A HIERARCHY OF BROUWER CONSTRUCTIVE FUNCTIONALS

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1. In the mathematical literature several approaches are considered to make more precise L. E. J. Brouwer's idea of the concept of arithmetic functional defined on unary arithmetic functions and computable at any such function from a finite collection of its values. In particular, in Weyl's paper [1] the "initial stage" was outlined (in the form of isolated examples) of a certain hierarchy of concrete types of functionals corresponding to this idea. The "hierarchic" approach was developed on the basis of the concepts of classical mathematics in Kalmar's paper [2]. In recent years considerable attention has been given in the literature to an approach based on a generalized inductive definition (GLD) of a special class K of total recursive (in other terminology — general recursive) functions — cf., in particular, [3, 4]. However, there is considerable difficulty involved in the following question: in what sense the formulation called generalized inductive definition, e.g., the definition of the class K, is a definition of a certain concept, in what sense this formulation "distinguishes" certain total recursive functions among all possible potentially realizable total recursive functions?

In the opinion of the author of this paper, not only the formulations called in the literature generalized inductive definitions, but also the formulations called inductive definitions of concrete types of words (having in mind "ordinary", not generalized inductive definitions - cf., e.g., [5, Sec. 53]), actually represented only descriptions of "intellectual concepts," only *formulations of requirements*, demanded of the original "sufficiently sharp" definitions of certain concepts.

If $\mathfrak{D}$ is an "ordinary" inductive definition of a certain type of word in the alphabet A, then one can always define the "most restricted" concept satisfying all the requirements of which the formulation of $\mathfrak{D}$ is made up, by means of "distinguishing conditions" of the form «X is a word in the alphabet A, derivable in $\mathfrak{P}$», where $\mathfrak{P}$ is some canonical calculus of E. Post in some extension B of the alphabet A (here it is intended that the concept «word in the alphabet B», used in the definition of the concept «derivation in the canonical calculus $\mathfrak{P}$» is characterized by the following *genetic* definition: by a word in B is meant first a special symbol not belonging to B and called the empty word in B, and second the result of developing in discrete steps and stopping after some step a potentially realizable process, the first step of which consists of constructing one of the letters of the alphabet B and each new step consists of constructing to the right of the last letter of the result of the immediately preceding step, alongside this letter one of the letters of the alphabet B) — cf. e.g., [6]. For a GID of the class K, not only is a "deciphering" of the indicated type impossible, but so is a "deciphering" with the help of considerably "stronger" languages by the methods of constructive mathematics. In the literature about intuitionistic mathematics, as a "deciphering" of a GID of the class K certain clarifications are offered, appealing in an essential way to the intuitionistic representation of "freely developing sequences" of corteges of natural numbers. Such clarifications are outside the domain of the constructive direction in mathematics.

In order to make the above-mentioned idea of Brouwer more precise in the realm of constructive mathematics, it is expedient to use appropriate constructive variants of the ideas of Weyl and Kalmar about the hierarchic approach to the problem considered. The basic goal

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of this paper consists of describing one such constructive variant. Another constructive variant, close to the one presented below in "basic tendency" but different from it in technical apparatus, can be "extracted" from the one written by using the concepts of the classical mathematical paper [7].

2. In the following account, the terminology and symbolism of [8] are used. In particular, the symbols θ_1, θ_2, and θ will be applied in the same sense as in [8]; here we shall assume that θ satisfies all the conditions enumerated in Subsecs. 3.1-3.3 of [8]. By G we shall denote the algorithm given in Sec. 3.3 of [9], which gives a one-to-one map of the set of all corteges of NN onto the set of all NN, and by $\bar{G}$ we denote the algorithm inverse to G. By the symbol L we denote КРФ(θ), which computes for any NN t_0 the number of terms of the cortege $\bar{G}_{\llcorner} t_{0\lrcorner}$. By $\prec$ we denote the binary КРФ(θ) such that $\prec(t_1, t_2)=0$ if and only if the cortege $\bar{G}_{\llcorner} t_{2\lrcorner}$ is an extension (possibly not proper) of the cortege $\bar{G}_{\llcorner} t_{1\lrcorner}$; the expression $T_1 \prec T_2$, where T_1 and T_2 are ПрТе(θ), will denote the formula $\prec(T_1, T_2)=0$.

The hierarchy of types of constructive operators of Brouwer will be defined on the basis of the hierarchy of two-parameter formulas of the language $\mathscr{L}_\theta$ (with objective parameters t_0 and t_1), called predicates below, by stopping corteges of NN, or, briefly, by stopping predicates (StPr). If β is a ConOr from some scale of ConOr such that we have been able to construct the hierarchy of stopping predicates "up to β inclusive," then the StPr, corresponding in this hierarchy to ConOr β will be written as follows: « the NN t_0 is the Gödel number of the total unary КРФ, stopping the cortege of NN $\bar{G}_{\llcorner} t_{1\lrcorner}$ at height not exceeding β». We shall use the following notation:

$$\Xi \leftrightharpoons \forall t_1 \exists t_2 \ulcorner \tau_1(t_0, t_1, t_2)\urcorner;$$
$$\mathcal{U} \leftrightharpoons (\Xi \,\&\, \forall t_2(\ulcorner \tau_1(t_0, t_1, t_2)\urcorner \longrightarrow \neg \ulcorner \nu(t_2)\urcorner));$$
$$\mathcal{V} = (\Xi \,\&\, \exists t_2 \forall t_3 \exists t_4 \ulcorner \chi^\circ(\langle t_i \rangle_0^4)\urcorner),$$

where χ° is a five-place КРФ(θ) constructed (with the aid of the algorithms $\mathfrak{E}$ and λ — cf. [8]) so that in $\mathcal{J}_\theta$ one can derive the formula

$$\forall t_2(\ulcorner \tau_1(t_0, t_1, t_2)\urcorner \longrightarrow \neg \ulcorner \nu(t_2)\urcorner) \leftrightarrow \exists t_2 \forall t_3 \exists t_4 \ulcorner \chi^\circ(\langle t_i \rangle_0^4)\urcorner;$$
$$\xi \leftrightharpoons \mathfrak{f}^5; \quad H \leftrightharpoons \exists t_2 \forall t_3 \exists t_4 \ulcorner \xi(\langle t_i \rangle_0^4)\urcorner; \quad H_\varphi \leftrightharpoons {}_{\llcorner} H \mathfrak{f}^{\xi}_{\varphi\lrcorner},$$

where φ is an arbitrary five-place КРФ(θ). The formula $\mathcal{V}$ is considered as the initial term of the hierarchy StPr. The inductive step will be realized with the aid of an algorithm $\mathfrak{H}$, defined on formulas of the form $(\Xi \,\&\, H_\varphi)$ and such that

$$\mathfrak{H}_{\llcorner}(\Xi \,\&\, H_\varphi)_{\lrcorner} \asymp (\Xi \,\&\, \exists t_5 \forall t_6 ((t_6 \succ t_1 \,\&\, L(t_6) \geqslant L(t_1)+t_5) \longrightarrow {}_{\llcorner} H_\varphi \mathfrak{f}^{t_1}_{t_6\lrcorner})).$$

Using Lemma 3 of [8] and the algorithms $\mathfrak{E}$ and λ we construct a five-place ФуТе(θ) Φ such that $(\xi /\!/ \Phi)$ and in $\mathcal{J}_\theta$ one can derive the formula

$$\exists t_5 \forall t_6 ((t_6 \succ t_1 \,\&\, L(t_6) \geqslant L(t_1)+t_5) \longrightarrow {}_{\llcorner} H \mathfrak{f}^{t_1}_{t_6\lrcorner}) \leftrightarrow \exists t_2 \forall t_3 \exists t_4 \ulcorner \Phi(\langle t_i \rangle_0^4)\urcorner.$$

We construct an algorithm $\bar{\mathfrak{H}}$ such that

$$\bar{\mathfrak{H}}_{\llcorner}(\Xi \,\&\, H_\varphi)_{\lrcorner} \asymp (\Xi \,\&\, \exists t_2 \forall t_3 \exists t_4 \ulcorner \Phi^*_\varphi(\langle t_i \rangle_0^4)\urcorner),$$

where Φ^*_φ denotes the values of the constant ФуТе ${}_{\llcorner}\Phi \mathfrak{f}^{\xi}_{\varphi\lrcorner}$. We denote by χ^ω and X^ω algorithms applicable to each NN and such that $\chi^\omega_0 \asymp \chi^\circ$ and for any j, the word χ^ω_{j+1} is the value of the constant ФуТе ${}_{\llcorner}\Phi \mathfrak{f}^{\xi}_{\chi^\omega_j\lrcorner}$ and

$$X_j^{\omega} \rightleftharpoons (\Xi \,\&\, \exists t_2 \forall t_3 \exists t_4 \ulcorner \chi_j^{\omega}(\langle t_i \rangle_0^4)\urcorner).$$

It is obvious that $X_0^{\omega} \rightleftharpoons \mathcal{V}$ and $X_{j+1}^{\omega} \rightleftharpoons \overline{\Phi}\eta_{\llcorner} X_{j\lrcorner}^{\omega}$ $(j=0,1,\ldots)$. We shall assume that the formula X_j^{ω} is the j-th term of the hierarchy StPr. To realize the ω step we construct a total unary КРФ g such that $g(j)=\{\chi_j^{\omega}\}$ for any j (cf. [8, Sec. 3.3]), and we introduce the following notation:

$$Y_j^{\omega} \Leftrightarrow (\ulcorner \tau_2(\{g\}, j, t_5)\urcorner \,\&\, \ulcorner \tau_5(\nu(t_5), \langle t_i \rangle_0^4, t_6)\urcorner \,\&\, \ulcorner \nu(t_6)\urcorner).$$

On the basis of Theorem XIX(a) of [5] for any j the closed formula

$$\ulcorner \chi_j^{\omega}(\langle t_i \rangle_0^4)\urcorner \longleftrightarrow \exists t_5 \exists t_6 Y_j^{\omega} \tag{1}$$

is true. With the aid of (C_{20}) of [8] and the algorithms and λ, we construct a six-place КРФ(θ) $\mathfrak{X}$ such that for any j in $\mathfrak{I}_{\theta}$ one can derive the formula

$$\exists t_4 \exists t_5 \exists t_6 Y_j^{\omega} \longleftrightarrow \exists t_4 \ulcorner \overline{\chi}_{\omega}(\langle t_i \rangle_0^4, j)\urcorner.$$

For each j in $\mathfrak{I}_{\theta}$ from (1) one can derive the formula

$$X_j^{\omega} \longleftrightarrow (\Xi \,\&\, \exists t_2 \forall t_3 \exists t_4 \ulcorner \overline{\chi}_{\omega}(\langle t_i \rangle_0^4, j)\urcorner).$$

For the construction of the hierarchy StPr the ω step consists of replacing the meta-linguistic variable j by the constructed by the formula

$$\exists t_5 (\Xi \,\&\, \exists t_2 \forall t_3 \exists t_4 \ulcorner \overline{\chi}_{\omega}(\langle t_i \rangle_0^4, t_5)\urcorner) \tag{2}$$

and then with the help of (C_{20}) of [8] and the algorithms $\mathfrak{X}$ and λ constructing a five-place КРФ(θ) $\chi_{\omega}^{\omega+1}$ such that formula (2) is equivalent in $\mathfrak{I}_{\theta}$ with the formula

$$(\Xi \,\&\, \exists t_2 \forall t_3 \exists t_4 \ulcorner \chi_{\omega}^{\omega+1}(\langle t_i \rangle_0^4)\urcorner). \tag{3}$$

We denote by $X^{\omega+1}$ the algorithm defined for each NN and for the symbol ω and such that $X_j^{\omega+1} \rightleftharpoons X_j^{\omega}$ $(j=0,1,\ldots)$ and $X_{\omega}^{\omega+1}$ represents (3). This formula will be considered the term of the hierarchy StPr having ordinal number ω.

3. The constructed hierarchy can be "equipped with a superstructure" according to the following definition. Let α be in ConOr from some scale of ConOr such that one can realize an algorithm conjugate with α (cf. [8, Sec. 5.3]); let k be some algorithm conjugate with α and χ be an algorithm transforming any ConOr β, preceding α, into some five-place КРФ(θ) χ_{β}. We shall say that χ is the (α, k) *spectrum of stopping characteristics* if first for any ConOr β such that $\beta+1<\alpha$, $\chi_{\beta+1}$ represents the value of the constant $\Phi\text{yTe}_{\llcorner}\Phi\{^{\tau}_{\chi_{\beta}\lrcorner}$, and second, for any limit ConOr γ, preceding α, the КРФ(θ) χ_{γ} can be constructed ("reproduced") by the same constructive method which constructed the КРФ(θ) $\chi_{\omega}^{\omega+1}$ above, but using in place of the КРФ g a total unary КРФ g^{γ}, such that $g^{\gamma}(j)=\{\chi_{k_{\llcorner}j\lrcorner}\}$ for any j. If is any (α, k) spectrum of stopping characteristics, then the algorithm X such that for any ConOr β preceding α,

$$X_{\beta} \rightleftharpoons (\Xi \,\&\, \exists t_2 \forall t_3 \exists t_4 \ulcorner \chi_{\beta}(\langle t_i \rangle_0^4)\urcorner),$$

will be called (α, k) *spectrum of stopping predicates*. About the construction of spectra of StPR one can say the same thing that was said in [8] (end of Sec. 5.5) about the construction of spectra of algorithms of КвРаз of standard formulas of the language $\mathcal{Y}_0$.

If β is a ConOr, k is an algorithm conjugate with $\beta+1$, and X* is an algorithm such that we have succeeded "sufficiently convincingly" in justifying the assertion that X* is a $(\beta+1, k)$ spectrum of StPr, then we shall say that the hierarchy StPr is constructed up to the ConOr β (inclusive). Let β, k and X* be just such constructive objects. Further, let F be a unary КРФ, such that the pair of NN $\mathcal{E}F3,0$ satisfy the predicate X^*_β. Then F defines by some special method a definite algorithm B_F, called *a constructive Brouwer functional of rank* β. The process of computing the value of the functional B_F at a given total unary ψ is the process developed in the following way: successively one computes the values of the function F on NN, $G_L\psi(0)_\lrcorner$, $G_L\psi(0),\psi(1)_\lrcorner$, $G_L\psi(0),\psi(1),\psi(2)_\lrcorner, \ldots$ and this process is stopped when for the first time one gets a NN different from 0. The natural number one less than that NN which is obtained in the way indicated at the time of stopping the process is the value of the functional B_F on ψ.

With the aid of the method of transfinite induction up to β, one can prove the following assertion.

THEOREM 1. If F is a total КРФ, giving a constructive Brouwer functiona of rank β, and ψ is a total unary КРФ, then one can realize an NN n such that

$$F(G_L \langle\psi(i)\rangle_0^n{}_\lrcorner) \neq 0.$$

THEOREM 2. If $\mathcal{R}$ is some law of a constructive finitary flow (cf. [9; 10, Point 15]) and F is a total КРФ, defining a constructive Brouwer functional of rank β, then one can realize a NN n such that for any total unary КРФ ψ, for which every cortege of the form $\langle\psi(i)\rangle_0^k$ is admissible by the law of flow $\mathcal{R}$, at least one of the NN

$$F(G_L\psi(0)_\lrcorner),\ F(G_L\psi(0),\psi(1)_\lrcorner), \ldots,\ F(G_L\psi(0),\ \psi(1), \ldots, \psi(n)_\lrcorner)$$

is different from zero.

Theorem 2 is a constructive variant of the theorem of L. E. J. Brouwer on uniform continuity of arithmetic functionals.

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Supplement. In [8] the following corrections should be made.

Page	Line	Printed	Should be
252	4 from the top	$\mathcal{P}^{\omega}_{a,i+1} \mathrel{\overline{\overline{\circ}}} (\mathcal{P}^{\circ}_{m+2,n} \circ \mathcal{P}^{\omega}_{a,i})$	$\mathcal{P}^{\omega}_{a,i+1} \mathrel{\overline{\overline{\circ}}} (\mathcal{P}^{\omega}_{m+2,n,i} \circ \mathcal{P}^{\circ}_{a})$
256	I6 from the top	$\mathcal{P}_{a,\beta+1} \mathrel{\overline{\overline{\circ}}} (\mathcal{P}^{\circ}_{m+2,n} \circ \mathcal{P}_{a,\beta})$	$\mathcal{P}_{a,\beta+1} \mathrel{\overline{\overline{\circ}}} (\mathcal{P}_{m+2,n,\beta} \circ \mathcal{P}^{\circ}_{a})$

One should make the analogous changes in the corresponding definitions of Sec. 6.3.

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