Proof search under Shanin

G. Mints
Department of Philosophy, Stanford University

May 23, 1999

This survey mainly follows section 3 of [7].

1 Natural Deduction

At the end of the fifties computers became accessible to logicians, and several proof search programs were written in the West. This trend continued in the sixties and influenced the development of proof theory, first in Leningrad and then in Kiev.

At Leningrad University, N. Shanin established a course of mathematical logic based on natural deduction and purely syntactic approach. Only finite models were considered. N. Shanin directed a logic seminar on proof theory of propositional and predicate calculus with the explicit aim of preparing for work in automated deduction. This line of investigationbecame known as the theory of logical deduction. Shanin's lecture courses and seminars gave rise to what later became known as Leningrad school in proof theory. His goal (fashionable even nowadays) was to obtain from a computer some human-friendly proofs of the theorems. Important features which now become universally accepted include:

(a) proofs by natural deduction or similar rules

(b) readability

(3) output in a natural language

A model task was the construction of natural deductions of propositional tautologies. All previous computer programs produced derivations in Gentzen-type sequent formulations dealing with multiple-succedent sequents and having rules for introduction of connectives both on the right of $\Rightarrow$ (into the conclusion) and on the left (into the premise). For example,

G. Davydov, S. Maslov, G. Mints, V. Orevkov, N.Shanin, A. Slisenko, N. Belyaeva, Sochilina, Vaksmanova (Hajkova), Kr*calova, Havel

$$\frac{X \Rightarrow Y, A \quad B, X \Rightarrow Y}{A \supset B, X \Rightarrow Y} \ (\supset\Rightarrow) \tag{1}$$

is the rule for implication on the left.

The transformation of such a derivation into a natural deduction suggested by Gentzen was rather complicated. It included insertion of numerous transitions from $X \Rightarrow Y, A$ to $\neg A, X \Rightarrow Y$ and back to leave at most one formula in the succedent Y, passage to a Hilbert-type derivation, and only then passage to natural deduction. The method applied in the first version of the Leningrad computer program described by [1] was to find first a Gentzen-type derivation g in a suitable sequent calculus, and then transform it into a natural deduction by inserting a series of natural deduction rules. Sequent calculus for proof search dealt with sequents with at most one succedent formula, so that the rules $(\supset\Rightarrow)$ and $(\Rightarrow \vee)$ were turned into

$$(\supset\Rightarrow)\ \frac{\neg A, X \Rightarrow Y \quad B, X \Rightarrow Y}{A \supset B, X \Rightarrow Y} \qquad \frac{\neg A, X \Rightarrow B}{X \Rightarrow A \vee B}\ (\Rightarrow \vee) \tag{2}$$

Natural deductions derived sequents (as in Gentzen's first consistency proof [4]) and a formula A depending on assumptions X was represented by $X \Rightarrow A$. Now natural deductions corresponding to rules (2) and using axioms $A \vee \neg A$ have the form

$$\dfrac{\Rightarrow A \vee \neg A \qquad \dfrac{A, A \supset B \Rightarrow B \quad \dfrac{B, X \Rightarrow Y}{X \Rightarrow B \supset Y}}{A, A \supset B, X \Rightarrow Y} \qquad \neg A, X \Rightarrow Y}{A \supset B, X \Rightarrow Y} \tag{3}$$

and

$$\dfrac{\Rightarrow A \vee \neg A \quad \dfrac{A, X \Rightarrow A}{A, X \Rightarrow A \vee B} \quad \dfrac{\neg A, X \Rightarrow B}{\neg A, X \Rightarrow A \vee B}}{X \Rightarrow A \vee B}$$

This rough schema had defects of two kinds. The Gentzen-type derivation obtained at the first stage contained a lot of similar and redundant branches, and natural deductions obtained at the second stage were not elegant, and also were difficult to read and interpret. Additional redundancies (like the pair of $\supset^+, \supset^-$ in (3) were introduced during transition to natural deduction. It was recognized that the whole proof-search process (not only final transformation into a natural deduction) should changed to obtain human-friendly end-product.

To overcome redundancy at the first stage, various kinds of pruning were introduced. The simplest of them were discovered by Kleene in his paper [5] on permutability of inferences in Gentzen's calculi (which was not known to Shanin at this time). If for example the uppermost B in the inference

$$\frac{X \Rightarrow Y, A \quad B, X \Rightarrow Y}{A \supset B, X \Rightarrow Y}$$

is not traceable to an axiom, then the sequent $X \Rightarrow Y$ obtained by deleting B is derivable, and the whole inference together with the left branch deriving

$\neg A, X \Rightarrow Y$ can be deleted. This pruning transformation was complemented by further transformations alllowing one to prune one of the A's in $A, A, X \Rightarrow Y$ and use this to obtain automatically the effect of simplifivations like

$$((A \vee B) \& (A \supset B)) \leftrightarrow B$$

The main tool of improving the final natural deduction was a new method of transforming a Gentzen-type derivation into a natural deduction discovered by Shanin in 1962 and improved by other members of the team, mainly Maslov and Slisenko. Its most significant part is similar to the transformation proposed independently by D. Prawitz in his monograph [8] on natural deduction. Consider for example the ($\supset\Rightarrow$) rule in (2) and suppose that the derivations of its premises are already transformed into natural deductions. Trace the B in the right premise to axioms (there can be several of them in several branches):

$$\begin{array}{c} B \Rightarrow B \\ \vdots \\ B, X \Rightarrow Y \end{array} \tag{4}$$

Now derive the conclusion $A \supset B, X \Rightarrow Y$ as in (3), but use the derivation of $A, A \supset B, X \Rightarrow Y$ obtained by replacing B on the left of (4) by $A \supset B$:

$$\dfrac{\Rightarrow A \vee \neg A \qquad \begin{array}{c} \dfrac{A \supset A \quad A \supset B \Rightarrow A \supset B}{A, A \supset B \Rightarrow B} \\ \vdots \\ A, A \supset B, X \Rightarrow Y \end{array} \qquad \neg A, X \Rightarrow Y}{A \supset B, X \Rightarrow Y}$$

The new derivation does not contain a redundant pair $\supset^{+}, \supset^{-}$ present in (3), and Prawitz established that it is normal if the original Gentzen-type derivation is cut-free. With respect to human-friendly qualities of the resulting natural deductions the Leningrad program is unsurpassed even today.

2 Predicate Logic

An essential feature of the rules used for proof search is invertibility: derivability of conclusion implies derivability of premises. Otherwise bottom-up search can produce an underivable subgoal from a derivable goal, and this could lead to a redundant backtracking in the propositional case and to infinite looping in the predicate case. The theoretical work on the Leningrad propositional program began with construction of several invertible calculi. At the same time invertible Gentzen-type systems for the predicate case (more or less similar to ones constructed in the West at the same time) were proposed by V. Matulis. This was done in the framework of a search for more efficient proof procedures for the predicate calculus. Work in this direction done in Leningrad was strongly influenced by the metavariable approach suggested in the early sixties by Shanin (unpublished, presented at Trakai conference in 1962) as well as by Kanger and Prawitz. It is concerned with bottom-up proof-search in the case of quantifier rules like

$$\frac{X \Rightarrow A(t), \exists y A(y), Y}{X \Rightarrow \exists y A(y), Y} \ (\Rightarrow \exists)$$

where the term t in the premise is to be found. Early methods included exhaustive search through all terms t in a given vocabulary. The metavariable approach suggests treating t as a variable of new kind to be instantiated (i.e. replaced by a suitable term) later in the process of proof-search, when one tries to turn all leaves of the proof-search tree into logical axioms by such substitution. In fact a kind of unification algorithm was proposed for finding such a substitution, but the first published formulation belongs to J.A. Robinson. The next step was made by S.Ju. Maslov who proposed more or less simultaneously with J. Robinson's resolution method) a new formulation of predicate logic and a corresponding proof search method which he called inverse method. Maslov's motivation was to start from metavariable form of logical axioms and proceed from the top down by cut-free rules using unification, i.e. deriving most general consequences. Both axioms and rules are restricted by those which can possibly occur in the cut-free derivation of the goal formula. A program for predicate logic based on the inverse method and developed by the Leningrad group [2] was of about the same stremgth as resolution provers developed at the same time in the West.

Instead of listing significant theoretical advances achieved by Leningrad school, let's make several remarks concerning interplay of proof theory and constructive ideology. All the work reported above was done in the framework of the Russian constructivist school of A.A. Markov, and N.A. shanin was its leading member. Ideology of this school required constructivization of all mathematics, including ptoof theory. Many results on completeness of proof strategies which had short and elegant indirect model-theoretiv proofs received new much longer but effective proofs. Some examples will be given by V. Lifschitz. Eventually this additional work turned out to be beneficial in many cases. It provided additional understanding of deductive structures and their interrelations which formed a foundation of further essential progress. Let us give two examples.

Maslov's inverse method was developed as a constructive systematic treatment of decidable subclasses of predicate logic.

Much less known version of Herbrand theorem for arithmetic due to N.A. Shanin [9] was developed to provide a constructive treatment of negative arithmetical formulas.

References

[1] G. Davydov, S. Maslov, G. Mints,V. Orevkov,N.Shanin, A. Slisenko) An Algorithm for a Machine Search of a Natural Logical Deduction in a Propositional Calculus. In: The Automation of Reasoning, Ed. J. Sieckmann, G. Wrightson, Springer Verlag, 1983, Russian original:1965.

[2] G. Davydov, S. Maslov,G. Mints, V. Orevkov, A. Slisenko, A Computer Algorithm for Establishing Deducibility Based on Inverse Method. Seminars in Mathematics, V.A. Steklov Math. Inst., Plenum Publishers, 1971, 16, 1 -16

[3] G. Gentzen, Untersuchungen über das logische Schliessen. Mathematische Zeitschrift, 39, 1934, 176-210, 405-431

[4] G. Gentzen. Die Wderspruchsfreiheit der reinen Zahlentheorie, Mathematische Annalen, 112, 1936, 493-565

[5] S. C. Kleene, Two Papers on the Predicate Calculus, Memoirs of the American Mathematical Society, 10, 1952

[6] G. Mints, Selected Papers in Proof Theory, North-Holland-Bibliopolis, 1992

[7] G. Mints, Proof Theory in the USSR (1925-1970), JSL, 56, no. 2, 1991, 385-422

[8] Prawitz D., Natural deduction, Almquist and Wiksell, 1965

[9] Shanin N., On a hierachy of ways of understanding sentences in constructive Mathematics, Proc. Steklov Inst. of Mathematics, 129, 1973, 203-266 (Russian); English translation by AMS