = 0$
- note that $f_{G,k}(x) = 1$ iff $x \geq v_S$ for some $S$, hence $f(A) = 1$
- if the graph satisfies the isolated point condition for $2k$, then we have the following intersection property: for any two subsets $S, T \subseteq V_1$ of size at most $k$,

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- it remains to verify that for every $R \in \mathcal{R}_{\text{mon}}(A, B)$, the submatrix $M_R$ has rank 1
- for each node $i \in V_1 \cup V_2$, let $R_i = \{v_S|v_S(i) = 1\} \times \{u_T|u_T(i) = 0\}$
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Summarizing

- Explicit bipartite graphs, satisfying the isolated neighbor condition for $k = \Omega(\log n)$ are known.
- Such are, for example, Paley graphs.
- Thus, the corresponding Boolean function $f_{G,k}$ requires monotone formula size at least $\binom{n}{k} = n^{\Omega(\log n)}$. 
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Paley Graphs

- A bipartite Paley graph is a bipartite graph $G_q = (V_1, V_2, E)$ with parts $V_1 = V_2 = \mathbb{F}_q$ for $q$ odd prime congruent to 1 modulo 4; two nodes, $x \in V_1$ and $y \in V_2$, are joined by an edge iff $x - y$ is a non-zero square in $\mathbb{F}_q$.

- The condition $q \equiv 1 \pmod{4}$ is only to ensure that $-1$ is a square in the field, so that the resulting graph is undirected.

- Given two disjoint sets of nodes $A, B \subseteq V_1$, let $v(A, B)$ denote the number of nodes in $V_2$ joined to each node of $A$ and to no node of $B$. 


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Paley Graphs
Theorem

Let $G_q = (V_1, V_2, E)$ be a bipartite Paley graph with $q > 9$, and $A, B$ be disjoint sets of nodes in $V_1$ such that $|A| + |B| = k$. Then

$$\left| v(A, B) - 2^{-k}q \right| \leq k\sqrt{q}.$$ 

In particular, $v(A, B) > 0$ as long as $k2^k, \sqrt{q}$. 


Disjointness Matrix

Let $k \leq n$ be natural numbers, and $X$ be a set of $n$ elements. A $k$-disjointness matrix over $X$ is a 0/1-matrix $D = D(n, k)$ whose rows and columns are labeled by subsets of $X$ of size at most $k$; the entry $D_{A,B}$ in the $A$-th row and $B$-th column is defined by:

$$D_{A,B} = \begin{cases} 0 & \text{if } A \cap B \neq \emptyset, \\ 1 & \text{if } A \cap B = \emptyset. \end{cases}$$

**Theorem (Razborov 1987)**

The $k$-disjointness matrix $D = D(n, k)$ has full rank over $\mathbb{F}_2$, that is,

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Proof

- let \( N = \sum_{i=0}^{k} \binom{n}{i} \)
- we must show that the rows of \( D \) are linearly independent over \( \mathbb{F}_2 \), i.e., that for any non-zero vector \( \lambda = (\lambda_{I_1}, \lambda_{I_2}, \ldots, \lambda_{I_N}) \) in \( \mathbb{F}_2^N \) we have \( \lambda \cdot D \neq 0 \)
- consider the following polynomial:

\[
f(x_1, \ldots, x_n) = \sum_{|I| \leq k} \prod_{i \in I} x_i.
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- since \( \lambda \neq 0 \), at least one of the coefficients \( \lambda_I \) is nonzero, and we can find some \( I_0 \) such that \( \lambda_{I_0} \neq 0 \) and \( I_0 \) is maximal in that \( \lambda_I = 0 \) for all \( I \subset I_0 \)
- assume w.l.o.g. that \( I_0 = \{1, \ldots, t\} \), and make in the polynomial \( f \) the substitution \( x_i := 1 \) for all \( i \not\in I_0 \)
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