A short introduction to site

An artless method for calculating approximate values of zeros of Riemann's zeta function

(http://logic.pdmi.ras.ru/~yumat/personaljournal/artlessmethod)

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1 Notation

We use notation:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \tag{1}$$

$$\xi(s) = g(s)\zeta(s), \tag{2}$$

where

$$g(s) = \pi^{-\frac{s}{2}}(s-1)\Gamma\left(\tfrac{s}{2}+1\right), \tag{3}$$

$$\Xi(t) = \xi\left(\tfrac{1}{2}+\mathrm{i}t\right). \tag{4}$$

Respectively,

$$\Xi(t) = \sum_{n=1}^{\infty} \alpha_n(t) \tag{5}$$

where

$$\alpha_n(t) \quad = \quad g\left(\tfrac{1}{2}+\mathrm{i}t\right) n^{-\left(\frac{1}{2}+\mathrm{i}t\right)}. \tag{6}$$

By the functional equation also

$$\Xi(t) = \Xi(-t) = \sum_{n=1}^{\infty} \alpha_n(-t), \tag{7}$$

so defining

$$\beta_n(t) \quad = \quad \frac{\alpha_n(-t) + \alpha_n(t)}{2} \tag{8}$$

we can formally write

$$\Xi(t) = \sum_{n=1}^{\infty} \beta_n(t), \tag{9}$$

but the series doesn't converge anywhere.

Assuming the Riemann hypothesis and the simplicity of all zeroes of $\Xi(t)$, let them be denoted $\pm\gamma_1, \pm\gamma_2, \ldots$, with $0 < \gamma_1 < \gamma_2 < \ldots$.

The main object of study are the determinants

$$\Delta_N(t) = \begin{vmatrix} \beta_1(\gamma_1) & \ldots & \beta_1(\gamma_{N-1}) & \beta_1(t) \\ \vdots & \ddots & \vdots & \vdots \\ \beta_N(\gamma_1) & \ldots & \beta_N(\gamma_{N-1}) & \beta_N(t) \end{vmatrix}. \tag{10}$$

2 Calculation of zeroes of the zeta-function

Evidently, $\gamma_1, \ldots, \gamma_{N-1}$ are zeroes of $\Delta_N(t)$ but caclulations show that $\Delta_N(t)$ has also zeroes surprizingly close (in spite of the divergence of (9)) to $\gamma_N, \ldots, \gamma_{N+k}$ up to some k increasing with increasing N. Just a few examples:

$$\begin{aligned} \Delta_{220}(\mathbf{427.208825084074}) &= -1.92776\ldots \cdot 10^{-17793} < 0 \\ \gamma_{220} &= \mathbf{427.208825084074}58052814\ldots \\ \Delta_{220}(\mathbf{427.208825084075}) &= +9.85564\ldots \cdot 10^{-17794} > 0 \end{aligned}$$

$$\begin{aligned} \Delta_{400}(\mathbf{741.75733557294167327}) &= -1.55647\ldots \cdot 10^{-52004} < 0 \\ \gamma_{447} &= \mathbf{741.75733557294167327}58611620\ldots \\ \Delta_{400}(\mathbf{741.75733557294167328}) &= +2.91156\ldots \cdot 10^{-52005} > 0 \end{aligned}$$

$$\begin{aligned} \Delta_{910}(\mathbf{1610.0032641900379443104707872025446}&\mathbf{35}) = \\ &-5.11614\ldots \cdot 10^{-233887} < 0 \\ \gamma_{1166} = \mathbf{1610.0032641900379443104707872025446}&\mathbf{35}373\ldots \\ \Delta_{910}(\mathbf{1610.0032641900379443104707872025446}&\mathbf{36}) = \\ &+3.74072\ldots \cdot 10^{-233886} > 0 \end{aligned}$$

3 Calculation of the values of the zeta function

We have

$$\Delta_N(t) = \sum_{n=1}^{N} \tilde{\delta}_{N,n} \beta_n(t) \tag{11}$$

where

$$\tilde{\delta}_{N,n} = (-1)^{N+n} \begin{vmatrix} \beta_1(\gamma_1) & \dots & \beta_1(\gamma_{N-1}) \\ \vdots & \ddots & \vdots \\ \beta_{n-1}(\gamma_1) & \dots & \beta_{n-1}(\gamma_{N-1}) \\ \beta_{n+1}(\gamma_1) & \dots & \beta_{n+1}(\gamma_{N-1}) \\ \vdots & \ddots & \vdots \\ \beta_N(\gamma_1) & \dots & \beta_N(\gamma_{N-1}) \end{vmatrix}. \tag{12}$$

We make normalization

$$\delta_{N,n} = \frac{\tilde{\delta}_{N,n}}{\tilde{\delta}_{N,1}}. \tag{13}$$

and define

$$\eta_N(s) = \sum_{n=1}^{N} \delta_{N,n} n^{-s}, \tag{14}$$

$$\frac{\eta_N(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \mu_{N,n} n^{-s}, \tag{15}$$

$$\nu_{N,M}(s) = \sum_{n=1}^{M} \mu_{N,n} n^{-s}. \tag{16}$$

It turns out that for large N and small M the ratio $\frac{\eta_N(s)}{\nu_{N,M}(s)}$ gives very good approximation to $\zeta(s)$ for s different from the poles of $\zeta(s)$. Just a few examples:

$$\text{for } s = \frac{1}{4} + 1000\mathrm{i} \qquad \left| \frac{\eta_N(s)}{\nu_{3000,8}(s)\zeta(s)} - 1 \right| = 2.13729... \cdot 10^{-681}, \tag{17}$$

$$\text{for } s = \frac{1}{4} + 5000\mathrm{i} \qquad \left| \frac{\eta_N(s)}{\nu_{6000,11}(s)\zeta(s)} - 1 \right| = 2.27557... \cdot 10^{-1569}. \tag{18}$$

$$\text{for } s = \frac{1}{4} + 10000\mathrm{i} \qquad \left| \frac{\eta_N(s)}{\nu_{12000,14}(s)\zeta(s)} - 1 \right| = 3.11101... \cdot 10^{-2908}. \tag{19}$$

4 "Almost linear" relations

For sufficiently large N, the values of $\mu_{N,n}$ and $\nu_{N,M}(1)$ for many initial values of n and M are very small, for example:

M	$\mu_{3000,M}$	$\nu_{3000,M}(1)$
2	$-2-4.9... \cdot 10^{-126}$	$-2.46855... \cdot 10^{-126}$
3	$-7.40566... \cdot 10^{-126}$	$-7.14726... \cdot 10^{-285}$
4	$2.85891... \cdot 10^{-284}$	$3.71565... \cdot 10^{-412}$
5	$-1.85782... \cdot 10^{-411}$	$4.26945... \cdot 10^{-510}$
6	$-2.56167... \cdot 10^{-509}$	$7.17431... \cdot 10^{-586}$
7	$-5.02202... \cdot 10^{-585}$	$-5.49626... \cdot 10^{-642}$
8	$4.39701... \cdot 10^{-641}$	$-5.73467... \cdot 10^{-681}$
9	$1.08444... \cdot 10^{-681}$	$-5.61417... \cdot 10^{-681}$
10	$1.90599... \cdot 10^{-716}$	$-5.61417... \cdot 10^{-681}$

This implies that there are "almost linear" relations between $\delta_{3000,1}, \ldots, \delta_{3000,M}$. In the case of $\mu_{N,n}$ such relations have coefficients $+1$ and -1 because according to (15)

$$\mu_{N,n} = \sum_{k|n} \mu\left(\tfrac{n}{k}\right) \delta_{N,k}, \tag{20}$$

where $\mu(m)$ is the Möbius function; in the case of $\nu_{N,M}(1)$ the coefficients are rational numbers with small denominators.

5 Connections with primes

On Figures 1 and 2 the color is magenta, if $\mu_{3000,n} > 0$, and green otherwise.

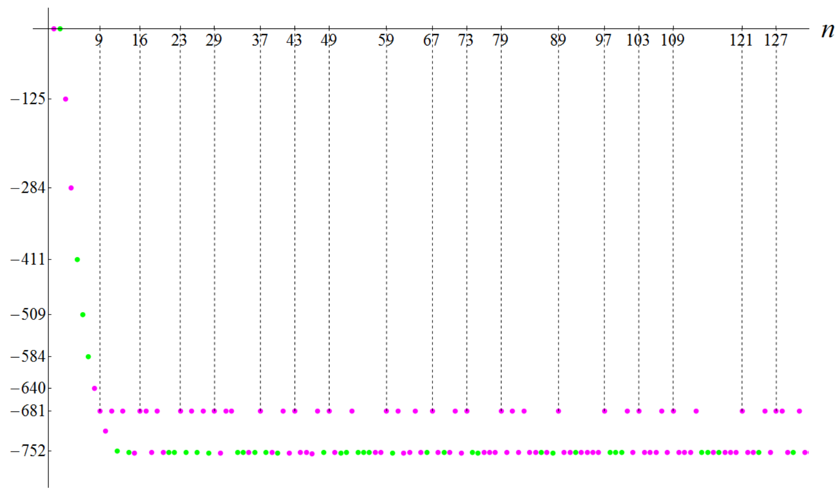

Figure 1: $\log_{10}|\mu_{3000,n}|$

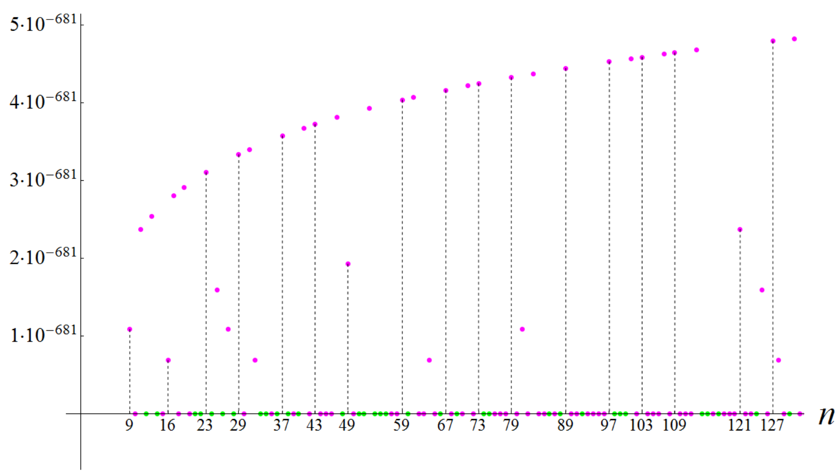

Figure 2: $\mu_{3000,n}$

Let $\omega_{3000} = \mu_{3000,13}/\ln(13) = 9.895811... \cdot 10^{-682}$. We have:

$$\left|\frac{\mu_{3000,p^k}}{\ln(p)} - \omega_{3000}\right| < 3.81... \cdot 10^{-754} \text{ for } 13 \leq p^k \leq 419, p \text{ is a prime} \qquad (21)$$

In other words, for $13 \leq n \leq 419$ the values of $\mu_{3000,n}$ are very close to $\omega_{3000}\Lambda(n)$, a multiple of the von Mangoldt function.

6 Calculation of $\zeta'(s)$ via $\zeta(s)$

Let us ajust $\mu_{3000,n}$:

$$\tilde{\mu}_{3000,n} = \mu_{3000,n} - \omega_{3000}\Lambda(n). \tag{22}$$

Then (21) "explains" the following approximation: for $s = \frac{1}{4} + 1000\mathrm{i}$

$$\left| \frac{\eta_{3000}(s) - \left(\sum_{n=1}^{11} \tilde{\mu}_{3000,n} n^{-s}\right)\zeta(s)}{-\omega_{3000}\zeta'(s)} - 1 \right| = 6.44... \cdot 10^{-73}. \tag{23}$$

Similar,

$$\left| \frac{\eta_{3000}\left(\frac{1}{2} + \mathrm{i}\gamma_{500}\right)}{-\omega_{3000}\zeta'\left(\frac{1}{2} + \mathrm{i}\gamma_{500}\right)} - 1 \right| = 2.786... \cdot 10^{-74}. \tag{24}$$

7 Simultaneous calculation of $\zeta(s)$ and $\zeta'(s)$

Solving the system

$$\eta_{3000}(s) - \left(\sum_{n=1}^{11} \tilde{\mu}_{3000,n} n^{-s}\right)\zeta(s) \approx -\omega_{3000}\zeta'(s) \tag{25}$$

$$\eta_{3000}(1-s) - \left(\sum_{n=1}^{11} \tilde{\mu}_{3000,n} n^{s-1}\right)\zeta(1-s) \approx -\omega_{3000}\zeta'(1-s) \tag{26}$$

$$g(s)\zeta(s) = g(1-s)\zeta(1-s) \tag{27}$$

$$g'(s)\zeta(s) + g(s)\zeta'(s) = -g'(1-s)\zeta(1-s) - g(1-s)\zeta'(1-s) \tag{28}$$

for $s = \frac{1}{4} + 1000\mathrm{i}$ produces 752 correct decimal digits for $\zeta(s)$ and 72 correct decimal digits for $\zeta'(s)$.

8 Special values

Let

$$\phi(N) = \frac{1}{2N} \cdot \cfrac{1}{1 + \cfrac{1}{2N - 1 + \cfrac{1}{\frac{2N}{2} + \cfrac{1}{\frac{2N}{3} + \cfrac{1}{\frac{2N}{4} + \cfrac{1}{\frac{2N}{5} + \cfrac{1}{\frac{2N}{6} + \ddots}}}}}}} \tag{29}$$

$$= \frac{1}{2}\psi\left(\frac{N}{2} + 1\right) - \frac{1}{2}\psi\left(\frac{N+1}{2}\right), \tag{30}$$

where

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{31}$$

then

$$\frac{\nu_{3000,3000}(1)}{\phi(3000)} = 1 - 2.46827839... \cdot 10^{-126}, \tag{32}$$

$$\frac{\nu_{6000,6000}(1)}{\phi(6000)} = 1 + 1.09736771... \cdot 10^{-165}. \tag{33}$$