ON THE QUANTIFIER OF LIMITING REALIZABILITY

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In the searches for "contentwise"-interesting constructive analogs of the theorems of classical mathematics, there occur useful logical connectives occupying an intermediate position between $\underset{\cdot}{\exists}$ and $\exists$ and between $\underset{\cdot}{\vee}$ and $\vee$ [$\underset{\cdot}{\exists} x F$ denotes $\neg\forall x \neg F$, and $(F_1 \underset{\cdot}{\vee} F_2)$ denotes $\neg(\neg F_1 \& \neg F_2)$] Two logical connectives of this types, suggested by the theory of limitedly computable (semicomputable) functions and defined in terms of the basic logical connectives of constructive logic, viz., the quantifier $\underset{\rightarrow}{\exists}$ of limiting realizability and the quantifier $\underset{\rightarrow}{\vee}$ of limiting disjunction, are introduced into consideration in the article. A number of properties are established for these logical connectives.

1. In constructive mathematics when considering the question of the truth of some statement H or other, formulated in one of the languages of constructive mathematics (for example, in the logicoarithmetic language $\mathcal{L}_\theta$; see Sec. 3 in [1]), the following situation emerges in certain cases: Inference H is refutable (i.e., inference $\neg H$ is provable), and at the same time also provable is a certain inference H° obtained by substituting into H the metalingual symbols $\exists$ and $\vee$ in the place of certain positive occurrences of the logical symbols $\underset{\cdot}{\exists}$ and $\underset{\cdot}{\vee}$, respectively; the former symbols denote definite derived (i.e., expressible in terms of the original ones) logical connectives, namely, $\underset{\cdot}{\exists} z F \leftrightharpoons \neg\forall z \neg F$, $(F_1 \underset{\cdot}{\vee} F) \leftrightharpoons \neg(\neg F_1 \& \neg F_2)$, where F , F_1, and F_2 are formulas and z is an objective variable (here and below we use without explanation the symbology and terminology from Secs. 1-4 of [1]). In such a situation the passage from H to H° can be treated as a certain correction of a false conjecture H , i.e., a correction by means of "far going" weakenings of certain positive occurrences in H of subformulas of form $\exists z F$ and of form $(F_1 \vee F_2)$.

In the situation being examined now it can happen that by means of weakenings, less significant then when passing to H° , of the mentioned positive occurrences in H of subformulas of the two forms indicated, we can obtain corrections of conjecture H° less "crude" and more interesting "in intension" than H . In particular, it can happen that for corrections of conjecture H it is sufficient to substitute in the place of the mentioned positive occurrences in H of subformulas of forms $\exists z F$ and $(F_1 \vee F_2)$ the formulas $(\&_{i=1}^{n} (P_i \vee \neg P_i) \to \exists z F)$ and $(\&_{i=1}^{n} (P_i \vee \neg P_i) \to (F_1 \vee F_2))$, respectively, where $P_1, \ldots, P_n$ are certain "conjectured" formulas having a comparatively simple logical form (for example, having the form $\exists x G$ or the form $\exists x \forall y G$, where G is a quantifier-free formula) and not having parameters other than the parameters of formulas $\exists z F$ and $(F_1 \vee F_2)$, respectively. In constructive

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mathematical analysis and in other sections of constructive mathematics many examples are known of obtaining interesting "in intension" provable constructive analogs of certain theorems of classical mathematics by means of correcting precisely by this method the refutable literal constructive analogs of the theorems of classical methematics being examined or the refutable assertions close to the literal constructive analogs.

Another approach to the search for corrections of conjecture H° less "crude" and more interesting "in intension" than H is based on the use of derived logical connectives taking an intermediate position between $\underset{\cdot}{\exists}$ and $\exists$ and between $\underset{\cdot}{\vee}$ and $\vee$. Such logical connectives can be introduced, in particular, in the following way (for definiteness we restrict the consideration to language $\mathcal{L}_{\theta}$). Let P and Q be formulas in language $\mathcal{L}_{\theta}$. Regarding the formula pair P,Q we shall say that it is admissible if P is a one-parameter formula, Q is a two-parameter formula, the single parameter of P (we denote it u) is also a parameter of formula Q (we denote the second parameter of formula Q by v), and the following three statements are true:

$$\forall u (P \rightarrow \forall v \forall w ((Q \,\&\, \llcorner Q \updownarrow^{v}_{w} \lrcorner) \rightarrow (w = v))) \quad | w \not\rightsquigarrow Q,$$

$$\forall u (P \rightarrow \underset{\cdot}{\exists} v Q), \qquad \forall v \exists u (P \,\&\, Q)$$

(here and below we have in mind truth in the sense of majorant semantics described in [1]). These statement we denote, respectively, by $\mathfrak{A}_1(P,Q)$, $\mathfrak{A}_2(P,Q)$, and $\mathfrak{A}_3(P,Q)$.

<u>Remark.</u> If statements $\mathfrak{A}_1(P,Q)$ and $\mathfrak{A}_2(P,Q)$ are true, then the binary predicate characterized by formula $(P \,\&\, Q)$, in which the first parameter is taken to be parameter u and the second to be v, is a unary pseudofunction (see [2], for example) specified on positive integers satisfying condition P, and the pair P,Q can be looked upon as some variant of specifying this pseudofunction. This variant is convenient in that it permits us to achieve a simplicity, essentially greater than under the direct use of the pseudofunction mentioned, of the logical structure of the conditions characterizing the derived logical connectives to be examined below (these connectives have the concept of a pseudofunction as their own ideological origin).

Let formulas P and Q be such that the pair P,Q is admissible; let F, F_1, and F_2 be formulas and z be an ObVa (objective variable). We introduce the following notation:

$$\underset{P,Q}{\exists} z F \leftrightharpoons \exists y (\llcorner \tilde{P} \updownarrow^{u}_{y} \lrcorner \,\&\, \forall z (\llcorner \tilde{Q} \updownarrow^{u,v}_{y,z} \lrcorner \rightarrow F)),$$

$$(F_1 \underset{P,Q}{\vee} F_2) \leftrightharpoons \underset{P,Q}{\exists} x (((x = 0) \rightarrow F_1) \,\&\, (\neg (x = 0) \rightarrow F_2));$$

here $\tilde{P}$ and $\tilde{Q}$ are formulas, while y and x are ObVa such that the following conditions are fulfilled: $\tilde{P}$ is congruent with P (see Sec. 33 in [3]), $\tilde{Q}$ is congruent with Q, $y \not\rightsquigarrow F$, $y \not\equiv z$, $y \underset{u}{\not\rightsquigarrow} \tilde{P}$, $y \underset{u}{\not\rightsquigarrow} \tilde{Q}$, $z \underset{v}{\not\rightsquigarrow} \tilde{Q}$, $x \not\rightsquigarrow F_1$, $x \not\rightsquigarrow F_2$. In particular, if $u \not\rightsquigarrow F$ and $z \mathrel{\dot{\equiv}} v$, then $\underset{P,Q}{\exists} z F \leftrightharpoons \exists u (P \,\&\, \forall v (Q \rightarrow F))$. The symbols $\underset{P,Q}{\exists}$ and $\underset{P,Q}{\vee}$ can be looked upon as the notation for certain derived logical connectives. It is easy to see that the formulas

$$(\exists z F \rightarrow \underset{P,Q}{\exists} z F), \qquad (\underset{P,Q}{\exists} z F \rightarrow \underset{\cdot}{\exists} z F),$$

$$((F_1 \vee F_2) \rightarrow (F_1 \underset{P,Q}{\vee} F_2)), \quad ((F_1 \underset{P,Q}{\vee} F_2) \rightarrow (F_1 \underset{\cdot}{\vee} F_2))$$

are deducible in the calculus resulting from adding the formulas $\mathfrak{A}_1(P,Q)$, $\mathfrak{A}_2(P,Q)$, and $\mathfrak{A}_3(P,Q)$ on to the calculus $\mathfrak{I}_\theta$, (see Sec. 4 in [1]) as additional postulates. Calculus $\mathfrak{I}_\theta$ is admissible from the point of view of majorant semantics, i.e., from the results in [4] it follows immediately that the closure of any formula deducible in this calculus has a true majorant (even a true majorant of finite rank). Consequently, the logical connective $\underset{P,Q}{\exists}$ takes an intermediate position between $\underset{\cdot}{\exists}$ and $\exists$, while the logical connective $\underset{P,Q}{\vee}$, between $\underset{\cdot}{\vee}$ and $\vee$.

In the exposition that follows, our attention will be centered on one concrete admissible formula pair P°, Q° prompted by the theory of limitedly computable (in another terminology, semicomputable) functions (see [5, 6, 7]) and on the two derived logical connectives of the type mentioned above, corresponding to this pair.

2. We introduce some notation. If a positive integer k is the code of an n-ary canonic recursive function φ (i.e., $k = \mathcal{E}\varphi\mathcal{3}$; see Paragraph 3.3 in [1]), then the expression $\{k\}_n$ will be denoted CRF φ; however, if k is not the code of any n-ary CRF, then $\{k\}_n$ will denote some isolated n-ary CRF not applicable to even one n-term cortege of positive integers. On the basis of Theorem XIX(a) from [3] we have
we have

$$\{k\}_n(\langle t_i\rangle_1^n) \simeq \nu\,(\mathcal{M}_{n+1}[\tau_n](k,\langle t_i\rangle_1^n))$$

(for an explanation of the notation see Paragraphs 3.3 and 3.2 in [1]; the symbol $\simeq$ is used as a synonym of the notation eqv, i.e., in the same sense as in Sec. 63 of [3]). In what follows we shall often use the fact that the condition $\ulcorner\tau_n(t_0,\langle t_i\rangle_1^n, t_{n+1})\urcorner$ admits (in accord with the method of constructing CRF τ_n) of the following interpretation: "t_0 is the code of an n-ary CRF, and t_{n+1} is the code of the text of the construction of the value of CRF $\{t_0\}_n$ on the cortege $\langle t_i\rangle_1^n$." Below we shall most often be dealing with unary CRF, and instead of the notation τ_1 and $\{k\}_1$ we shall use the notation τ and $\{k\}$, respectively. Letters u, v, w, x, y, z and these same letters with subscripts and superscripts will be used as metalingual variables whose admissible values are reckoned to be objective variables. When formulating definitions and statements with the use of such metavariables we shall always understand that only objective variables differing from each other are permitted to be substituted for metavariables differing from each other.

We shall use as well the following notation (the "way of reading" the notation introduced is indicated within the brackets):

(u appl. x) $\leftrightharpoons \underset{\cdot}{\exists} w \ulcorner\tau(u,x,w)\urcorner$ [CRF $\{u\}$ is applicable to x];

(v val. u, x) $\leftrightharpoons \underset{\cdot}{\exists} w(\ulcorner\tau(u,x,w)\urcorner \& (v = \nu(w)))$ [v is the value of CRF $\{u\}$ at point x];

(x pt.ps-st. u) $\leftrightharpoons \forall y \forall w_1 \forall w_2((\ulcorner\tau(u,x,w_1)\urcorner \& \ulcorner\tau(u,(x+y),w_2)\urcorner) \rightarrow (\nu(w_1) = \nu(w_2)))$ [x is a point of pseudostabilization of CRF $\{u\}$];

(x pt.rel-st u) $\leftrightharpoons$ ((x pt.ps-st. u) $\&$ (u appl. x)) [x is a point of relative stabilization of CRF $\{u\}$ (a point of stabilization relative to the set of those positive integers to which CRF $\{u\}$ is applicable)];

$(\mathfrak{v}$ lim.val. $\mathfrak{u}) \leftrightharpoons \exists x\, ((x$ pt.ps-st. $\mathfrak{u})\ \&\ (\mathfrak{v}$ pt.ps-st. $\mathfrak{u}, x))$ val. [$\mathfrak{v}$ is the limit value of CRF $\{\mathfrak{u}\}$];

$(\mathfrak{u}$ rel.stab. $\leftrightharpoons \exists x\,(x$ rel-st. . $\mathfrak{u})$ [$\{\mathfrak{u}\}$ is a relatively stabilized CRF];

$(\mathfrak{u}$ stab. $\leftrightharpoons (\exists y\,(x$ pt.ps-st. $\mathfrak{u})\ \&\ \forall y\ (\mathfrak{u}$ appl. $y))$ [$\{\mathfrak{u}\}$ is a stabilized CRF];

$(\mathfrak{u}_1$ lim.eq. $\mathfrak{u}_2) \leftrightharpoons (((\mathfrak{u}_1$ rel.st.$) \leftrightarrow (\mathfrak{u}_2$ rel.st.$))\ \&\ \forall \mathfrak{v}((\mathfrak{v}$ lim.val. $\mathfrak{u}_1) \leftrightarrow (\mathfrak{v}$ lim.val. $\mathfrak{u}_2)))$ [CRF $\{\mathfrak{u}_1\}$ equals CRF $\{\mathfrak{u}_2\}$ in the limit];

$(F$ stable $\mathfrak{u}) \leftrightharpoons \forall \mathfrak{u} \forall \mathfrak{u}'((\mathfrak{u}'$ lim.eq. $\mathfrak{u}) \to (F \to \llcorner F \S^{\mathfrak{u}}_{\mathfrak{u}'} \lrcorner))\ |\mathfrak{u}' \not\rightsquigarrow F$ [formula F is stable with respect to variable $\mathfrak{u}$];

$$P^\circ \leftrightharpoons (\mathfrak{u} \text{ stab.}); \qquad Q^\circ \leftrightharpoons (\mathfrak{v} \text{ lim.val. } \mathfrak{u}).$$

From Lemma B (see below) it follows that statements $\mathfrak{A}_1(P^\circ, Q^\circ)$, $\mathfrak{A}_2(P^\circ, Q^\circ)$, and $\mathfrak{A}_3(P^\circ, Q^\circ)$ are true and, consequently, the formula pair P°, Q° is admissible. We shall denote the derived logical connectives $\exists_{P^\circ, Q^\circ}$ and $\vee_{P^\circ, Q^\circ}$ by, respectively, $\underset{\rightarrow}{\exists}$ and $\underset{\rightarrow}{\vee}$ and we shall call them the quantifier of limiting realizability and the limiting disjuction. In other words,

$$\underset{\rightarrow}{\exists} z F \leftrightharpoons \exists y\,((y \text{ stab.})\ \&\ \forall z((z \text{ lim.val. } y) \to F)) \quad | y \not\rightsquigarrow F,$$
$$(F_1 \underset{\rightarrow}{\vee} F_2) \leftrightharpoons \underset{\rightarrow}{\exists} x\,(((x=0) \to F_1)\ \&\ (\neg(x=0) \to F_2))\,| x \not\rightsquigarrow F_1, x \not\rightsquigarrow F_2.$$

In the subsequent exposition we shall use (where this will not cause misunderstanding) an abbreviated way of writing formulas resulting from the omission of certain pairs of parentheses, viz., the parentheses enclosing atomic formulas or the above-introduced notation for certain predicates, the brackets enclosing conjunctions (disjunctions) that are graphic units of conjunctions (respectively, of disjunctions), and the extreme parentheses in formulas starting off with a parenthesis.

3. The main purpose of the exposition that follows is to establish certain properties of the logical connectives $\underset{\rightarrow}{\exists}$ and $\underset{\rightarrow}{\vee}$. The formulations of the properties implicit here have the form of statements of the deducibility in calculus $\mathfrak{J}^*_\theta$ of formulas of specific types written in abbreviated form with the use of the symbol $\underset{\rightarrow}{\exists}$ or of the symbol $\underset{\rightarrow}{\vee}$. The expression $\mathfrak{J}^*_\theta$ denotes the calculus resulting from the union of the postulates of calculi $\mathbf{E}_\theta$ and $\mathfrak{J}_\theta$ (see Paragraphs 3.2 and 4.1 in [1]) and from the addition of an induction rule for formulas of language $\mathcal{L}_\theta$. Here, in Lemmas C and D and in Sec. 4 we shall imply that θ includes the base of the operator outcome of primitive recursive functions. From the results of [4] it follows immediately that calculus $\mathfrak{J}^*_\theta$ is admissible from the point of view of majorant semantics. In the subsequent exposition the combination of letters "ERF" will be used as an abbreviation for the term "elementary (in the sense of L. Kalmar) recursive function." Here we formulate certain auxiliary statements.

We note first of all that the ERF τ and ν figuring in the equality condition $\{\mathfrak{u}\}(x) \simeq \nu(\mathcal{M}_2[\tau](\mathfrak{u}, x))$ (Theorem XIX(a) in [3] for $n=1$) are constructed such that the formula

$$(\ulcorner \tau(\mathfrak{u}, x, w_1) \urcorner\ \&\ \ulcorner \tau(\mathfrak{u}, x, w_2) \urcorner) \to w_2 = w_1$$

is deducible in $\mathfrak{J}^*_\theta$ and, in addition, the following assertion is provable:

<u>LEMMA A.</u> ERF q_0, q_1, and q_2 can be constructed such that the translations into the language of calculus $\mathfrak{J}^*_\theta$ of the expressions

A.0. $\{q_0(\mathfrak{v})\}(x) \simeq \mathfrak{v}$, A.1. $\{q_1(u_1,u_2)\}(x)\simeq\{u_2\}(\{u_1\}(x))$,

A.2. $\{q_2(u)\}(x)\simeq\{\{u\}(x)\}(x)$

are deducible in this calculus.

In this lemma we have in mind translations of equality conditions realizable with the aid of τ and ν by an algorithm resulting from trivial simplifications for the language of arithmetic of the algorithms described in Sec. 5 of [8] (also see Sec. 7 of [9]). For example, the translation of expression A.0 has the form $(\dot{\exists} w \ulcorner\tau(q_0(v),x,w)\urcorner \& \forall w(\ulcorner\tau(q_0(v),x,w)\urcorner\to\nu(w)=v))$.

LEMMA B. The following formulas as deducible in calculus $\mathcal{J}_\theta^*$:

B.1. u appl. $x\longrightarrow(x$ pt. ps-st. $u\longleftrightarrow x$ pt. rel-st. $u)$,

B.2. u stab. $\longrightarrow u$ rel. stab.,

B.3. $\forall x\,(u$ appl. $x\,)\longrightarrow(u$ rel. stab. $\longleftrightarrow u$ stab.),

B.4. $(x$ pt. rel-st. $u\,\&\,u$ appl. $(x+y))\longrightarrow(x+y)$ pt. rel-st. u,

B.5. $(x$ pt. rel-st. $u\,\&\,\ulcorner\tau(u,x,w)\urcorner)\longrightarrow\nu(w)$ lim. val. u,

B.6. u appl. $x\longrightarrow(v$ val. $u,x\longleftrightarrow\forall w(\ulcorner\tau(u,x,w)\urcorner\to v=\nu(w)))$,

B.7. u rel. stab. $\longrightarrow(v$ lim. val. $u\longleftrightarrow\forall x\forall w((x$ pt. ps-st. $u\,\&\,\ulcorner\tau(u,x,w)\urcorner)\longrightarrow v=\nu(w)))$,

B.8. $(v_1$ lim. val. $u\,\&\,v_2$ lim. val. $u\,)\longrightarrow v_2=v_1$,

B.9 u rel. stab. $\longleftrightarrow\dot{\exists} v\,(v$ lim. val. $u\,)$,

B.10 $\forall v\exists u(u$ stab. $\&$ v lim. val. $u\,)$,

B.11. $(v$ lim. val. $u\,\&\,z$ appl. $v\,)\longrightarrow(x$ val. $z,v\longleftrightarrow x$ lim. val. $q_1(u,z))$,

B.12. $(v$ lim. val. $u\,\&\,z$ appl. $v\,\&$ u' val. $z,v\,)\longrightarrow$ $q_2(q_1(u,z))$ lim. eq. u'.

A.0 (respectively, A.1, A.2) is used in the deduction of formula B.10 (formula B.11, formula B.12).

COROLLARY. Statements $\mathfrak{N}_1(P^\circ,Q^\circ)$, $\mathfrak{N}_2(P^\circ,Q^\circ)$, and $\mathfrak{N}_3(P^\circ,Q^\circ)$ are deducible in $\mathcal{J}_\theta^*$.

LEMMA C. An ERF q_3 can be constructed such that the formula

C.1. $\forall u((\dot{\exists} x(u$ appl. $x)\longrightarrow\forall x(q_3(u)$ appl. $x))\,\&\,q_3(u)$ lim. eq. $u)$

is deducible in $\mathcal{J}_\theta^*$ and, consequently, the formula scheme

C.2. $\underset{\to}{\exists} z F\longleftrightarrow\exists y((y$ rel. stab. $\&\forall z\,(z$ lim. val. $y\to F))\,|y\rightsquigarrow F$

is deducible in $\mathcal{J}_\theta^*$.

The ERF q_3 can be constructed, for example, in the following manner. We construct in succession the CRF p, L, ℓ, d, and φ such that the translations into the language of calculus $\mathcal{J}_\theta^*$ of the expressions

$$p(u,w)=\tau(u,\varkappa_1(w),\varkappa_2(w)),\ L(u)\simeq\mathcal{M}_1[p](u),\ \ell(u,w)\simeq 1\dot{-}p(u,L(u)+w+1),$$

$$d(u,w,v)\simeq\nu(\varkappa_2(L(u)+w+1)\cdot\ell(u,w)+v\cdot(1\dot{-}\ell(u,w)),$$

$$\varphi(u,0)\simeq\nu(\varkappa_2(L(u))),\ \varphi(u,w+1)\simeq d(u,w,\varphi(u,w))$$

are deducible in this calculus. We note that if CRF L has a value at a positive integer (PI) m, then

$$\varphi(m,0)=\nu(\varkappa_2(L(m))),\qquad \varphi(m,w+1)=\begin{cases}\nu(\varkappa_2(L(m)+w+1)) & \text{if } p(m,L(m)+w+1)=0,\\ \varphi(m,w) & \text{if } p(m,L(m)+w+1)\neq 0.\end{cases}$$

Finally, we construct an ERF q_3 such that $q_3(\mathfrak{u}) = \{{}^0\mathcal{K}_{1,1}[\varphi, \mathfrak{u}]\}$. This is the desired ERF.

In the study of the logical connectives $\underset{\rightarrow}{\exists}$ and $\underset{\rightarrow}{\vee}$ we often have to examine formulas of the form $\forall \mathfrak{v} (\mathfrak{v}$ lim.val. $\mathfrak{u} \rightarrow F)$ in the presence of some information or other on formula F. The next lemma says something about certain connections of formulas of this sort with formulas of other sorts.

<u>LEMMA D.</u> The following formula schemes are deducible in calculus $\mathfrak{J}_\theta^*$:

D.1. $(\mathfrak{u}$ stab. $\&(\neg\neg F \rightarrow F)) \rightarrow (\forall \mathfrak{v}(\mathfrak{v}$ lim.val. $\mathfrak{u} \rightarrow F) \leftrightarrow F)$ $|\mathfrak{v} \nrightarrow F$;

D.2. $(\mathfrak{u}$ stab. $\rightarrow (\forall \mathfrak{v}(\mathfrak{v}$ lim.val. $\mathfrak{u} \rightarrow \exists \mathfrak{x} F_0) \rightarrow \underset{\rightarrow}{\exists} \mathfrak{x} \forall \mathfrak{v}(\mathfrak{v}$ lim.val. $\mathfrak{u} \rightarrow F_0))$;

D.3. $(\mathfrak{u}$ stab. $\rightarrow (\forall \mathfrak{v}(\mathfrak{v}$ lim.val. $\mathfrak{u} \rightarrow \exists \mathfrak{x} F_0) \rightarrow \underset{\rightarrow}{\exists} \mathfrak{x} \neg\neg F_0))$ $|\mathfrak{v} \nrightarrow F_0$;

D.4. $(\mathfrak{u}$ stab. $\&(F'$ stable $\mathfrak{u}')) \rightarrow (\forall \mathfrak{v}(\mathfrak{v}$ lim.val. $\mathfrak{u} \rightarrow \exists \mathfrak{u}' (\mathfrak{u}'$ stab. $\& F')) \leftrightarrow \exists \mathfrak{u}'(\mathfrak{u}''$ stab. $\& \forall \mathfrak{v} (\mathfrak{v}$ lim.val. $\mathfrak{u} \rightarrow F')))$.

Formulas B.2 and B.9 are used in the deduction of formula D.1.

Passing to formula D.2, let us outline the deduction of formula $\underset{\rightarrow}{\exists} \mathfrak{x} \forall \mathfrak{v}$ $(\mathfrak{v}$ lim.val. $\mathfrak{u} \rightarrow F_0)$ from the hypotheses $\mathfrak{u}$ stab. and $\forall \mathfrak{v}(\mathfrak{v}$ lim.val. $\mathfrak{u} \rightarrow \exists \mathfrak{x} F_0)$ with the conditions of the deduction theorem being observed. From the second hypothesis, in calculus $\mathfrak{J}_\theta^*$ is deducible the formula $\exists \mathfrak{z} \mathfrak{L}$, where $\mathfrak{L} \leftrightharpoons \forall \mathfrak{v}$ $(\mathfrak{v}$ lim.val. $\mathfrak{u} \rightarrow (\mathfrak{z}$ appl. $\mathfrak{v} \& \forall \mathfrak{x}(\mathfrak{x}$ val. $\mathfrak{z}, \mathfrak{v} \rightarrow F_0))$). On the basis of C.2 it is sufficient to deduce from hypotheses $\mathfrak{u}$ stab. and $\mathfrak{L}$ the formula $\exists \mathfrak{u}'(\mathfrak{u}'$ rel.stab. $\& \forall \mathfrak{x} (\mathfrak{x}$ lim.val. $\mathfrak{u}' \rightarrow \forall \mathfrak{v}(\mathfrak{v}$ lim.val. $\mathfrak{u} \rightarrow F_0)))$. From the hypotheses indicated, with the aid of B.11, we can deduce: $\mathfrak{v}$ lim.val. $\mathfrak{u} \rightarrow \mathfrak{z}$ appl. $\mathfrak{v}$, $q_1(\mathfrak{u}, \mathfrak{z})$ rel.stab.; from those same hypotheses and the additional hypotheses $\mathfrak{x}$ lim.val. $q_1(\mathfrak{u}, \mathfrak{z})$ and $\mathfrak{v}$ lim.val. $\mathfrak{u}$ we can, with the aid of B.11, successively deduce: $\mathfrak{z}$ appl. $\mathfrak{v}, \mathfrak{x}$ val. $\mathfrak{z}, \mathfrak{v}$, $\forall \mathfrak{x}(\mathfrak{x}$ val. $\mathfrak{z}, \mathfrak{v} \rightarrow F_0)$, F_0. Consequently, from the hypotheses $\mathfrak{u}$ stab. and $\mathfrak{L}$ we can deduce the formula $(q_1(\mathfrak{u}, \mathfrak{z})$ rel.stab. $\& \forall \mathfrak{x} \forall \mathfrak{v}(\mathfrak{x}$ lim.val. $q_1(\mathfrak{u}, \mathfrak{z}) \& \mathfrak{v}$ lim.val. $\mathfrak{u}) \rightarrow F_0))$. the deducibility of D.2.

Passing to the deduction of formula D.4, let us outline the deduction of formula $\exists \mathfrak{u}' (\mathfrak{u}'$ stab. $\& \forall \mathfrak{v} (\mathfrak{v}$ lim.val. $\mathfrak{u} \rightarrow F'))$ from the hypotheses $\mathfrak{u}$ stab., F' stable $\mathfrak{u}'$ and $\forall \mathfrak{v} (\mathfrak{v}$ lim.val. $\mathfrak{u} \rightarrow \exists \mathfrak{u}' (\mathfrak{u}'$ stab. $\& F'))$ with the conditions of the deduction theorem being observed. We denote these hypotheses, respectively, by $\mathfrak{D}_1$, $\mathfrak{D}_2$, and $\mathfrak{D}_3$. In vies of the deducibility of formula D.1 in $\mathfrak{J}_\theta^*$ it is sufficient to prove the deducibility from $\mathfrak{D}_1$, $\mathfrak{D}_2$ and $\mathfrak{D}_3$ of the formula $\exists \mathfrak{u}' \forall \mathfrak{v}$ $(\mathfrak{v}$ lim. val. $\mathfrak{u} \rightarrow (\mathfrak{u}'$ stab. $\& F'))$. In $\mathfrak{J}_\theta^*$ from $\mathfrak{D}_3$ is deducible the formula $\exists \mathfrak{z} \mathcal{L}$, where

$$\mathcal{L} \leftrightharpoons \forall \mathfrak{v}(\mathfrak{v} \text{ lim.val. } \mathfrak{u} \rightarrow (\mathfrak{z} \text{ appl. } \mathfrak{v} \& \forall \mathfrak{u}' (\mathfrak{u}' \text{ val. } \mathfrak{z}, \mathfrak{v} \rightarrow (\mathfrak{u}' \text{ stab.} \& F')))).$$

From the hypotheses $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathcal{L}$, and $\mathfrak{v}$ lim.val. $\mathfrak{u}$, it is sufficient to deduce the formula ($\mathfrak{a}'$ stab. $\& \llcorner F' \wr^{\mathfrak{u}'}_{\mathfrak{a}'} \lrcorner$) , where $\mathfrak{a}' \leftrightharpoons q_3(\mathfrak{a}), \mathfrak{a} \leftrightharpoons q_2(q_1(\mathfrak{u}, \mathfrak{z}))$ [note that $\{\mathfrak{a}\}(\mathfrak{x}) \simeq \{\{\mathfrak{z}\}(\{\mathfrak{u}\}(\mathfrak{x}))\}(\mathfrak{x})$] . From the hypotheses $\mathfrak{D}_1$, $\mathcal{L}$, and $\mathfrak{v}$ lim.val. $\mathfrak{u}$ we can deduce in succession (using, in particular, A.1, A.2, B.12, and C.1): $\mathfrak{z}$ appl. $\mathfrak{v}$, $\mathfrak{u}'$ val. $\mathfrak{z}, \mathfrak{v} \rightarrow \mathfrak{u}'$ stab. $\mathfrak{a}$ rel.stab. $\mathfrak{a}'$ stab. $\mathfrak{u}'$ val. $\mathfrak{z}, \mathfrak{v} \rightarrow \mathfrak{a}$ lim.eq. $\mathfrak{u}'$, $\mathfrak{u}'$ val. $\mathfrak{z}, \mathfrak{v} \rightarrow \mathfrak{a}'$ lim.eq. $\mathfrak{u}'$, $\mathfrak{u}'$ val. $\mathfrak{z}, \mathfrak{v} \rightarrow F'$ By bringing in the hypothesis $\mathfrak{D}_2$ as well, we can successively deduce: $\mathfrak{u}'$ val. $\mathfrak{z}, \mathfrak{v} \rightarrow \llcorner F' \wr^{\mathfrak{u}'}_{\mathfrak{a}'} \lrcorner$, $\exists \mathfrak{u}' (\mathfrak{u}'$ val. $\mathfrak{z}, \mathfrak{v}) \rightarrow \llcorner F' \wr^{\mathfrak{u}'}_{\mathfrak{a}'} \lrcorner$, and, finally, $\llcorner F' \wr^{\mathfrak{u}'}_{\mathfrak{a}'} \lrcorner$.

4. Below we shall use the following metalingual relation. We shall say that an ObVa is isolated from an ObVa $\mathbf{z}'$ in formula F', and we shall write $(\mathbf{z}\ \text{isol.}\ \mathbf{z}', F)$, if not even one free occurrence of $\mathbf{z}$ in F is found in the domain of action of some occurrence of the quantifier connecting $\mathbf{z}'$.

<u>THEOREM.</u> The following formula schemes are deducible in calculus $\mathfrak{Y}^*_\theta$:

1. $\exists\mathbf{z}F \longrightarrow \underset{\to}{\exists}\mathbf{z}F$;
2. $\underset{\to}{\exists}\mathbf{z}F \longrightarrow \underset{\bullet}{\exists}\mathbf{z}F$;
3. $\forall\mathbf{z}(F\vee\neg F) \longrightarrow (\underset{\to}{\exists}\mathbf{z}F \longleftrightarrow \exists\mathbf{z}F)$;
4. $\forall\mathbf{z}\forall x(F\vee\neg F) \longrightarrow (\underset{\to}{\exists}\mathbf{z}\exists xF \longleftrightarrow \exists\mathbf{z}\exists xF)$;
5. $\forall\mathbf{z}\forall x(F\vee\neg F) \longrightarrow (\underset{\to}{\exists}\mathbf{z}\forall xF \longleftrightarrow \underset{\bullet}{\exists}\mathbf{z}\forall xF)$;
6. $(\underset{\to}{\exists}\mathbf{z}F \longrightarrow \neg\neg F) \quad \big|\ \mathbf{z} \rightsquigarrow F$;
7. $(\neg\neg F \longrightarrow F) \longrightarrow (\underset{\to}{\exists}\mathbf{z}F \longleftrightarrow F) \quad \big|\ \mathbf{z} \rightsquigarrow F$;
8. $(\underset{\to}{\exists}\mathbf{z}\exists xF \longrightarrow \underset{\to}{\exists}x\neg\neg F) \quad \big|\ \mathbf{z} \rightsquigarrow F$;
9. $\underset{\to}{\exists}\mathbf{z}F \longleftrightarrow \underset{\to}{\exists}\mathbf{z}'\llcorner F\downarrow^{\mathbf{z}}_{\mathbf{z}'}\lrcorner \quad \big|\ \mathbf{z}' \rightsquigarrow F,\ (\mathbf{z}\ \text{isol.}\ \mathbf{z}', F)$;
10. $\underset{\to}{\exists}\mathbf{z}_1\underset{\to}{\exists}\mathbf{z}_2F \longleftrightarrow \underset{\to}{\exists}\mathbf{z}_2\underset{\to}{\exists}\mathbf{z}_1F$;
11. $\underset{\to}{\exists}\mathbf{z}_1\underset{\to}{\exists}\mathbf{z}_2F \longleftrightarrow \underset{\to}{\exists}\mathbf{z}\llcorner F\downarrow^{\mathbf{z}_1,\ \mathbf{z}_2}_{\varkappa_1(\mathbf{z}),\varkappa_2(\mathbf{z})}\lrcorner \quad \big|\ \mathbf{z} \not\rightsquigarrow F,\ (\mathbf{z}\ \text{isol.}\ \mathbf{z}, F), (\mathbf{z}_2\ \text{isol.}\ \mathbf{z}, F)$;
12. $\neg\underset{\to}{\exists}\mathbf{z}F \longleftrightarrow \forall\mathbf{z}\neg F$;
13. $\neg\neg\underset{\to}{\exists}\mathbf{z}F \longleftrightarrow \underset{\bullet}{\exists}\mathbf{z}F$;
14. $(\neg\neg G \longrightarrow G) \longrightarrow ((\underset{\to}{\exists}\mathbf{z}F \,\&\, G) \longleftrightarrow \underset{\to}{\exists}\mathbf{z}(F \,\&\, G)) \quad \big|\ \mathbf{z} \rightsquigarrow G$;
15. $\forall\mathbf{z}(F \longrightarrow G) \longrightarrow (\underset{\to}{\exists}\mathbf{z}F \longrightarrow \underset{\to}{\exists}\mathbf{z}G)$;
16. $(\neg\neg G \longrightarrow G) \longrightarrow ((\underset{\to}{\exists}\mathbf{z}F \longrightarrow G) \longleftrightarrow \forall\mathbf{z}(F \longleftrightarrow G)) \quad \big|\ \mathbf{z} \rightsquigarrow G$;
17. $\forall x(\neg\neg G \longrightarrow G) \longrightarrow ((\underset{\to}{\exists}\mathbf{z}F \longrightarrow \underset{\to}{\exists}xG) \longleftrightarrow \forall\mathbf{z}(F \longrightarrow \underset{\to}{\exists}xG)) \quad \big|\ \mathbf{z} \rightsquigarrow G$;
18. $(F\vee\neg F) \longrightarrow ((F \longrightarrow \underset{\to}{\exists}xG) \longleftrightarrow \underset{\to}{\exists}x(F \longrightarrow G)) \quad \big|\ x \rightsquigarrow F$;
19. $\mathfrak{u}\ \text{stab.} \longrightarrow (\forall\mathfrak{v}(\mathfrak{v}\ \text{lim.val.}\ \mathfrak{u} \longrightarrow \underset{\to}{\exists}\mathbf{z}F) \longleftrightarrow \underset{\to}{\exists}\mathbf{z}\forall\mathfrak{v}(\mathfrak{v}\ \text{lim.val.} \longrightarrow F))$;
20. $(F\vee G) \longrightarrow (F \underset{\to}{\vee} G)$;
21. $(F \underset{\to}{\vee} G) \longrightarrow (F \underset{\bullet}{\vee} G)$;
22. $(F\vee\neg F) \longrightarrow ((F \underset{\to}{\vee} G) \longrightarrow (F\vee\neg\neg G))$;
23. $(F \underset{\to}{\vee} G) \longleftrightarrow (G \underset{\to}{\vee} F)$;
24. $(F \underset{\to}{\vee} (G \underset{\to}{\vee} H)) \longleftrightarrow ((F \underset{\to}{\vee} G) \underset{\to}{\vee} H)$;
25. $(F \underset{\to}{\vee} F) \longrightarrow \neg\neg F$;
26. $(\neg\neg F \longrightarrow F) \longrightarrow ((F \underset{\to}{\vee} F) \longleftrightarrow F)$;
27. $(\neg\neg G \longrightarrow G) \longrightarrow ((\underset{\to}{\exists}\mathbf{z}F \underset{\to}{\vee} G) \longleftrightarrow \underset{\to}{\exists}\mathbf{z}(F \underset{\to}{\vee} G)) \quad \big|\ \mathbf{z} \rightsquigarrow G$;
28. $\forall\mathbf{z}(F\vee\neg F) \longrightarrow (\exists\mathbf{z}F \underset{\to}{\vee} \neg\exists\mathbf{z}F)$.

See Sec. 1 regarding the deducibility of formulas 1, 2, 20, and 21; formulas 3 and 4 are deducible from 1 and 2, which can be proved with the aid of the following auxiliary assertion: the formula scheme

$$\forall\mathbf{z}(F\vee\neg F) \longrightarrow (\underset{\bullet}{\exists}\mathbf{z}F \longrightarrow \exists\mathbf{z}F)$$

is deducible in $\mathfrak{Y}^*_\theta$.

Let us sketch the deduction of formula 5. In view of Theorem IV and Example 1 from Sec. 57 in [3], it is essentially sufficient to consider the case when $F \rightleftharpoons \ulcorner\varphi(x,\mathbf{z})\urcorner$, where φ is an ERF. We construct the PRF (primitive recursive functions) η, ψ, and χ such that in $\mathbf{E}_\theta$ are deducible the equalities $\psi(0)=0$, $\chi(0)=0$, $\eta(y)=\varphi(\chi(y),\psi(y))$ and the equalities interpreting the following entries:

$$\psi(y+1)=\begin{cases}\psi(y) & \text{if } \eta(y)=0,\\ \psi(y)+1 & \text{if } \eta(y)\neq 0,\end{cases}\quad \chi(y+1)=\begin{cases}\chi(y)+1 & \text{if } \eta(y)=0,\\ 0 & \text{if } \eta(y)\neq 0.\end{cases}$$

We assume: $\mathfrak{c} \leftrightharpoons \{\psi\}$. The formula

$$\exists z \forall x \ulcorner \varphi(x,z)\urcorner \to (\mathfrak{c}\ \text{stab.} \& \forall z (z\ \text{lim.val.}\mathfrak{c} \to \forall x \ulcorner \varphi(x,z)\urcorner))$$

is deducible in $\mathfrak{I}^*_\theta$.

Formula 8 is deducible with the aid of D.3. In the deductions of certain formulas, beginning with formula 10 (in particular, in the deductions of formulas 17, 19, and 27), we use D.4. Let us outline further the deduction of formula 28. In view of Theorem IV and Example 1 from Sec. 57 in [3], it is essentially sufficient to consider the case when $F \leftrightharpoons \ulcorner\alpha(z)\urcorner$, where α is an ERF. We construct an ERF β such that the equality $\beta(z)=1 \dot{-} \sum_{x=0}^{z}(1\dot{-}\alpha(x))$ is deducible in $\mathbf{E}_\theta$. It is obvious that $\forall z(\alpha(z)\neq 0)\to\forall z(\beta(z)=1)$ and $\ulcorner\alpha(y)\urcorner\to\forall z(z\geqslant y\to\ulcorner\beta(z)\urcorner)$. We assume $\mathfrak{e}\leftrightharpoons\{\beta\}$. The formula

$$(\mathfrak{e}\ \text{stab.}\&\forall y(y\ \text{lim. val.}\mathfrak{e}\to((y=0\to\exists z F)\&(y\neq 0\to\forall z \neg F))))$$

is deducible in $\mathfrak{I}^*_\theta$.

Let us note one example of the use of formula scheme 28. With its aid it is easy to prove that $\forall X \forall Y (X \underset{\mathcal{B}}{=} Y \underset{\to}{\vee} \neg(X \underset{\mathcal{B}}{=} Y))$. Here X and Y are variables for realduplexes, and $\underset{\mathcal{B}}{=}$ is an equality relation for them (see Secs. 3 and 4 in [10]).

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