The Four Colour Theorem as a possible corollary of binomial summation

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Abstract

The Four Colour Conjecture is reformulated as a statement about non-divisibility of certain binomial coefficients. This reformulation opens a (hypothetical) way of proving the Four Colour Theorem by taking advantage of recent progress in finding closed forms for binomial summations.

1 Introduction

The famous Four Color Conjecture (4CC) has, as many other great problems have, numerous equivalent reformulations (see, for example, [4], [5], [6], [7], [8], [9], [10], [11], [13], [18] and further references in these papers). Now that we have the remarkable work of K. Appel, W. Haken and J. Koch [1, 2, 3], these reformulation can be viewed as corollaries of the Four Colour Theorem (4CT). Nevertheless, still there is some interest in such reformulations in the original sense, i.e. as possible approaches to other proofs of the 4CT. That is why both terms, 4CC and 4CT are used in this paper. Namely, 4CC is treated here as a mathematical statement the validy of which is not supposed to be known. This is done just to emphasize the fact the proofs in this papers do not depend on the result of Appel and Haken (the Main Lemma below trivially follows from the Four Colour Theorem, the interest in the Lemma is due only to the fact that its independent proof is simple enough).

The proof of the 4CT given in above cited famous works of K. Appel, W. Haken and J. Koch can be characterized as a *computer-proof-for-computers* in the sense that checking it requires an amount of computation of the same order as the amount of computation used to establish the proof. Very recently a shorter proof was given by N. Robertson, D. Sanders, P. Seymour and R. Thomas [16]. However, this new proof is based on the same general approach, in particular, uses extensive computations.

It still seems to be desirable to find a *computer-proof-for-human-beings* proof, i.e., to get, possibly after many CPU-hours, a printed proof would occupy a reasonable number of pages and which could be read, understood and verified by a human being.

This paper suggests a possible approach to finding such a new proof of the Four Colour Theorem via a new reformulation of the 4CC. The idea of the suggested approach is to try to take advantage of the recent impressive progress (see, for example, [15, 19]) in computer search for closed forms of sums of the form

$$\sum_{c_1} \dots \sum_{c_q} E(a_1, \dots, a_p, c_1, \dots, c_q) \tag{1}$$

in the case when all the ratios

$$\frac{E(a_1, \dots, a_p, c_1, \dots, c_{r-1}, c_r + 1, c_{r+1}, \dots, c_q)}{E(a_1, \dots, a_p, c_1, \dots, c_{r-1},\ c_r,\ c_{r+1}, \dots, c_q)} \tag{2}$$

are rational functions. At first sight, the idea looks crazy but we'll see that it is not quite so. Namely, the following relation between the 4CC and binomial coefficients is established in this paper.

Main Lemma *There are natural numbers p and q, and $4q$ linear functions $A_k(m, c_1, \dots, c_p)$, $B_k(m, c_1, \dots, c_p)$, $C_k(m, c_1, \dots, c_p)$, $D_k(m, c_1, \dots, c_p)$, $k = 1, \dots, q$, such that the* 4CC *is equivalent to the following statement*:

$$\forall nm \exists c_1 \dots c_p E(n, m, c_1, \dots, c_p) \not\equiv 0 \pmod 7 \tag{3}$$

where

$$E(n, m, c_1, \dots, c_p) = \prod_{k=1}^{q} \binom{A_k(m, c_1, \dots, c_p) + 7^n B_k(m, c_1, \dots, c_p)}{C_k(m, c_1, \dots, c_p) + 7^n D_k(m, c_1, \dots, c_p)}, \tag{4}$$

all variables ranging over non-negative integers.

The choice of p, q, A's, B's, C's and D's is not unique; a particular example with $p = 24$ and $q = 986$ is exhibited below in the proof.

Having chosen A's, B's, C's and D's satisfying the Main Lemma, one can try to prove the 4CT in the following way:

- select an integer-valued function $F(n, m, c_1, \ldots, c_p)$;
- find closed form
$$\sum_{c_1} \ldots \sum_{c_p} E(n, m, c_1, \ldots, c_p) F(n, m, c_1, \ldots, c_p) = G(n, m); \tag{5}$$
- prove that
$$\forall nm G(n, m) \not\equiv 0 \pmod 7. \tag{6}$$

The success of this approach heavily depends on the lucky choice of the functions. The full freedom in selection of the F raises some hope which can be stated as

Hypothesis *There are functions A's, B's, C's and D's satisfying the Main Lemma and functions F and G satisfying* (5) *and* (6).

2 Proof of the Main Lemma

The proof consists of a sequence of reformulations of the 4CC. Each reformulation has the same logical form $\forall\alpha\exists\beta\Gamma(\alpha, \beta)$ where α can be interpreted as map, β as four-colouring and $\Gamma(\alpha, \beta)$ as statement that β is indeed a four-coloring of α.

There will be 9 successive reformulation given in subsections 2.1–2.9 and numbered 1, …, 9 respectively. The reformulations would require definitions and redefinitions of similar objects, for example, maps. Such (re)definitions will be numbered by two digits, the last digit will coincide with the last digit in the number of subsection.

2.1 Original formulation

By a *geographical map* we mean a map of an imaginary island in the ocean. The island is divided into finitely many *countries* each of which is connected.

4CC, Original formulation 1 *Countries of every geographical map can be coloured by* 4 *colours in such a way that every two neighbouring countries are coloured differently.*

2.2 Standard reformulation for graphs

There is a well-known way to reformulate the 4CC in terms of colouring vertices of a planar graph. To pass from a geographical map to a graph, we select inside each country a *capital* and make *roads* connecting the capitals of neighbouring countries. The capitals and the roads are, respectively, the vertices and the edges of corresponding graph.

4CC, Reformulation 2 *The vertices of any planar graph $G = \langle V, E\rangle$ can be coloured in* 4 *colours in such a way that the ends of each edge are coloured differently.*

2.3 New reformulation for graphs

We are going to modify the above presented standard representation of geographical maps by planar graphs. Namely, in each country we can select now several *towns* rather than only the capital. The towns will be connected by roads of two types:

(I) *internal* roads, connecting two towns inside the same country;

(X) *external* roads, connecting two towns across the border between two neighbouring countries.

As usual, the roads do not cross one another outside the towns. One should be able to travel between any two towns of the same country by internal roads, and there should be at least one external road between every two neighbouring countries. In this way we obtain a planar graph $G = \langle V, E_{\mathrm{I}}, E_{\mathrm{X}}\rangle$ with edges *ranked* into *internal* (those from E_{I}) and *external* (those from E_{X}). The connected components of graph $\langle V, E_{\mathrm{I}}\rangle$ can be viewed as the "countries". The standard reformulation of the 4CC corresponds to the case of empty E_{I}.

By a *colouring* of graph with ranked edges we mean an assignment of colours to its vertices such that

(I) the ends of each internal edge are coloured in the same colour;

(X) the ends of each external edge are coloured in different colours.

Reformulation of the 4CC, a wrong one *The vertices of every planar graph* $G = \langle V, E_{\mathrm{I}}, E_{\mathrm{X}} \rangle$ *with ranked edges can be coloured by* 4 *colours.*

As a counterexample to this wrong reformulation of the 4CC we can consider the graph $G = \langle V, E_{\mathrm{I}}, E_{\mathrm{X}} \rangle$ where

$$V = \{v_1, v_2, v_3\}, \tag{7}$$

$$E_{\mathrm{I}} = \{\{v_1, v_2\}, \{v_2, v_3\}\}, \quad E_{\mathrm{X}} = \{\{v_1, v_3\}\}. \tag{8}$$

Here, on the one hand all vertices should have the same colour, on the other hand vertices v_1 and v_3 should be coloured differently. In the standard reformulation this situation corresponds to loops which are not allowed. Of course, we should forbid such "extended loops" too. We will do it in the following way.

4CC, Reformulation 3 *If a planar graph* $G = \langle V, E_{\mathrm{I}}, E_{\mathrm{X}} \rangle$ *with ranked edges can be coloured in some number of colours, then it can be coloured in* 4 *colours as well.*

Clearly, this excludes from consideration all graphs with "extended loops", and only them, because any "loopless" graph can be evidently coloured in as many colours as is the number of its countries.

2.4 Colouring as a definition of a map

The last reformulation of the 4CC suggests that we should deal with colouring as a primary object.

Definition 1.4 *A* discrete map *is a couple* $\langle G, \mu \rangle$ *where* $G = \langle E, V \rangle$ *is a planar graph and* μ *is an arbitrary assignment of colours to its vertices.*

Interpretation *A discrete map* $\langle\langle V, E\rangle, \mu\rangle$ induces *a graph with ranked edges* $\langle V, E_{\mathrm{I}}, E_{\mathrm{X}}\rangle$ *where* E_{I} *and* E_{X} *are the set of all edges from* E *the ends of which are coloured by* μ*, respectively, in equal or in different colours.*

Definition 2.4 *Two discrete maps* $\langle G, \mu\rangle$ *and* $\langle G, \lambda\rangle$ *are* equivalent *if they induce the same graph with ranked edges.*

4CC, Reformulation 4 *For every discrete map* $\langle G, \mu\rangle$ *there is an equivalent discrete map* $\langle G, \lambda\rangle$ *in which* λ *uses only* 4 *colours.*

2.5 Standardization of graphs

The idea behind the introduction of the notion of an internal edge is as follows: *now we need not consider all possible planar graphs.* Namely, it is sufficient to consider only graphs from some class $\mathcal{K}$ of planar graphs which is rich enough in the following sense: for every planar graph G there is a discrete map $\langle G', \mu\rangle$ with graph G' from $\mathcal{K}$ such that deleting some vertices and contracting all internal edges results in graph G. Clearly, in this case any colouring of graph G' produces a colouring of graph G.

For such a class $\mathcal{K}$ we can take a class of graphs with very regular structure, for example, the class of all rectangular grids. This is rather evident fact, however, a formal proof is given in Appendix 1. Another proof can be found in [17]. The difference between the two proofs is as follows. The proof in [17] is based on quite non-trivial theorem of H. Whitney about the existence of Hamiltonian circles in planar graphs. The use of this deep result in [17] is quite justified because there the authors are interested in obtaining a good bound for the size of the required grid. This size is of no importance for our purpose, and the proofin Appendix 1 is based just on the intuitevely clear fact that any curve can be well approximated by another curve lying on sufficiently dense grid.

From now on we restrict ourself to considering only those discrete maps $\langle G, \mu\rangle$ in which G is a subgraph of a rectangular grid. Such a graph G is also a subgraph of one of infinitely many infinite graphs defined as follows.

Definition *A* spiral graph S_n *has infinitely many vertices numbered by natural numbers* 0, 1,. . . , *two vertices numbered* i *and* j *being connected by an edge if and only if one of the following conditions holds:*

(S) $\mid i - j \mid = 1$ (spiral *edges*);

(R) $\mid i - j \mid = n$ (radial *edges*).

Redefinition 1.5 *A* discrete map *is a couple $\langle n, \mu \rangle$ where n is a non-negative integer and μ is an arbitrary function defined on all natural numbers with natural number values such that all but finitely many of its values are equal to* 0*.*

Interpretation Take for G (from the original Definition 1.4) the graph resulting from S_n by removing all vertices coloured by μ in colour 0.

The n in the statement of the Main Lemma corresponds exactly to the n in the above definition of a discrete map. Clearly, by taking sufficiently large values for n we can represent any geographical map by a discrete map in the sense of Redefinition 1.5.

Redefinition 2.5 *Two discrete maps $\langle n, \lambda \rangle$ and $\langle n, \mu \rangle$ are* equivalent *if*

(Z) $\lambda(k) = 0 \Leftrightarrow \mu(k) = 0$;

(S) $\lambda(k) = \lambda(k+1) \Leftrightarrow \mu(k) = \mu(k+1)$;

(R) $\lambda(k) = \lambda(k+n) \Leftrightarrow \mu(k) = \mu(k+n)$.

We say that a colouring λ uses c colours if $\lambda(i) \leq c$ for all i.

4CC, Reformulation 5 *For every discrete map $\langle n, \mu \rangle$ there is an equivalent discrete map $\langle n, \lambda \rangle$ in which λ uses only* 4 *colours.*

2.6 Six Colour Theorem

So far to represent a geographical map we constructed a discrete map with a colouring which uses as many colours as is the number of contries. However, it is very easy to prove that every geographical map can be coloured in 6 colours (see, for example, [18]). In our terminology this means that for every discrete map $\langle n, \mu \rangle$ there is an equivalent discrete map $\langle n, \mu' \rangle$ in which μ'

uses only 6 colours. So we can restrict ourself only discrete maps defined by such colourings.

Redefinition 1.6 *A* discrete map *is a couple* $\langle n, \mu \rangle$ *where* n *is a non-negative integer and* μ *is an arbitrary function defined on all natural numbers with values from the set* $\{0, 1, 2, 3, 4, 5, 6\}$ *such that all but finitely many of its values are equal to* 0.

Definition 3.6 *A* four-colouring *of a discrete map* $M = \langle n, \mu \rangle$ *is an arbitrary function* λ *defined on all non-negative integers with values from the set* $\{0, 1, 2, 3, 4\}$ *such that:*

(Z) $\lambda(k) = 0 \Leftrightarrow \mu(k) = 0$;

(S) $\lambda(k) = \lambda(k+1) \Leftrightarrow \mu(k) = \mu(k+1)$;

(R) $\lambda(k) = \lambda(k+n) \Leftrightarrow \mu(k) = \mu(k+n)$.

4CC, Reformulation 6 *Every discrete map* $\langle n, \mu \rangle$ *has a four-colouring* λ.

Note that we do not exclude the degenerate case $n = 0$ in which all spiral edges are loops and hence inevitably are ranked as internal.

2.7 Coding of discrete maps and four-colourings

Thanks to the fact that all but finitely many values of μ in the (re)definition of a discrete map are equal to zero, we can easily and in a natural way repesent such a μ by a single integer

$$m = \sum_{k=0}^{\infty} \mu(k) 7^k. \tag{9}$$

Clearly, this number uniquely determines μ because the values of the latter are nothing else but the digits of m in base-7 notation. Having this in mind, we give

Redefinition 1.7 *A* discrete map *is a couple* $\langle n, m \rangle$ *where* n *and* m *are non-negative integers.*

In this sense the quantifier $\forall nm$ in the Main Lemma means "for every map".

Respectively, we can code in a similar way a four-colouring λ by the number

$$l = \sum_{k=0}^{\infty} \lambda(k) 7^k. \tag{10}$$

Pay attention that we use again base-7 notation rather than (possible for four-colouring) base-5 notation. The reason is that we need be able to express in a simple way what it means that l is a code of a four-colouring of a given discrete map.

Redefinition 3.7 *A number l is a four-colouring of a discrete map $M = \langle n, m \rangle$ if*

(F) *the base-7 notation of l does not contain digits "5" or "6";*

(Z) *the k-th digit of l is equal to zero if and only if the k-th digit of m is equal to zero;*

(S) *the k-th digit of l is equal to the $(k+1)$-st digit of l if and only if the k-th digit of m is equal to the $(k+1)$-st digit of m;*

(R) *the k-th digit of l is equal to the $(k+n)$-th digit of l if and only if k-th digit of m is equal to the $(k+n)$-th digit of m.*

4CC, Reformulation 7 *Every discrete map $\langle n, m \rangle$ has a four-colouring l.*

2.8 Another coding of discrete maps and four-colourings

The above introduced method of representing μ by a single number m is convenient because every nonnegative integer m corresponds to some μ. However, this method is not very suitable for describing the desired relation between λ and μ in terms of l and m. For the latter purpose we will represent μ also by 6 numbers m_1, m_2, m_3, m_4, m_5, m_6 defined by

$$m_i = \sum_{\mu(k)=i} 7^k. \tag{11}$$

These numbers can be viewed as characteristic functions: the k-th digit of m_i is equal to 1 or 0 depending on whether the vertex numbered k has colour i or not.

There is a natural relation between m and m_1, m_2, m_3, m_4, m_5, m_6, namely,

$$m = 1 \times m_1 + 2 \times m_2 + 3 \times m_3 + 4 \times m_4 + 5 \times m_5 + 6 \times m_6. \tag{12}$$

Moreover, this equality uniquely defines m_1, m_2, m_3, m_4, m_5, m_6 for given m under some additional conditions.

Definition *A non-negative integer b is* Boolean (written Bool(b)) *if all its base-7 digits of are either* "0" *or* "1".

Definition *Two boolean numbers a and b are said to be* orthogonal (written $a \perp b$) *if they never have* "1" *in the same position.*

Clearly, we have

$$\mathrm{Bool}(m_i), \quad 1 \le i \le 6, \tag{13}$$

$$m_i \perp m_j, \quad 1 \le i < j \le 6 \tag{14}$$

and the 22 conditions (12), (13) and (14) uniqely determine m_1, m_2, m_3, m_4, m_5, m_6 for given value of m.

In a similar way we could represent a four-colouring l by 4 pairwise orthogonal Boolean numbers l_1, l_2, l_3, l_4 such that

$$l = 1 \times l_1 + 2 \times l_2 + 3 \times l_3 + 4 \times l_4. \tag{15}$$

However, instead of separate representations of μ and λ we will jointly resepresent them by 24 numbers

$$\begin{array}{cccc} c_{1,1}, & c_{1,2}, & c_{1,3}, & c_{1,4}, \\ c_{2,1}, & c_{2,2}, & c_{2,3}, & c_{2,4}, \\ c_{3,1}, & c_{3,2}, & c_{3,3}, & c_{3,4}, \\ c_{4,1}, & c_{4,2}, & c_{4,3}, & c_{4,4}, \\ c_{5,1}, & c_{5,2}, & c_{5,3}, & c_{5,4}, \\ c_{6,1}, & c_{6,2}, & c_{6,3}, & c_{6,4} \end{array} \tag{16}$$

defined by

$$c_{i,j} = \sum_{\substack{\mu(k)=i \\ \lambda(k)=j}} 7^k. \tag{17}$$

Clearly, these numbers are Boolean and pairwise orthogonal, i.e., they satisfy conditions

$$\mathrm{Bool}(c_{i,j}), \quad i = 1, 2, 3, 4, 5, 6, \quad j = 1, 2, 3 \tag{18}$$

and

$$c_{i',j'} \perp c_{i'',j''}, \quad i', i'' = 1, 2, 3, 4, 5, 6, \quad j', j'' = 1, 2, 3, 4, \quad \langle i', j' \rangle \neq \langle i'', j'' \rangle. \tag{19}$$

There are evident relations between the c's, m's and l's:

$$m_i = c_{i,1} + c_{i,2} + c_{i,3} + c_{i,4}, \quad i = 1, 2, 3, 4, 5, 6, \tag{20}$$

$$l_j = c_{1,j} + c_{2,j} + c_{3,j} + c_{4,j} + c_{5,j} + c_{6,j}, \quad j = 1, 2, 3, 4. \tag{21}$$

Respectively, we have a counterpart of (12):

$$\begin{aligned} m \;=\; & 1 \times (c_{1,1} + c_{1,2} + c_{1,3} + c_{1,4}) + \\ & 2 \times (c_{2,1} + c_{2,2} + c_{2,3} + c_{2,4}) + \\ & 3 \times (c_{3,1} + c_{3,2} + c_{3,3} + c_{3,4}) + \\ & 4 \times (c_{4,1} + c_{4,2} + c_{4,3} + c_{4,4}) + \\ & 5 \times (c_{5,1} + c_{5,2} + c_{5,3} + c_{5,4}) + \\ & 6 \times (c_{6,1} + c_{6,2} + c_{6,3} + c_{6,4}) \end{aligned} \tag{22}$$

and a counterpart of (15):

$$\begin{aligned} l \;=\; & 1 \times (c_{1,1} + c_{2,1} + c_{3,1} + c_{4,1} + c_{5,1} + c_{6,1}) + \\ & 2 \times (c_{1,2} + c_{2,2} + c_{3,2} + c_{4,2} + c_{5,2} + c_{6,2}) + \\ & 3 \times (c_{1,3} + c_{2,3} + c_{3,3} + c_{4,3} + c_{5,3} + c_{6,3}) + \\ & 4 \times (c_{1,4} + c_{2,4} + c_{3,4} + c_{4,4} + c_{5,4} + c_{6,4}). \end{aligned} \tag{23}$$

Clearly, condition (F) from Redefinition 3.7 is implied by (18), (19) and (23). Similar, condition (Z) is implied by (18), (19), (22) and (23). Conditions (S) and (R) can be expressed in terms of the c's as some orthogonalities.

Redefinition 3.8 *Numbers* (16) *are a four-colouring of a discrete map* $M = \langle n, m \rangle$ *if they satisfy conditions* (18), (19), (22) *and*

(S′) $7c_{i,j}\perp c_{i,j'}$, $i=1,2,3,4,5,6$, $j=1,2,3,4$, $j'=1,\dots,j-1,j+1,\dots,4$;

(S″) $7c_{i,j}\perp c_{i',j}$, $i=1,2,3,4,5,6$, $i'=1,\dots,i-1,i+1,\dots,6$, $j=1,2,3,4$;

(R′) $7^n c_{i,j}\perp c_{i,j'}$, $i=1,2,3,4,5,6$, $j=1,2,3,4$, $j'=1,\dots,j-1,j+1,\dots,4$;

(R″) $7^n c_{i,j}\perp c_{i',j}$, $i=1,2,3,4,5,6$, $i'=1,\dots,i-1,i+1,\dots,6$, $j=1,2,3,4$.

4CC, Reformulation 8 *Every discrete map $\langle n,m\rangle$ has a four-colouring* (16).

2.9 Kummer's theorem

Both the previous reformulation of the 4CC and the Main Lemma have the same quantifier prefix $\forall nm\exists c$'s but the former contains the equality (22) and a number of orthogonality conditions rather than only congruence modulo 7 as the latter. The equality (22) can be rewritten in the desired form as

$$\binom{m}{C}\binom{C}{m}\not\equiv 0 \pmod 7 \tag{24}$$

where C is an abbreviation for the right-hand side of (22). To rewrite the orthogonality conditions we need some magic from the old Number Theory:

Theorem (E. E. Kummer [12]) *A prime number p comes into the factorization of the binomial coefficient*

$$\binom{a+b}{a} \tag{25}$$

with the exponent equal to the number of carries performed when adding $a+b$ in base-p positional notation.

This beautiful theorem had been originally published in 1852 (the year of the first documented mentioning of the 4CC!—see, e.g. [14]) but has not become widely known and was several times rediscovered. To make this paper self-contained, a proof of the theorem is reproduced in Appendix 2.

Corollary 1 *All base-7 digits of a number a are less or equal to 3 if and only if*

$$\binom{2a}{a} \not\equiv 0 \pmod 7. \tag{26}$$

If all base-7 digits of a number a are less or equal to 3, then the digits of the number $2a$ are doubles of the digits of a, so applying the above Corollary twice we get

Corollary 2

$$\mathrm{Bool}(a) \Longleftrightarrow \binom{2a}{a}\binom{4a}{2a} \not\equiv 0 \pmod 7. \tag{27}$$

Clearly, two Boolean numbers a and b are orthogonal if and only if their sum $a+b$ is a Boolean number as well. Hence, we have

Corollary 3

$$\mathrm{Bool}(a)\&\mathrm{Bool}(b) \Longrightarrow \left[a \perp b \Longleftrightarrow \binom{2(a+b)}{a+b}\binom{4(a+b)}{2(a+b)} \not\equiv 0 \pmod 7\right]. \tag{28}$$

The above representation for the $\perp$ relation can be a bit simplified thanks to the fact that the sum of two Boolean numbers never contains digits greater than 2:

Corollary 4

$$\mathrm{Bool}(a)\&\mathrm{Bool}(b) \Longrightarrow \left[a \perp b \Longleftrightarrow \binom{4(a+b)}{2(a+b)} \not\equiv 0 \pmod 7.\right] \tag{29}$$

It is evident that the numbers $7c_{i,j}$ and $7^n c_{i,j}$ occuring in the Reformulation 3.8 are Boolean.

Putting all things together, we get the last reformulation of the 4CC which is just an explicit form of the Main Lemma.

4CC, Reformulation 9 *For every nonnegative integers n and m there are 24 nonnegative integers*

$$\begin{array}{cccc} c_{1,1}, & c_{1,2}, & c_{1,3}, & c_{1,4}, \\ c_{2,1}, & c_{2,2}, & c_{2,3}, & c_{2,4}, \\ c_{3,1}, & c_{3,2}, & c_{3,3}, & c_{3,4}, \\ c_{4,1}, & c_{4,2}, & c_{4,3}, & c_{4,4}, \\ c_{5,1}, & c_{5,2}, & c_{5,3}, & c_{5,4}, \\ c_{6,1}, & c_{6,2}, & c_{6,3}, & c_{6,4} \end{array} \tag{30}$$

such that none of the following 986 binomial coefficients is divisible by 7:

$$\binom{2c_{i,j}}{c_{i,j}}, \quad \binom{4c_{i,j}}{2c_{i,j}}, \quad i = 1, \ldots, 6, \quad j = 1, \ldots, 4; \tag{31}$$

$$\binom{4(c_{i',j'} + c_{i'',j''})}{2(c_{i',j'} + c_{i'',j''})}, \quad i', i'' = 1, \ldots, 6, \quad j', j'' = 1, \ldots, 4, \quad \langle i', j' \rangle \neq \langle i'', j'' \rangle; \tag{32}$$

$$\binom{4(7c_{i,j} + c_{i,j'})}{2(7c_{i,j} + c_{i,j'})}, \quad i = 1, \ldots, 6, \quad j = 1, \ldots, 4, \quad j' = 1, \ldots, j{-}1, j{+}1, \ldots, 4; \tag{33}$$

$$\binom{4(7c_{i,j} + c_{i',j})}{2(7c_{i,j} + c_{i',j})}, \quad i = 1, \ldots, 6, \quad i' = 1, \ldots, i{-}1, i{+}1, \ldots, 6, \quad j = 1, \ldots, 4; \tag{34}$$

$$\binom{4(7^n c_{i,j} + c_{i,j'})}{2(7^n c_{i,j} + c_{i,j'})}, \quad i = 1, \ldots, 6, \quad j = 1, \ldots, 4, \quad j' = 1, \ldots, j{-}1, j{+}1, \ldots, 4; \tag{35}$$

$$\binom{4(7^n c_{i,j} + c_{i',j})}{2(7^n c_{i,j} + c_{i',j})}, \quad i = 1, \ldots, 6, \quad i' = 1, \ldots, i{-}1, i{+}1, \ldots, 6, \quad j = 1, \ldots, 4; \tag{36}$$

$$\binom{m}{C}, \qquad \binom{C}{m} \tag{37}$$

where C is an abbreviation for the sum

$$\begin{aligned} &1 \times (c_{1,1} + c_{1,2} + c_{1,3} + c_{1,4}) + 2 \times (c_{2,1} + c_{2,2} + c_{2,3} + c_{2,4}) + \\ &3 \times (c_{3,1} + c_{3,2} + c_{3,3} + c_{3,4}) + 4 \times (c_{4,1} + c_{4,2} + c_{4,3} + c_{4,4}) + \\ &5 \times (c_{5,1} + c_{5,2} + c_{5,3} + c_{5,4}) + 6 \times (c_{6,1} + c_{6,2} + c_{6,3} + c_{6,4}). \end{aligned}$$

3 Discussion

The constructed system (31)–(37) is just one posible explicit formulation of the Main Lemma. Below we discuss possible variations which might influence success in proving the hypothesis on the existence of functions A–G satisfying the Main Lemma and (5)–(6).

3.1 Choice of the modulus

It can be seen from the proof that one could easily take any other prime number greater than 6 for the role of the modulus; however, this would immediately result in an increase of the number of binomial coefficients.

It is a bit more tricky, but still possible, to use also numbers 2, 3 or 5 as the modulus.

3.2 Reduction of the number of binomial coefficients

In the above given proof of the Main Lemma almost no care was taken to have the total number of binomial coefficients small. However, this number can be considerably reduced using the following tricks.

All Boolean numbers $a_1, \dots, a_k$ are pairwise orthogonal provided that

$$\begin{aligned} a_2 &\perp a_1, \\ a_3 &\perp a_1 + a_2, \\ &\vdots \\ a_k &\perp a_1 + \ldots + a_{k-1}, \end{aligned} \tag{38}$$

so to express (19) we need only 23 binomial coefficients instead of 552 used in (32).

Similarly, 72 conditions (S′) from Redefinition 3.8 can be rewritten as 24 conditions

$$7c_{i,j} \perp \sum_{j' \neq j} c_{i,j'}, \qquad i = 1,2,3,4,5,6, \quad j = 1,2,3,4. \tag{39}$$

Analogous reduction can be applied to conditions (S″), (R′) and (R″) from the same redefinition.

The equality (22) can be used for expressing $c_{1,1}$ in terms of other 23 c's and m thus eliminating the need for (37).

The above ideas lead to a proof of the Main Lemma with $p = 23$ and $q = 215$.

3.3 Use of multinomial coefficients

The multinomial version of Kummer's theorem enables one to express the statement that non-negative integers $a_1, \ldots, a_k$ are pairwise orthogonal Boolean numbers compactly as

$$\binom{4(a_1 + \ldots + a_k)}{a_1; a_1; a_1; a_1; \ldots; a_k; a_k; a_k; a_k} \not\equiv 0 \pmod 7. \qquad (40)$$

3.4 Combining symmetric binomial coefficients

It follows from Kummer's theorem that

$$\binom{2a}{a}\binom{2b}{b} \not\equiv 0 \pmod 7 \Longleftrightarrow \binom{2(7^k a + b)}{7^k a + b} \not\equiv 0 \pmod 7 \qquad (41)$$

where k is so large that $7^k > b$. However, it is not clear whether this method of combining many symmetric binomial coefficient non-divisibility conditions into a single non-divisibility condition with extra 7^k factors has any advantages.

3.5 Hunting the Hypothesis

The main difficulty in attacking the 4CC in the proposed way is connected with the great freedom in the choice of the function F. It is subjected to only two conditions:

- to allow closed summation (5);
- the sum $G(m, n)$ should never be divisible by 7.

Respectively, the search for a suitable F can go in two directions:

- based on an analysis of the known summation algorithms, select for F a function giving closed summation in the hope that the sum will be not divisible by 7;

- select for F a function which for several trial values of n and m would give a sum non-divisible by 7 in the hope of finding afterwards a closed summation with required properties for arbitrary n and m.

At first sight it seems that the multiple sum in (5) can be calculated in reasonable time only for very small values of n and m, i.e., only for very primitive maps. However, much more could be done taking into account that we need to know the sum only modulo 7.

Namely, it follows from the proof of the Main Lemma that for given n and m almost all values of $E(n, m, c_1, \dots, c_p)$ are divisible by 7, and those which are not divisible are in one-to-one correspondence with 4-colourings of the discrete map $\langle n, m \rangle$. In other words, calculating $G(n, m)$ modulo 7, we can replace summation over all $c_1, \dots, c_p$ by summation over all 4-colourings of the map.

3.6 The mystery of 7^n

The binomial coefficients in (36) and (35) contain the exponential factor 7^n. This does not prevent a computer search for a closed form (5) for any *fixed* value of n; however, to prove the 4CT a human being should then generalize a number of particular examples to the case of *arbitrary* n.

The role of the factor 7^n in (36) and (35) is clear from a graph-theoretical point of view—it corresponds to radial edges. However, it is not clear at all that this special form of the factor should be of any importance for the success of closed summation, i.e., it seems to be equally likely that we could find a more general closed summation

$$\sum_{c_1} \dots \sum_{c_p} E'(l, m, c_1, \dots, c_p) F'(l, m, c_1, \dots, c_p) = G'(l, m) \tag{42}$$

where

$$E'(l, m, c_1, \dots, c_p) = \prod_{k=1}^{q} \binom{A_k(m, c_1, \dots, c_p) + lB_k(m, c_1, \dots, c_p)}{C_k(m, c_1, \dots, c_p) + lD_k(m, c_1, \dots, c_p)}. \tag{43}$$

Now to prove the 4CT it would be sufficient to check that

$$\forall nm G'(7^n, m) \not\equiv 0 \pmod 7. \tag{44}$$

Again it is not clear whether the special form of the first argument of G' above is essential for the non-divisibility by 7. It might well turn out to be true that

$$\forall lm G'(7l, m) \not\equiv 0 \pmod 7. \tag{45}$$

If this is so, it would be a striking generalization of the 4CT.

4 Appendix 1: Representation of planar graphs by discrete maps

Here we show that for every planar graph G there is a discrete map $\langle G', \mu \rangle$ (in the sense of Definition 1.4) such that graph G' is a rectangular grid and graph G can be obtained by deleting some vertices and contracting all internal edges of graph G'.

Let graph $G = \langle V, E \rangle$ with n vertices and m edges be drawn on a plane $\mathcal{P}$. We identify the verticies $v_1, \ldots, v_m$ with their cartesian coordinates $\langle x_1, y_1 \rangle, \ldots, \langle x_n, y_n \rangle$ and treat the edges $e_1, \ldots, e_m$ as simple Jordan curves, i.e. continuous one-to-one mappings $e_k : [0, 1] \to \mathcal{P}$, such that:

- the end-points of the edges are vertices, i.e.

$$e_k(0) \in V, \quad e_k(1) \in V; \tag{46}$$

- no other point of an edge can be a vertex, i.e.

$$e_k(t) = v_i \Longrightarrow t = 0 \vee t = 1; \tag{47}$$

- there are no loops, i.e.

$$e_k(0) \neq e_k(1); \tag{48}$$

- two different edges can have in common only the end-points,

$$k \neq l \ \& \ e_k(s) = e_l(t) \Longrightarrow s = 0 \vee s = 1. \tag{49}$$

By an abuse of the language, sometimes we will also identify the edges with the images $e_k([0, 1])$.

Let d_1 and d_2 denote respectively the minimal distance between two different vertices and the minimal distance between non-incident a vertex and an edge. Let

$$d_3 = \frac{\min\{d_1, d_2\}}{10}. \tag{50}$$

Square area

$$A_i = \{\langle x, y\rangle : |x - x_i| \le d_3, |y - y_i| \le d_3\} \tag{51}$$

will belong to the country with capital v_i. This enables us to deal with *cut-off edges* $e'_1, \ldots, e'_n$ defined as follows. Let $e_k(0) = v_i$, $e_k(1) = v_j$. The mapping e'_k is defined as the restriction of the edge e_k to the interval $[s_k, t_k]$ where

$$s_k = \sup\{s : e_k(s) \in A_i\}, \qquad t_k = \inf\{t : s_k < t \ \&\ e_k(t) \in A_j\}. \tag{52}$$

By the continuity of e_k, $0 < s_k$ and $t_k < 1$, so different cut-off edges have no common points.

Let us introduce a notation for the end-points of the cut-off edges, namely, let

$$e'_k(s_k) = e_k(s_k) = \langle x'_k, y'_k\rangle \tag{53}$$

and

$$e'_k(t_k) = e_k(t_k) = \langle x''_k, y''_k\rangle. \tag{54}$$

Let d_4 be the minimal distance between two different cut-off edges, i.e. the minimal distance between two points on different cut-off edges; by the above remark $d_4 > 0$. Let further

$$d_5 = \min\{d_3, \frac{d_4}{10}\}. \tag{55}$$

Let

$$x_{\min} = \min\{x_1, \ldots, x_m\}, \qquad x_{\max} = \max\{x_1, \ldots, x_m\}, \tag{56}$$

$$y_{\min} = \min\{y_1, \ldots, y_m\}, \qquad y_{\max} = \max\{y_1, \ldots, y_m\}. \tag{57}$$

Let D be an integer so large that

$$\delta x = \frac{x_{\max} - x_{\min}}{D} < d_5, \qquad \delta y = \frac{y_{\max} - y_{\min}}{D} < d_5. \tag{58}$$

Let

$$\begin{aligned} X \ = \ & \{x_1, \ldots, x_n\} \cup & (59)\\ & \{x_1 - d_3, \ldots, x_n - d_3\} \cup & (60)\\ & \{x_1 + d_3, \ldots, x_n + d_3\} \cup & (61)\\ & \{x'_1, x''_1, \ldots, x'_m, x''_m\} \cup & (62)\\ & \{x_{\min}, x_{\min} + \delta x, x_{\min} + 2\delta x, \ldots, x_{\max}\}, & (63) \end{aligned}$$

$$
\begin{aligned}
Y \quad = \quad & \{y_1, \ldots, y_n\} \cup & (64)\\
& \{y_1 - d_3, \ldots, y_n - d_3\} \cup & (65)\\
& \{y_1 + d_3, \ldots, y_n + d_3\} \cup & (66)\\
& \{y'_1, y''_1, \ldots, y'_m, y''_m\} \cup & (67)\\
& \{y_{\mathrm{min}}, y_{\mathrm{min}} + \delta y, y_{\mathrm{min}} + 2\delta y, \ldots, y_{\mathrm{max}}\}. & (68)
\end{aligned}
$$

The sets X and Y contain coordinates of all the veritices, the end-points of all cut-off edges, and some extra numbers. Let us introduce a uniform notation for the elements of these sets:

$$X = \{\bar{x}_0, \ldots, \bar{x}_p\}, \tag{69}$$

$$Y = \{\bar{y}_0, \ldots, \bar{y}_q\} \tag{70}$$

where

$$\bar{x}_0 < \bar{x}_1 < \ldots < \bar{x}_p, \tag{71}$$

$$\bar{y}_0 < \bar{y}_1 < \ldots < \bar{y}_q. \tag{72}$$

Let us consider the grid $G' = \langle V', E' \rangle$ having $(p+1)(q+1)$ vertices in $V' = X \times Y$ and $(p+1)q + p(q+1)$ edges connecting vertices $\langle x_i, y_{j-1} \rangle$ and $\langle x_i, y_j \rangle$ and vertices $\langle x_{i-1}, y_j \rangle$ and $\langle x_i, y_j \rangle$. Clearly, V' containes all vertices of the original graph G and the end-points of all cut-off edges.

Rectangular areas

$$C_{ij} = \{\langle x, y \rangle : \bar{x}_{i-1} \leq x \leq \bar{x}_i, \bar{y}_{j-1} \leq y \leq \bar{y}_j\} \tag{73}$$

will be called *cells*. We shall say that a cut-off edge e'_k *visits* a cell C_{ij} if their intersection is not empty.

Let a *fat edge* f_k be the union of all cells visited by the cut-off edge e'_k. Thanks to our choice of the d's, the fat edges have some propeties similar to that of the edges and cut-off edges.

- a fat edge f_k does not intersect an area A_i unless v_i is an end-point of the edge e_k;
- two different fat edges have no common points.

Moreover, the fat edge f_k has the following two properties:

- it is a connected set;
- it contains the end-points of the cut-off edge e'_k.

These two properties will be kept even if we remove from f_k all points both coordinates of which do not belong to the sets X and Y respectively. The resulting *enlighted edge* h_k is a subgraph of G'.

Vertices belonging neither to the A's no to the h's should be deleted. Let us now colour the remaining vertices of G'.

First, the vertices lying in an area A_i are coloured in the colour i (this definition is not contradictory because these areas are mutually disjoint thanks to our choice of the d's).

By colouring the areas A_i we have also coloured the end-points of e'_k which belong to h_k and, maybe, some of their nearby vertices as well. We shall continue colouring the vertices of h_k one by one keeping the following rule: a vertex of h_k can be coloured in some colour provided that this vertex is an end-point of an edge from h_k the other end-point of which has already been coloured in this colour.

It is not difficult to see that in this way we obtain a desired colouring μ.

5 Appendix 2: Proof of Kummer's Theorem

Note that the identity

$$\binom{a+b}{a} = \frac{(a+b)!}{a!\,b!} \tag{74}$$

implies that

$$v_p\left(\binom{a+b}{a}\right) = v_p\left((a+b)!\right) - v_p(a!) - v_p(b!), \tag{75}$$

where $v_p(k)$ is the exponent of p in the prime factorization of k. It is not difficult to see that

$$v_p(k!) = \left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{k}{p^2} \right\rfloor + \dots, \tag{76}$$

because among the numbers $1, \dots, k$, there are exactly $\lfloor \frac{k}{p} \rfloor$ numbers divisible by p, exactly $\lfloor \frac{k}{p^2} \rfloor$ numbers divisible by p^2, and so on. Thus,

$$v_p\left(\binom{a+b}{a}\right) = \sum_{l\geq 1}\left(\left\lfloor \frac{a+b}{p^l} \right\rfloor - \left\lfloor \frac{a}{p^l} \right\rfloor - \left\lfloor \frac{b}{p^l} \right\rfloor\right). \tag{77}$$

Now it suffices to note that in this sum, the lth summand is equal to either 1 or 0, depending on whether or not there is a carry from the $(l-1)$th digit.

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