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CANONICAL RECURSIVE FUNCTIONS AND OPERATIONS

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A series of properties of canonical recursive functions and operations are established, allowing the possibility of extrapolating to these functions and operations the method proposed by R. L. Goodstein for constructing equational calculi.

Section 1

Certain attractive singularities of the concept of partial recursive function (we have in mind the definition of this concept formulated in [1, Sec. 63] for the case when the function being characterized is previously unknown and $l = 0$) impel many authors to rest their choice precisely on this concept, when the requirement arises in using any precisely defined equivalent of the general (descriptively characterized) concept of arithmetic algorithm (computable arithmetic function). In particular, in the search for an answer to the question of the feasibility of an arithmetic algorithm having given properties the method itself of defining partial recursive functions (PRF) in many cases "suggests" a fruitful path to the construction of the algorithm sought. Here we have in mind the following.

Various examples indicate that in many cases the problem of constructing an arithmetic algorithm having given properties $U_1, \ldots, U_m$ is successfully solved as a result of reducing it to a problem of constructing a certain finite collection of mutually "interacting" arithmetic algorithms, among which the algorithm sought is found (the remaining algorithms occurring in the collection are considered as auxiliary), moreover such a collection, for which the conditions imposed on the algorithms (in particular, on the connections of the algorithms with one another) and sufficing together for proving that the algorithm sought has the properties $U_1, \ldots, U_m$, can be expressed in the form of a certain consistent system $E_1, \ldots, E_k$ of conditional equalities. (If F and G are functional expressions (objective terms), constituted by the usual method for arithmetic languages from the number 0, the sign |, used for constructing for a natural number N the natural number N| directly following it (we have in mind natural numbers of a single number system), functional constants denoting PRF, objective and functional variables, then the expression $F \simeq G$ is called conditional equality (CE) and is used as a notation for the expression "if one of the terms F, G has a value (i.e., is computable), then the other term also has a value and the values of these terms are equal." Let us assume that each functional variable occurring in the CE $F \simeq G$ corresponds to some PRF with the same capacity as the given variable (i.e., with the same number of argument places); one says of the collection of PRF obtained that it satisfies the CE $F \simeq G$, if after substitution in this CE in place of all functional variables of the PRF corresponding to them one gets a CE, true for all values of the objective variables. The system (i.e., list) of conditional equalities $E_1, \ldots, E_k$ is called consistent if in the calculus whose axioms are $E_1, \ldots, E_k$, and whose rules of inference are R1 and R2 from [1, Sec. 54], formulated with the replacement of the sign = by the sign $\simeq$, and also the rules $\frac{\mathfrak{r} \simeq \mathfrak{s}}{\mathfrak{s} \simeq \mathfrak{r}}$ and $\frac{\mathfrak{r} \simeq \mathfrak{s},\ \mathfrak{s} \simeq \mathfrak{t}}{\mathfrak{r} \simeq \mathfrak{t}}$, it is impossible to derive a CE of the form $M \simeq N$, where M and N are distinct natural numbers.)

A consistent system of CE $E_1, \ldots, E_k$, expressing the requirements of the collection of algorithms sought, in many cases (but not always!) is not only a constituent part of the formulation of the problem to which the original problem on the feasibility of an algorithm satisfying given conditions $U_1, \ldots, U_m$ has been reduced, but also a "prepared" solution of this problem (and simultaneously of the original problem). The system $E_1, \ldots, E_k$ can be considered as a constructive object, giving a definite collection of PRF (to each functional variable occurring in this system, distinguished as main functional variable, corresponds its PRF); in order to be able to directly use the definition of the concept of PRF from [1, Sec. 63], it suffices in the system considered to replace the sign $\simeq$ used in this definition by the sign = and, possibly, to change the order of terms

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of the system considered. If this collection of PRF satisfies all the CE from the system $E_1, \ldots, E_k$ (and this system usually tends to be constructed so that such conditions are satisfied), we get a solution of the problem together with certain (in many cases quite useful) initial data for joint investigation of various properties of the PRF sought and auxiliary PRF.

At the same time the concept of PRF has singularities which evoke considerable inconvenience in the use of this concept.

Firstly, partial recursive functions are, according to their definition, calculations of definite type, and not algorithms. To this remark one can object that the calculations with which we are concerned transform in an obvious way into algorithms – to this end one formulates a certain (one for all such calculations) appropriate method of developing a determinate process of successive "expansions" of derivations in any concrete calculus – such a process in which for any potentially feasible derivation W in the calculus considered there occurs a step, as a result of which there turns out to be constructed a derivation with the same last term as the derivation W. Here one also intends the following rule for terminating the process for a given sequence of natural numbers (i.e., for that sequence for which one calculates the value of the PRF considered): the process stops at that step at which the conditional equality expressing the value of the PRF considered on the given sequence turns out to be associated with the derivation constructed earlier. However, algorithms defined by this method are unsuccessful in many relations. In them the determinate process develops independently of the given sequence of natural numbers (this sequence serves only for recognizing the concluding step of the process). In view of this: (a) it is quite far in this type from algorithms actually constructed in mathematics and its applications for solving concrete problems, (b) it is unfit from the practical point of view in view of its extreme complexity and the clear "uneconomicalness" of the computational processes, (c) in studying properties of these algorithms the approach consisting of the analysis "step by step" of the singularities of the development of the algorithmic process is inapplicable (this approach often turns out to be the resultant in applications to algorithms of other types, at least in seeking formulations of hypotheses about properties of the concrete algorithm considered).

Secondly, it is known that the set of consistent systems of CE is nondenumerable, and consequently undecidable.

Thus, from the point of view of the classification of concepts with respect to logical complexity characterizing their conditions, the concept of PRF turns out to be more complex than, e.g., the concept of Turing–Post algorithm (machine), normal algorithm of A. A. Markov, algorithm of A. N. Kolmogorov. [It is known that for modeling the algorithms of the types mentioned here by means of PRF a certain (suitable) decidable subset of the set of all PRF suffices.]

Thirdly, any consistent system of CE uniquely determines in the sense indicated above a certain collection of PRF; however, it is not always true that this collection satisfies all the CE of the given system. For example, the system of CE $g(x|) \simeq x$, $g(0) \simeq g(0)|$, $f(g(x)) \simeq 0$ is consistent, to the functional variable f corresponds the PRF identically equal to zero, to the functional variable g corresponds a PRF defined on all natural numbers, except 0, however, this pair of PRF does not satisfy the third CE. Here the CE $g(0) \simeq g(0)|$ does not allow one to extend somehow the second of these PRF to the number 0, and consequently, such a pair of PRF which satisfies all the CE from the consistent system considered is impossible.

What was said above extends also to the concept of recursive operator (we have in mind the definition of this concept formulated in [1, Sec. 63]).

It is known that Kleene's theorem on the normal form of PRF (cf. [1, Theorem XIX(a)]) and certain of its modifications allowed the possibility of isolating certain decidable subsets of the set of all PRF, each of which, on the one hand, is representative (this means that it is sufficient for the representation up to the relation $\simeq$ of any PRF), and, on the other hand, does not have the above-mentioned deficiencies of the set of all PRF. The representative subsets used in the literature are usually determined in the following way: their elements are considered to be those and only those PRF which can be obtained from certain PRF by giving, generally speaking, comparatively simple (short) systems of CE and so-called initial functions, by means of chains of applications of certain recursive operations (e.g., the operation of regular substitution, the operation of primitive recursion, and the operation of constructing the least root of an equation, also called μ-operation). Some authors associate the term "PRF" not with all PRF in the sense of Kleene, but only with the elements of one of the representative subsets, and expound the theory of algorithms on the basis of such a choice of a standard type of arithmetic algorithms (cf., e.g., [2, 3]). In those versions of the choice of representative subset of the set of all PRF, which are actually used in the literature for standardization of the concept of arithmetic algorithm, it emerges that the system of CE characterizing the PRF from the distinguished subset have certain singularities, making it possible to place at the foundation of the definition of algorithms a completely different method of determinate development of the process of calculation of the value of the function considered at a given sequence of natural numbers than in the usual case – such a method for which the process from the very start "is sent" to that sequence of NN K, for which one calculates the value of the function, and consists of the successive transitions from a term of the form h(K), where h is the functional symbol corresponding to the function considered, to newer and newer terms by means of substitutions, "suggested" by the system of CE considered. Moreover, any such system of CE is consistent and the collection of PRF corresponding to it satisfies all the CE occurring in the system. This includes the essential merits of the approach mentioned here to the choice of a standard type of arithmetic algorithm.

The inconvenience of this approach consists of the imposition of exceedingly rigid restrictions on the types of those systems of CE which are intended for the given algorithms of standard type. For example, in the version most prevalent in the literature of the approach considered, in which the role of initial functions is played by the PRF corresponding to the conditional equalities $f(t_0) \simeq t_0|$, $g(t_0) \simeq 0$, $h_{n,k}(t_1, \ldots, t_n) \simeq t_k$ ($n = 1, 2, \ldots; 1 \leq k \leq n$), and as admissible operations one takes the operations of regular substitution, primitive recursion, and the construction of the least root of an equation, any system of CE in which there figures recurrent recursion or joint recursion for several functions, or recursion with respect to several variables, or transfinite recursion for ordered natural numbers with respect to the type of any constructive ordinal different from ω turns out to be unsuitable for the goal indicated. In view of this, the approach considered now allows one to use only partially those merits of the general concept of PRF with which we were concerned at the beginning of this paper.

However, in order to get a representative subset of the set of all PRF, free from the deficiencies listed above of the set of all PRF and having the merits of certain representative sets mentioned, there is no necessity to resort to so radical a restriction of the types of systems of CE, intended for giving arithmetic algorithms of standard type, as is characteristic for the versions of the construction of a representative subset actually used in the literature. The indicated goal can be achieved by means of the insertion in the definition of the concept of PRF formulated in [1, Sec. 63], of only those changes which in essentially (from the point of view mentioned at the beginning of this paper of applications of the general concept of PRF) restrict the class of systems of CE, intended for giving algorithms of standard type.

In this paper it is shown that as suitable modification for this end of the concept of PRF one can take the concept of canonically recursive function (CRF), defined in [5, Sec. 2]. In particular, it is proved below that any system of CE, suitable for the direct construction with respect to it of CRF, is such that the collection of CRF corresponding to the considered system satisfies all the CE from this system. In [5] simultaneously with the concept of CRF and on the basis of analogous compelling considerations, there is introduced the concept of canonically recursive operator (CRO). Canonically recursive operators are sufficient for modeling all possible recursive operators over completely recursive (in other terminology general recursive) functions and many (but not all!) recursive operators over PRF (we note that the μ-operator, considered as an operator over PRF, is a canonically recursive operator). It is proved below that CRO are "good" operators in the same sense that CRF are "good" functions.

The singularities of the concepts of CRF and CRO (in particular, those which distinguish these concepts from the concepts of PRF and recursive operator over PRF in the sense of Kleene) make it possible to extrapolate in appropriate form to the set of all CRF and CRO the method of construction of equational calculi proposed by Goodstein (cf. [4]) on the example of the class of primitive recursive functions (and applied by a series of other authors to certain other classes of complete recursive functions).

Section 2

In the following account we assume known Sections 0.2, 1.1-1.4, 2.1-2.6, 3.1-3.3 of [5]. For comparison of the account in this paper with the account of the theory of PRF in [1-3] and in many other monographs and papers it is necessary to keep the following in mind.

In [5] there are clearly separated languages of two types: algorithmic languages, intended for giving concrete CRF and CRO (Sec. 2.1), and logicomathematical (arithmetic) languages, intended for formulating statements about CRF and CRO, and also conditions imposed on CRF and CRO (Secs. 2.5 and 3.1). [The absence of a clear distinction of languages of these two types in certain accounts of the theory of PRF is essentially an appeal (not always explicit) to the set-theoretic version of the theory of recursive functions, in which by recursive functions is meant arithmetic functions in the set-theoretic sense, satisfying together with certain auxiliary arithmetic functions specific conditions.]

Symbolic expressions giving concrete CRF (concrete CRO) are canonically recursive functions (CRF) [respectively, canonically recursive operators (CRO)]. With respect to any CRF one constructs in a trivial way PRF equivalent to it (the construction with respect to a given PRF of a CRF equivalent to it in a wide class of cases covering practically all those "entering into use" in the theory of algorithms of concrete classes of PRF, is also accomplished in a trivial way; beyond the realms of this class it is accomplished with the help of the theorem of Kleene mentioned above). However the process of calculating the value of a given n-ary CRF φ on a given n-termed sequence of NN K (in contrast with the process of calculating the value of the corresponding PRF on the same sequence K) is not a process of successive generation of certain equalities or conditional equalities, but is a process of another type (a process of successive generation of network terms), with its very start "directed" by the sequence K and analogous in its type to the "natural" process of calculating the value of a primitive-recursive function. In the algorithmic language intended for giving CRF and CRO, the signs $\simeq$ and $=$ do not figure (these signs figure in correspondence with their common meaning in suitable logicomathematical languages). At the same time, for any system of CE $E_1, \ldots, E_k$, which is a canonical system of CE without parameters (cf. below), one constructs in a single-valued and trivial way a basis of reductions (cf. [5, Sec. 2.1]), determining in the standard way a certain collection of CRF, whose terms are equivalent (in the sense of the relation $\simeq$) with the corresponding terms of the collection of PRF, directly defined (in the sense of Kleene's definition) by the system $E_1, \ldots, E_k$. In this sense we also apply in this paper expressions like "the collection of CRF corresponding to a given (suitable with respect to its type) system of CE," "direct construction of a collection of CRF with respect to a given system of CE," etc.

The logicomathematical (arithmetic) languages $\mathfrak{L}_\theta$, $\mathfrak{L}_\theta^1$, and $\mathfrak{L}_\theta^2$ are introduced in Sec. 3.1 of [5] under the assumption that there is fixed a certain basis of operator generation (BOG) θ, satisfying specific conditions, where it is essential that among these conditions there also figure conditions (a), (b), and (c), formulated at the beginning of Sec. 3.1. That the last conditions hold guarantees that all the CRF of type θ (i.e., the CRF which are values of constant functors of terms of type θ) are complete CRF. The restrictions (a) and (b), imposed on θ, exclude, in particular, the case when as distinguished BOG there figures a pair Θ, consisting of the set of all possible CRF Θ_1 and the set of all possible CRO Θ_2. However, precisely this case is especially important from the point of view of the goals pursued in the present paper. Hence in the following account it is assumed that as BOG there is chosen the above-mentioned pair Θ, not satisfying conditions (a) and (b) of Sec. 3.1, and we shall use logicomathematical languages $\mathfrak{L}^1$ and $\mathfrak{L}^2$, different in certain respects from $\mathfrak{L}_\theta^1$ and $\mathfrak{L}_\theta^2$. The descriptions of these languages are based on the definitions of the concepts "objective term" (ObTe), "functorial term" (FuTe), and "term" (Te) formulated in Sec. 2.6 of [5]. The alphabet of the new languages is obtained as a result of adjoining to the alphabet of the languages mentioned the signs ! and $\simeq$. By atomic formulas of the languages $\mathfrak{L}^1$ and $\mathfrak{L}^2$ are meant words of the form !T, where T is an ObTe, and words of the form $(T_1 \simeq T_2)$, where T_1 and T_2 are ObTe. The atomic formula !T is read thus: "the term T has value." The atomic formula $(T_1 \simeq T_2)$ is applied in the same sense as the metalinguistic notation T_1 equ T_2 in [5] (Sec. 2.6).

Formulas of the language $\mathfrak{L}^1$ are constructed from atomic formulas with the help of the logical connectives $\neg, \&, \vee, \rightarrow$ and quantifier complexes of the form $\forall x$ and of the form $\exists x$, where x is an ObVa, by the usual rules. By formulas of the language $\mathfrak{L}^2$ are meant formulas of the language $\mathfrak{L}^1$ and words of the form $\forall \xi_1 \ldots \forall \xi_k F$, where $\xi_1, \ldots, \xi_k$ are FuVa and F is a formula of the language $\mathfrak{L}^1$. In contrast with the languages $\mathfrak{L}_\theta^1$ and $\mathfrak{L}_\theta^2$, it will be assumed that the admissible values of any n-ary FuVa are all possible n-ary CRF. In Sec. 3.1 of [5] in connection with formulas of the languages $\mathfrak{L}_\theta^1$ and $\mathfrak{L}_\theta^2$ there is defined a series of concepts and relations. These definitions carry over word for word to the formulas of the languages $\mathfrak{L}^1$ and $\mathfrak{L}^2$. It is known (cf. [5, Sec. 6.2]) that formulas of the languages $\mathfrak{L}^1$ and $\mathfrak{L}^2$ can be interpreted with the help of the theorem of Kleene mentioned above in the language $\mathfrak{L}_\theta$ for suitable choice of BOG θ (e.g., if θ is the BOG of primitive-recursive functions). However, in the following account there will be no reason to appeal to this possibility.

Section 3

In this section we use the following notation: $\mathfrak{B}$ denotes a certain BaRed; $\mathbb{E} \leftrightharpoons$, list of all ScheFu occurring in $\mathfrak{B}$; $\bar{m} \leftrightharpoons \delta_{\llcorner} \mathbb{E}_{\lrcorner}$, $\varepsilon_m \leftrightharpoons \underline{\mathbb{E}}_m$; and $d_m \leftrightharpoons$, equiplacedness of the ScheFu ε_m $(1 \le m \le \bar{m})$; $B_1, \ldots, B_{\bar{p}}$ denote ScheTe; $\mathfrak{N}_1, \ldots, \mathfrak{N}_{\bar{p}}$, sequences of numberlike terms; and $r_1, \ldots, r_{\bar{p}}$ are positive integers such that the BaRed $\mathfrak{B}$ is representable in the form

$$\varepsilon_{\nu_1}(\mathfrak{U}_1) \succ B_1, \ldots, \varepsilon_{\nu_{\bar p}}(\mathfrak{U}_{\bar p}) \succ B_{\bar p};$$

$\Xi \leftrightharpoons$, list of all functional parameters of $\mathfrak{X}$; $\bar\kappa \leftrightharpoons \delta_{\llcorner}\Xi_{\lrcorner}$, $\xi_\kappa \leftrightharpoons \Xi_\kappa$ $(1 \leqslant \kappa \leqslant \bar\kappa)$; $\mathbb{P} \leftrightharpoons$, list of all objective parameters of $\mathfrak{X}$, $\bar l \leftrightharpoons \delta_{\llcorner}\mathbb{P}_{\lrcorner}$, $x_l \leftrightharpoons \mathbb{P}_l$ $(1 \leqslant l \leqslant \bar l)$. Φ, any sequence of FuTe, of the same type as Ξ, and Δ, any sequence of ObTe, having the same number of terms as the list $\mathbb{P}$, is the quality of the parts. $\pi_m \leftrightharpoons \{\varepsilon_m; \mathfrak{X}; \Xi, \mathbb{P}\}$ $(1 \le m \le \bar m)$; here, if $\mathfrak{B}$ is an autonomous BaRed (and consequently the lists Ξ and $\mathbb{P}$ and the sequences Φ and Δ are empty), then the expressions $\{\varepsilon_m; \mathfrak{B}; \Xi, \mathbb{P}\}$ and $\{\varepsilon_m; \mathfrak{B}; \Xi, \mathbb{P}\}[\Phi, \Delta]$ will be identified in meaning with the CRF $\{\varepsilon_m; \mathfrak{B}\}$.

Let α be any variable and $\mathfrak{S}$ be any sequence of terms. We shall say that α is a parameter of the sequence $\mathfrak{S}$ if α is the parameter of at least one term of this sequence.

We shall use the metalinguistic algorithms $\overline{\mathrm{el}}\,\mathfrak{B}$ and $\mathfrak{z}\mathfrak{B}$ introduced in [5] (Sec. 2.1) in connection with ScheTe. The algorithm of elementary transformation of constant terms, defined in [5] (Sec. 2.5), is called in this paper the algorithm of quasielementary transformation and is denoted by $\overline{\mathrm{quel}}\,+$. The algorithm calculating the value of constant terms we denote by $\mathfrak{z}^+$. In formulating the metalinguistic definitions and assertions containing the notation for metalinguistic algorithms, we shall use metalinguistic expressions of the form $!\mathfrak{T}$ and of the form $(\mathfrak{T}_1 \simeq \mathfrak{T}_2)$, where $\mathfrak{T}$, $\mathfrak{T}_1$, and $\mathfrak{T}_2$ are metalinguistic terms, in the same senses in which inside the languages $\mathfrak{X}^1$ and $\mathfrak{X}^2$ one uses, respectively, atomic formulas of the form $!\,T$ and of the form $(T_1 \simeq T_2)$.

THEOREM 1. For each p $(1 \le p \le \bar p)$ the assertion

$$\tilde{\forall}(\pi_{\nu_p}[\Xi, \mathbb{P}](\mathfrak{U}_p) \simeq {}_{\llcorner}B_p \downarrow^{\langle \varepsilon_m \rangle_1^{\bar m}}_{\langle \pi_m[\Xi, \mathbb{P}] \rangle_1^{\bar m}}{}_{\lrcorner})$$

is true.

COROLLARY. For each p $(1 \le p \le \bar p)$ the assertion

$$\tilde{\forall}(\pi_{\nu_p}[\Phi, \Delta](\mathfrak{U}_p) \simeq {}_{\llcorner}B_p \downarrow^{\langle \varepsilon_m \rangle_1^{\bar m}, \Xi, \mathbb{P}}_{\langle \pi_m[\Phi, \Delta] \rangle_1^{\bar m}, \Phi, \Delta}{}_{\lrcorner})$$

is true.

THEOREM 2. If X is a sequence of FuTe, of the same type as the list $\mathbb{E}$, and the set of proper objective variables of the BaRed $\mathfrak{B}$ does not intersect the set of parameters of the sequence X, Φ, Δ, then for each s $(1 \le s \le \bar m)$ the assertion

$$\tilde{\forall}\Big(\forall z_1 \ldots z_{\bar q} \mathop{\&}_{p=1}^{\bar p} \Big(\chi_{\nu_p}(\mathfrak{U}_p) \simeq {}_{\llcorner}B_p \downarrow^{\langle \varepsilon_m \rangle_1^{\bar m}, \Xi, \mathbb{P}}_{\langle \chi_m \rangle_1^{\bar m}, \Phi, \Delta}{}_{\lrcorner}\Big) \rightarrow \swarrow$$
$$(!\,\pi_s[\Phi, \Delta](y_1, \ldots, y_{d_s}) \rightarrow (\chi_s(y_1, \ldots, y_{d_s}) \simeq \pi_s[\Phi, \Delta](y_1, \ldots, y_{d_s})))\Big)$$

is true. Here $\chi_s \leftrightharpoons \underline{X}_s$, $z_1, \ldots, z_{\bar q}$ is the list of all proper objective variables of the BaRed $\mathfrak{B}$, and $y_1, \ldots, y_{d_s}$ are distinct ObVa not belonging to the set of parameters of the sequence X, Φ, Δ.

The basis of the proofs of Theorems 1 and 2 is constituted by the lemmas formulated below. In these lemmas $\mathfrak{L}$ denotes any autonomous BaRed.

LEMMA 1. If $\langle C_i \rangle_1^n$ is a sequence of constant ScheTe, η is an n-ary ScheFu, and at least one of the ScheTe $C_1, \ldots, C_n$ is not atomic, then

$$\overline{\mathrm{el}}_{\mathfrak{L}\llcorner}\,\eta(\langle C_i \rangle_1^n) \mathrel{\overline{=}} \eta(\langle \overline{\mathrm{el}}_{\mathfrak{L}\llcorner} C_i{}_{\lrcorner} \rangle_1^n).$$

To prove the lemma it suffices to note that any occurrence of any CanScheTe K in the ScheTe $\eta(\langle C_i \rangle_1^n)$ arises from some occurrence of K in one of the ScheTe $C_1, \ldots, C_n$.

LEMMA 2. If C is a constant ScheTe, $K_1, \ldots, K_m$ are constant canonical ScheTe, $\bar K_1, \ldots, \bar K_m$ are the result of reduction of the ScheTe $K_1, \ldots, K_m$ (respectively) by means of $\mathfrak{L}$, $V_1, \ldots, V_m$ are pairwise nonoverlapping occurrences in C of words $K_1, \ldots, K_m$ (respectively), and $\bar C$ is the result of the simultaneous substitution in C in place of $V_1, \ldots, V_m$ of the ScheTe $\bar{K}_1, \ldots, \bar{K}_m$ (respectively), then $\mathfrak{z}_{\mathfrak{L}} \llcorner \bar{C} \lrcorner \simeq \mathfrak{z}_{\mathfrak{L}} \llcorner C \lrcorner$. In particular, $\mathfrak{z}_{\mathfrak{L}} \llcorner \overline{\mathrm{el}}_{\mathfrak{L}} \llcorner C \lrcorner \lrcorner \simeq \mathfrak{z}_{\mathfrak{L}} \llcorner C \lrcorner$.

LEMMA 3. If $C, C_1, \ldots, C_n$ are constant ScheTe, $W_1, \ldots, W_n$ are pairwise nonoverlapping occurrences in C of words $C_1, \ldots, C_n$ (respectively), and C^* is the metalinguistic term obtained as a result of the simultaneous substitution in $\mathfrak{L}$ in place of $W_1, \ldots, W_n$ of the expressions $\mathfrak{z}_{\mathfrak{L}} \llcorner C_1 \lrcorner, \ldots, \mathfrak{z}_{\mathfrak{L}} \llcorner C_n \lrcorner$ (respectively), then $\mathfrak{z}_{\mathfrak{L}} \llcorner C^* \lrcorner \simeq \mathfrak{z}_{\mathfrak{L}} \llcorner C \lrcorner$.

Remark. Here and in the following account it is assumed that the processes of calculating the values of constant metalinguistic terms are developed according to the same rule as for constant terms of the languages $\mathfrak{X}^1$ and $\mathfrak{X}^2$. Hence the following assertion is true: $!C^*$ if and only if $!C_1$ and ... and $!C_n$ and $!C^+$, where C^+ denotes the result of simultaneous substitution in C in place of the occurrences $W_1, \ldots, W_n$, respectively, of natural numbers $M_1, \ldots, M_n$ such that $M_i \simeq \mathfrak{z}_{\mathfrak{L}} \llcorner C_i \lrcorner$ $(i = 1, \ldots, n)$.

The proofs of Lemmas 2 and 3 consist (in correspondence with the meaning of the relation $\simeq$) of constructing algorithms transforming the texts or collections of texts of the construction of values in the BaRed $\mathfrak{L}$ of one of the terms mentioned in the lemmas into the texts of the construction of values in $\mathfrak{L}$ of other suitable terms. For Lemma 2 these algorithms are obvious; for Lemma 3 they are constructed by induction on the steps of the processes of generating schematic terms.

LEMMA 4. If $\langle S_m \rangle_1^n$ is a sequence of constant ObTe, χ is a constant n-ary FuTe, and at least one of the terms $\chi, S_1, \ldots, S_n$ is not atomic, then

$$\overline{\mathrm{quel}}^{+} \llcorner \chi(\langle S_m \rangle_1^n) \lrcorner \simeq \overline{\mathrm{quel}}^{+} \llcorner \chi \lrcorner (\langle \overline{\mathrm{quel}}^{+} \llcorner S_m \lrcorner \rangle_1^n).$$

LEMMA 5. If $\mathfrak{O}$ is a CRO, $\langle \Phi_i \rangle_1^k$ a sequence of constant FuTe, coherent with $\mathfrak{O}$, $\langle T_j \rangle_1^l$ is a sequence of constant ObTe, coherent with $\mathfrak{O}$, and at least one of the terms $\Phi_1, \ldots, \Phi_k, T_1, \ldots, T_l$ is not atomic, then

$$\overline{\mathrm{quel}}^{+} \llcorner \mathfrak{O}[\langle \Phi_i \rangle_1^k, \langle T_j \rangle_1^l] \lrcorner \simeq \mathfrak{O}[\langle \overline{\mathrm{quel}}^{+} \llcorner \Phi_i \lrcorner \rangle_1^k, \langle \overline{\mathrm{quel}}^{+} \llcorner T_j \lrcorner \rangle_1^l].$$

LEMMA 6. If $\Gamma, \Gamma_1, \ldots, \Gamma_n$ are constant terms, $\Omega_1, \ldots, \Omega_n$ are pairwise nonoverlapping occurrences in Γ of the terms $\Gamma_1, \ldots, \Gamma_n$ (respectively), and Γ^* is the metalinguistic term obtained as a result of the simultaneous substitution in Γ in place of $\Omega_1, \ldots, \Omega_n$ of the expressions $\mathfrak{z}^+ \llcorner \Gamma_1 \lrcorner, \ldots, \mathfrak{z}^+ \llcorner \Gamma_n \lrcorner$ (respectively), then $\mathfrak{z}^+ \llcorner \Gamma^* \lrcorner \simeq \mathfrak{z}^+ \llcorner \Gamma \lrcorner$.

Lemma 6 is proved with the help of Lemmas 4 and 5 by induction on the steps of the process of generating terms.

Let Φ^0 be a sequence of CRF, of the same type as the list $\varphi_k \leftrightharpoons \underline{\Phi}_k^0$ $(1 \le k \le \bar{k})$. We construct CRF $\psi_1, \ldots, \psi_{\bar{k}}$ such that the sequence $\langle \psi_i \rangle_1^{\bar{k}}$ satisfies conditions (1)-(3), formulated in [5] (Sec. 2.3) for the definition of the process of application of a canonical recursive operator to initial data of suitable type. We denote by η_k and $\mathfrak{B}_k$, respectively, the ScheFu and BaRed such that $\psi_k \rightleftharpoons \{\eta_k; \mathfrak{B}_k\}$ $(1 \le k \le \bar{k})$. The sequence of CRF $\langle \psi_k \rangle_1^{\bar{k}}$ (list of ScheFu $\langle \eta_k \rangle_1^{\bar{k}}$) will be called the proper version (respectively, proper trace) of the sequence $\langle \varphi^k \rangle_1^{\bar{k}}$ for $\mathfrak{B}$.

Let us assume that besides Φ^0 there is given another sequence of NN Δ^0 such that $\delta \llcorner \Delta^0 \lrcorner = \bar{l}$. We introduce the notation:

$$\overline{\mathrm{clos}} \llcorner \mathfrak{B}; \Phi^0, \Delta^0 \lrcorner \leftrightharpoons \llcorner \mathfrak{B} \dagger_{H^0, \Delta^0}^{\Xi, \mathbb{P}} \lrcorner, \langle \mathfrak{B}_k \rangle_1^{\bar{k}};$$

here $H^0 \leftrightharpoons \langle \eta_k \rangle_1^{\bar{k}}$. The autonomous BaRed $\overline{\mathrm{clos}} \llcorner \mathfrak{B}; \Phi^0, \Delta^0 \lrcorner$ will be called the closure of $\mathfrak{B}$ by means of the sequence Φ^0, Δ^0. The result of applying the CRO $\{\varepsilon_m; \mathfrak{B}; \Xi, \mathbb{P}\}$ to the sequence Φ^0, Δ^0 is (by definition) the CRF $\{\varepsilon_m; \overline{\mathrm{clos}} \llcorner \mathfrak{B}; \Phi^0, \Delta^0 \lrcorner\}$.

We note that for any k $(1 \le k \le \bar{k})$ the i-th step of the process of calculating the values of φ_k on a given sequence of NN Q coincides with the i-th step of the process of calculating the value of the constant ScheTe $\eta_k(Q)$ in the BaRed $\overline{\mathrm{clos}} \llcorner \mathfrak{B}; \Phi^0, \Delta^0 \lrcorner$.

In the following account we preserve the above-indicated meaning of the notations Φ^0, Δ^0, and H^0. Moreover, $\mathfrak{B}^* \leftrightharpoons \overline{\mathrm{clos}} \llcorner \mathfrak{B}; \Phi^0, \Delta^0 \lrcorner$.

LEMMA 7. If C is a ScheTe such that the ScheFu and FuVa occurring in C also occur in the list $\mathbb{E}, \Xi$, then for all values of ObVa:

$$\mathfrak{z}^{+}_{L} C \dagger^{\langle \varepsilon_m \rangle_1^{\bar m}, \equiv}_{\langle \pi_m[\Phi^\circ, \Delta^\circ]\rangle_1^{\bar m}, \Phi^\circ \lrcorner} \simeq \mathfrak{z}_{\mathfrak{X}^*} {}_L C \dagger^{\equiv}_{H^\circ \lrcorner}.$$

Lemma 7 is proved with the help of Lemmas 3 and 6 by induction on the steps of the processes of generating ScheTe.

LEMMA 8. For all values occurring in $\mathfrak{U}_p$:

$$\mathfrak{z}_{\mathfrak{X}^*} {}_L \varepsilon_{\tau_p}(\mathfrak{U}_p)_\lrcorner \simeq \mathfrak{z}_{\mathfrak{X}^*} {}_L B_p \dagger^{\equiv, \mathbb{P}}_{H^\circ, \Delta^\circ \lrcorner} \quad (p = 1, \ldots, \bar p).$$

This lemma follows directly from Lemma 2.

To complete the proof of Theorem 1 we note that for all values of ObVa occurring in $\mathfrak{U}_p$ one has the assertion:

$$\mathfrak{z}^{+}_{L} \pi_{\tau_p}[\Phi^\circ, \Delta^\circ](\mathfrak{U}_p)_\lrcorner \simeq \mathfrak{z}_{\mathfrak{X}^*} {}_L \varepsilon_{\tau_p}(\mathfrak{U}_p)_\lrcorner.$$

From this assertion and from Lemmas 7 and 8 it follows that for all values of ObVa:

$$\mathfrak{z}^{+}_{L} \pi_{\tau_p}[\Phi^\circ, \Delta^\circ](\mathfrak{U}_p)_\lrcorner \simeq \mathfrak{z}^{+}_{L} B_p \dagger^{\langle \varepsilon_m \rangle_1^{\bar m}, \equiv, \mathbb{P}}_{\langle \pi_m[\Phi^\circ, \Delta^\circ]\rangle_1^{\bar m}, \Phi^\circ, \Delta^\circ \lrcorner}.$$

Passing to the proof of Theorem 2, we formulate some definitions and lemmas. The letter $\mathfrak{L}$ will denote any autonomous BaRed. Moreover, $\mathbb{R} \rightleftharpoons$ denotes the list of all ScheFu occurring in $\mathfrak{L}$, $\bar r \rightleftharpoons \delta_L \mathbb{R}_\lrcorner$, $\rho_r \rightleftharpoons \underline{\mathbb{R}}_r$ $(1 \le r \le \bar r)$.

We fix a NN $\bar s$ such that $1 \le \bar s \le \bar r$. The sequence of ScheFu $\langle \rho_r \rangle_{\bar s+1}^{\bar r}$ will be called a distinguished sequence of ScheFu and we denote it by $\overline{\mathbb{R}}$. Let C be a constant ScheTe. By the process of quasielementary transformation of a ScheTe by means of $\mathfrak{L}$ with distinguished sequence $\overline{\mathbb{R}}$ we mean the following process. We make up the list $L_1, \ldots, L_m$ of all occurrences in C of CanScheTe and for each term of this list, starting with a schematic functor belonging to $\overline{\mathbb{R}}$, we develop the process of calculation of its value in $\mathfrak{L}$. If all these processes terminate (and, consequently, each of them reduces to the construction of some NN), then after this for each of the remaining terms of the list we construct the result of its reduction by means of $\mathfrak{L}$. Finally, we construct the ScheTe ${}_L C \dagger^{L_1, \ldots, L_m}_{D_1, \ldots, D_m \lrcorner}$, where D_i denotes the NN or some ScheTe obtained from L_i in the way indicated. We denote the algorithm described by $\overline{\text{quel}}_{\mathfrak{L}, \overline{\mathbb{R}}}$. If $\overline{\mathbb{R}}$ is the empty sequence (i.e., $\bar s = \bar r$), then $\overline{\text{quel}}_{\mathfrak{L}, \overline{\mathbb{R}}} {}_L C_\lrcorner \simeq \overline{\text{el}}_{\mathfrak{L}} {}_L C_\lrcorner$. We denote by $\overline{\text{el}}^{\infty}_{\mathfrak{L}}$ and $\overline{\text{quel}}^{\infty}_{\mathfrak{L}, \overline{\mathbb{R}}}$ the iterations of the algorithms $\overline{\text{el}}_{\mathfrak{L}}$ and $\overline{\text{quel}}_{\mathfrak{L}, \overline{\mathbb{R}}}$ (respectively), i.e., the algorithms satisfying the following conditions: $\overline{\text{el}}^{\infty}_{\mathfrak{L}} {}_L C, 0_\lrcorner \simeq \overline{\text{quel}}^{\infty}_{\mathfrak{L}, \overline{\mathbb{R}}} {}_L C, 0_\lrcorner \simeq C$, $\overline{\text{el}}^{\infty}_{\mathfrak{L}} {}_L C, n|_\lrcorner \simeq \overline{\text{el}}_{\mathfrak{L}} {}_L \overline{\text{el}}^{\infty}_{\mathfrak{L}} {}_L C, n_{\lrcorner\lrcorner}$, $\overline{\text{quel}}^{\infty}_{\mathfrak{L}, \overline{\mathbb{R}}} {}_L C, n|_\lrcorner \simeq \overline{\text{quel}}_{\mathfrak{L}, \overline{\mathbb{R}}} {}_L \overline{\text{quel}}^{\infty}_{\mathfrak{L}, \overline{\mathbb{R}}} {}_L C, n_{\lrcorner\lrcorner}$ (n is any NN).

LEMMA 9. (a) If the hypotheses of Lemma 1 hold:

$$\overline{\text{quel}}_{\mathfrak{L}, \overline{\mathbb{R}}} {}_L \eta(\langle B_i \rangle_1^n)_\lrcorner \simeq \eta(\langle \overline{\text{quel}}_{\mathfrak{L}, \overline{\mathbb{R}}} {}_L B_{i \lrcorner} \rangle_1^n);$$

(b) $\mathfrak{z}_{\mathfrak{L}} {}_L \overline{\text{quel}}_{\mathfrak{L}, \overline{\mathbb{R}}} {}_L C_{\lrcorner\lrcorner} \simeq \mathfrak{z}_{\mathfrak{L}} {}_L C_\lrcorner$;

(c) for any NN n,

$$\mathfrak{z}_{\mathfrak{L}} {}_L \overline{\text{el}}^{\infty}_{\mathfrak{L}} {}_L C, n_{\lrcorner\lrcorner} \simeq \mathfrak{z}_{\mathfrak{L}} {}_L C_\lrcorner, \quad \mathfrak{z}_{\mathfrak{L}} {}_L \overline{\text{quel}}^{\infty}_{\mathfrak{L}, \overline{\mathbb{R}}} {}_L C, n_{\lrcorner\lrcorner} \simeq \mathfrak{z}_{\mathfrak{L}} {}_L C_\lrcorner.$$

LEMMA 10. If the NN n is such that $\overline{\text{el}}^{\infty}_{\mathfrak{L}} {}_L C, n_\lrcorner$ is an NN, then $\overline{\text{quel}}^{\infty}_{\mathfrak{L}, \overline{\mathbb{R}}} {}_L C, n_\lrcorner$ is also an NN and $\overline{\text{quel}}^{\infty}_{\mathfrak{L}, \overline{\mathbb{R}}} {}_L C, n_\lrcorner \doteq \overline{\text{el}}^{\infty}_{\mathfrak{L}} {}_L C, n_\lrcorner \doteq \mathfrak{z}_{\mathfrak{L}} {}_L C_\lrcorner$.

Lemma 10 is proved by induction on the steps of the processes of generating ScheTe.

By the process of constructing the quasivalue of the ScheTe C in the BaRed $\mathfrak{L}$ with distinguished sequence $\overline{\mathbb{R}}$ we mean the process of successive construction of the values of the metalinguistic terms $\overline{\text{quel}}^{\infty}_{\mathfrak{L}, \overline{\mathbb{R}}} {}_L C, 0_\lrcorner$, $\overline{\text{quel}}^{\infty}_{\mathfrak{L}, \overline{\mathbb{R}}} {}_L C, 0|_\lrcorner$ etc., terminated after a step whose result is an NN. The algorithm, according to which process is realized, we denote by $\overline{\text{quva}}_{\mathfrak{L}, \overline{\mathbb{R}}}$. With the help of the preceding lemmas one proves the following lemma.

LEMMA 11. $\overline{\text{quva}}_{\mathcal{L},\bar{\mathbb{R}}}\llcorner C\lrcorner \approx \mathfrak{z}_{\mathcal{L}}\llcorner C\lrcorner$.

It suffices to prove Theorem 2 for the case when all terms of the sequences Φ, Δ, and X are constant terms. We shall start from this assumption and we assume in addition that the assertion constituting in the formulation of Theorem 2 the premise of the outer implication is true (we denote this assertion by $\mathcal{F}$). Let s be one of the numbers $1, \ldots, \bar{m}$, and let $e_1, \ldots, e_{d_s}$ be natural numbers such that $!\,\pi_s[\Phi, \Delta](E)$, where $E \rightleftharpoons \langle e_i\rangle_1^{d_s}$. We shall show that $\chi_s(E) \simeq \pi_s[\Phi, \Delta](E)$.

On the basis of Lemma 6 we have: $!\,\mathfrak{z}^+\llcorner\Phi_k\lrcorner\ (1 \le k \le \bar{k})$ and $!\,\mathfrak{z}^+\llcorner\Delta_l\lrcorner\ (1 \le l \le \bar{l})$. We introduce the notation: $\varphi_k \rightleftharpoons \mathfrak{z}^+\llcorner\Phi_k\lrcorner$, $m_l \rightleftharpoons \mathfrak{z}^+\llcorner\Delta_l\lrcorner$, $\Phi^\circ \rightleftharpoons \langle\varphi_k\rangle_1^{\bar{k}}$, $\Delta^\circ \rightleftharpoons \langle m_l\rangle_1^{\bar{l}}$, $\pi_s^* \rightleftharpoons \mathfrak{z}^+\llcorner\pi_s[\Phi^\circ, \Delta^\circ]\lrcorner$, $\mathfrak{B}^* \rightleftharpoons \overline{\text{clos}}\llcorner\mathfrak{B}; \Phi^\circ, \Delta^\circ\lrcorner$. We have: $\pi_s^* \doteq \{\varepsilon_s;\mathfrak{B}^*\}$. In view of the fact that $!\,\mathfrak{z}^+\llcorner\pi^*(E)\lrcorner$, there are realizable a unique NN M and a unique collection of values of ObVa occurring in the sequence of numberlike terms $\mathfrak{U}_M$, such that $\varepsilon_s \doteq \varepsilon_{v_M}$ and the sequence $\mathfrak{U}_M$ for the collection mentioned of values of ObVa goes into the sequence of NN E. By Lemma 7 we have: $\mathfrak{z}^+\llcorner\pi_s^*(E)\lrcorner \approx \mathfrak{z}_{\mathfrak{B}^*}\llcorner\varepsilon_s(E)\lrcorner$. Consequently, the process of calculating $\mathfrak{z}^+_{\mathfrak{B}^*}\llcorner\varepsilon_s(E)\lrcorner$ terminates and has as its result a certain NN w. We apply Lemma 11 to the case when as $\mathcal{L}$ and $\bar{\mathbb{R}}$ we have chosen, respectively, $\mathfrak{B}^*$ and H^0. We have: $\overline{\text{quva}}_{\mathfrak{B}^*,H^\circ}\llcorner\varepsilon_s(E)\lrcorner \doteq w$. We denote by N an NN for which $\overline{\text{quel}}\,{}^{\infty}_{\mathfrak{B}^*,H^\circ}\llcorner\varepsilon_s(E), N\lrcorner \doteq w$.

In addition: $G_n \rightleftharpoons \overline{\text{quel}}\,{}^{\infty}_{\mathfrak{B}^*,H^\circ}\llcorner\varepsilon_s(E), n\lrcorner$, $\widetilde{G}_n \rightleftharpoons \llcorner G_n \big|^{\langle\varepsilon_m\rangle_1^{\bar{m}},\ \Xi,\ \mathbb{P}}_{\langle\gamma_m\rangle_1^{\bar{m}},\ \Phi^\circ,\ \Delta^\circ}\lrcorner\ (0 \le n \le N)$. We have: $\widetilde{G}_0 \doteq \chi_s(E)$, $\widetilde{G}_N \doteq$ w. From the assumption $\mathcal{F}$ follows

LEMMA 12. $\mathfrak{z}^+\llcorner\widetilde{G}_n\lrcorner \approx \mathfrak{z}^+\llcorner\widetilde{G}_{n-1}\lrcorner \quad (1 \le n \le N)$.

From Lemma 12 we get the equality $\mathfrak{z}^+\llcorner\chi_s(E)\lrcorner \doteq w$. Theorem 2 is proved. In the case when $\mathfrak{B}$ is an autonomous BaRed, the proof given of Theorem 2 can be appreciably simplified.

Remark. The concepts of CRF and CRO extend in a natural way to the case when as initial data for the development of the algorithmic processes one takes arbitrary words or sequences of words in a given alphabet A. We note the basic changes in the system of definitions (it is assumed that the alphabet A does not contain the letters (,), [,], {, }, $\odot$, $\cdot$, $\succ$, $\vdash$). As admissible values of ObVa we will consider any A-words. Any word of the form $S_0 t_{i_1} S_1 \ldots t_{i_k} S_k$, where $S_0, S_1, \ldots, S_k$ are A-words and $k \ge 0$, will be called a word-form over A. We shall say that the word-form U contains the A-word S, if U goes into S under some collection of admissible values of the ObVa. We shall say that the word-forms U and V are disjoint if the sets of values of the word-forms U and V are disjoint. In the definition of the schematic terms we replace the generating rules (1) and (3), respectively, by the following rules: (1) S is an A-word $\vdash$S is a ScheTe, (3) C_1 is a ScheTe, C_2 is a ScheTe $\vdash C_1C_2$ is a ScheTe. The remaining definitions are brought into correspondence with those mentioned. Upon the introduction in this way of the extrapolation of the concepts of CRF and CRO, Theorems 1 and 2 extend together with their proofs.

Section 4

Let $\mathfrak{B}$ be an autonomous BaRed. For each m ($1 \le m \le \bar{m}$) we let correspond to the ScheFu ε_m a certain FuVa γ_m of the same type as it, in such a way that different ScheFu will be associated with different FuVa. We introduce notation: $\Gamma \rightleftharpoons \langle\gamma_m\rangle_1^{\bar{m}}$, $\widetilde{B}_m \rightleftharpoons \llcorner B_m \big|^{E}_{\Gamma}\lrcorner\ (1 \le m \le \bar{m})$. The system of functional equations

$$\gamma_{\iota_1}(\mathfrak{U}_1) \simeq \widetilde{B}_1, \ldots, \gamma_{\iota_{\bar{p}}}(\mathfrak{U}_{\bar{p}}) \simeq \widetilde{B}_{\bar{p}} \tag{*}$$

has the following properties: (a) the sequence of almost atomic ObTe $\langle\gamma_{\iota_p}(\mathfrak{U}_p)\rangle_1^{\bar{p}}$ is regular in the same sense in which in [5] the term "regular sequence of CanScheTe" is applied; (b) any ObVa occurring in the term $\widetilde{B}_p$ also occurs in $\gamma_{\iota_p}(\mathfrak{U}_p)\ (1 \le p \le \bar{p})$; (c) any FuVa occurring in the sequence $\langle\widetilde{B}_p\rangle_1^{\bar{p}}$ also occurs in the sequence $\langle\gamma_{\iota_p}(\mathfrak{U}_p)\rangle_1^{\bar{p}}$. We shall call systems of functional equations satisfying these conditions canonical systems of equations without parameters. [Here we do not consider the case when in (*) there figure notations for certain separately given CRF, since one can always adjoin (after suitable change of schematic functors) bases of reduction defining such CRF to the BaRed considered and arrive at a system of equations of the form (*), in which to the separately given CRF will also correspond its FuVa.]

It follows from Theorems 1 and 2 that the sequence of CRF $\langle\{\varepsilon_m;\mathfrak{B}\}\rangle_1^{\bar{m}}$ is a minimal solution of this system of equations – minimal from the point of view of the relation $\sqsubseteq$, defined for sequences of CRF $\langle U_m\rangle_1^{\bar{m}}$

and $\langle V_m \rangle_1^{\bar{m}}$ in the following way: $\langle \mathfrak{U}_m \rangle_1^{\bar{m}} \sqsubseteq \langle \mathfrak{V}_m \rangle_1^{\bar{m}}$ if and only if for each m ($1 \le m \le \bar{m}$) the CRF V_m is an extension (in a broad sense) of the CRF U_m. Theorems 1 and 2, considered together, can be interpreted as one theorem on the realizability and uniqueness of a minimal fixed point of the defined operator over sequences of CRF, closely connected with the BaRed $\mathfrak{B}$. Here we have in mind the operator $\mathfrak{V}$, defined in the following way. We denote by $\widetilde{\mathfrak{B}}$ the parametric BaRed

$$\varepsilon_{\nu_1}(\mathfrak{U}_1) \succ \widetilde{B}_1, \ldots, \varepsilon_{\nu_{\bar{p}}}(\mathfrak{U}_{\bar{p}}) \succ \widetilde{B}_{\bar{p}}$$

and we introduce the following CRO: $\mathfrak{V}_m \rightleftharpoons \{\varepsilon_m; \widetilde{\mathfrak{B}}_m; \Gamma\}$ $(1 \le m \le \bar{m})$, where $\widetilde{\mathfrak{B}}_m$ denotes the BaRed, composed of those terms of the BaRed $\mathfrak{B}$ on whose left sides occur the ScheFu ε_m. (It is easy to see that the CRO $\mathfrak{V}_m$ is equivalent with the CRO $\{\varepsilon_m; \widetilde{\mathfrak{B}}; \Gamma\}$) The sequence of CRO $\langle \mathfrak{V}_m \rangle_1^{\bar{m}}$ we denote by $\mathbb{E}$. This sequence is an operator, transforming any sequence of CRF, of the same type as the list $\mathfrak{V}$, into a sequence of CRF of the same type as this list. For the operator $\mathfrak{V}$ the sequence of CRF $\langle \{\varepsilon_m; \mathfrak{B}\} \rangle_1^{\bar{m}}$ is a minimal fixed point.

Each CRO over sequences of CRF can be defined by means of a certain BaRed of the form now considered. Consequently, Theorems 1 and 2 repeat, in connection with the concepts of CRF and CRO, Kleene's theorem on the realizability and uniqueness of a minimal fixed point of a recursive operator over PRF (cf. [1, Theorem XXVI]). However the proof of Kleene's theorem and the proofs of Theorems 1 and 2 suggest the construction of fixed points in the realms of algorithms of different types. These proofs appeal to a specific method of developing the process of calculating the values of PRF, on the one hand, and the process of calculating the values of CRF, on the other hand. The types of these processes differ from one another radically. However, in view of the fact that in point (ii) of the proof of Theorem XXVI of [1] the type of the arithmetic algorithm φ' does not play a role, and on the other hand, the proof given above of Theorem 2 carries over word for word to the case when $\chi_1, \ldots, \chi_{\bar{m}}$ are arithmetic algorithms of arbitrary type, we, combining Kleene's recursion theorem, point (iii) of the proof of this theorem, and Theorems 1 and 2, get the following assertion:

THEOREM 3. For any m ($1 \le m \le \bar{m}$) the CRF $\{\varepsilon_m; \mathfrak{B}\}$ is equivalent (as arithmetic algorithm) with the PRF defined (in the sense of Kleene) by the system of conditional equalities (*) with distinguished functional sign γ_m.

Section 5

Bases of reductions defining CRF and sequences of CRF are similar in certain respects to recursive programs. The term "recursive program" is applied here in the same sense as, e.g., in [6], Secs. 5.2 and 5.2.3. Using the symbolics and terminology adopted in [5] and used above, one can write a recursive program in the form of a list

$$h_1^n(\langle t_i \rangle_1^n) \succ \tau_1[\langle h_j^n \rangle_1^m](\langle t_i \rangle_1^n), \ldots, h_m^n(\langle t_i \rangle_1^n) \succ \tau_m[\langle h_j^n \rangle_1^m](\langle t_i \rangle_1^n),$$

where $\tau_1, \ldots, \tau_m$ are symbols of operators, given in the form of terms, each of which is constructed from the symbols of certain given functions and schematic functors, belonging to the list $h_1^n, \ldots, h_m^n$.

A basis of reductions, giving a sequence of CRF, can have a more complicated form – in it there can be several schemes of reductions for one and the same ScheFu. From the point of view expounded in [6] of the theory of recursive programs, this difference is inessential, since in the list of symbols of the given functions one is allowed to include the expression <u>if</u> . . . <u>then</u> $\not\succ$ . . . <u>else</u> (in the following account it is denoted by <u>ite</u>), which is given the role of three-placed functional symbol, used for writing in termlike form the operation of branching of a set of functions and from this point of view is suitable for the "gluing" of several schemes of reductions of the form mentioned above into one, having form admissible in recursive programs.

However, the customary operational understanding of <u>ite</u> as a specifically calculable prescription with specific sequence of calculations of values of three arguments leads, as is known, to the fact that linguistic expressions constructed on the model of terms using <u>ite</u> and symbols for algorithmically defined functions with infinite process of calculation of the value for certain values of the variables, "behave" in general not like terms of ordinary mathematical languages, and the operational (constructive) semantics of such linguistic expressions is awkward. On the other hand, passage by means of "gluing" with the help <u>ite</u> from a given BaRed to a recursive program is actually the transition from the direct writing down of a certain list of reduction schemes to the indirect writing down of the same list in a form impeding the clear description of the connection between the functions considered in a sufficiently simple form (e.g., in the form of conditional equalities). In the language of CRF and CRO the operation of branching of a set of two functions can be given in the form of the CRO $\{h^n; \mathfrak{L}_0; f_1^n, f_1^n, t_0\}$, where $\mathfrak{L}_0$ denotes the BaRed $h^n(\langle t_i\rangle_1^n) \succ h^{n+1}(t_0, \langle t_i\rangle_1^n)$, $h^{n+1}(0, \langle t_i\rangle_1^n) \succ f_1^n(\langle t_i\rangle_1^n)$, $\urcorner$ $h^{n+1}(t_0|, \langle t_i\rangle_1^n) \succ f_2^n(\langle t_i\rangle_1^n)$. Using the language of CRF and CRO we get the possibility of expressing many connections between functions of interest to us in the form of conditional equalities, in which the terms are understood in the usual sense.

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Essential Corrections to [5]. On p. 252, line 4 from the top: written: $(\mathfrak{P}^0_{m+2,n} \circ \mathfrak{P}^\omega_{a,i})$; should be $(\mathfrak{P}^\omega_{m+2,n,i} \circ \mathfrak{P}^0_a)$. On p. 256, line 16 from the bottom: written: $(\mathfrak{P}^0_{m+2,n} \circ \mathfrak{P}_{a,\beta})$; should be $(\mathfrak{P}_{m+2,n,\beta} \circ \mathfrak{P}^0_a)$.