TWO UNIVERSAL 3-QUANTIFIER REPRESENTATIONS OF RECURSIVELY ENUMERABLE SETS

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1. Let us agree on the following notation. Lower-case Latin letters from a to n (inclusively) with indices and without them will be used as variables for nonnegative integers, the rest lower-case Latin letters will be used as variables for integers. Analogously, lower case Greek letters from α to ν will be used as metavariables for nonnegative integers, and the rest will be used as metavariables for integers.

Upper case Latin letters will denote polynomials. Here as polynomials one means only polynomials with integer coefficients; this property of polynomials will not be stated explicitly later.

2. We say that a set $\mathfrak{R}$ of nonnegative integers is represented by an arithmetic formula $\mathfrak{F}$ with one free variable a if the equivalence $a \in \mathfrak{R} \Leftrightarrow \mathfrak{F}$ is true.

As K. Gödel showed, any recursively enumerable set is represented by some arithmetical formula. One can improve this result by putting various restrictions on the types of formulas. Such investigation was done in [2-13]. The aim of this paper is to show that every recursively enumerable set is represented by formulas of each of the two following types:

$$\exists b \exists c \mathop{\&}_{\iota=1}^{\varepsilon} \exists d [P_\iota(a,b,c) < D_\iota(a,b,c)d < Q_\iota(a,b,c)], \tag{1}$$

$$\exists b \exists c \forall f [f \leq F(a,b,c) \Rightarrow W(a,b,c,f) > 0]. \tag{2}$$

3. Let $\mathfrak{R}$ be a recursively enumerable set. Let us find formula of the type

$$\exists h_1 \ldots \exists h_\delta [R(a,h_1,\ldots,h_\delta) = 0], \tag{3}$$

which represents the set $\mathfrak{R}$ (existence of such formula is shown, for instance, in [6-9,14]).

Let us denote the degree of the polynomial R by λ. Without loss of generality we may assume that $\lambda \geq 1$.

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If formula (2) is equivalent to formula (3), then the pair $\langle b, c\rangle$, whose existence is stated in (2), must carry the whole information about the δ-tuple $\langle h_1, \dots h_\delta\rangle$, whose existence is stated in (3).

Many ways to code tuples of nonnegative integers by means of one or two nonnegative integers are known. We choose one nonstandard method, which allows us to check the truth of the relation

$$R(a, h_1, \dots, h_\delta) = 0, \tag{4}$$

by the code directly, without finding the numbers $h_1, \dots, h_\delta$.

Let us denote by $B(h_1, \dots, h_\delta, k)$ the polynomial

$$\sum_{\iota=1}^{\delta} h_\iota k^{(\lambda+1)^\iota}.$$

The "geometric sense" of this polynomial is the following: if k is greater than each of the numbers $1, h_1, \dots, h_\delta$, then $h_1, \dots, h_\delta$ are $(\lambda+1)$-th, ... , $(\lambda+1)^\delta$-th digits of the number $B(h_1, \dots, h_\delta, k)$ in the k-ary number system; in addition, all other digits are zeros.

One can easily check that the identity of the following type holds:

$$(1 + ak + B(h_1, \dots, h_\delta, k))^\lambda = \sum_{\alpha_0+\dots+\alpha_\delta \le \lambda} \kappa_{\alpha_0,\dots,\alpha_\delta} a^{\alpha_0} h_1^{\alpha_1} \dots h_\delta^{\alpha_\delta} k^{N(\alpha_0,\dots,\alpha_\delta)}, \tag{5}$$

where

$$N(l_0, \dots, l_\delta) = \sum_{\iota=0}^{\delta} l_\iota (\lambda+1)^\iota,$$

and $\kappa_{\alpha_0,\dots,\alpha_\delta}$ are positive integers. One can easily see that the polynomial N has the following property:

$$\left\{ \mathop{\&}_{\iota=0}^{\delta} ((l'_\iota \le \lambda) \,\&\, (l''_\iota \le \lambda) \,\&\, N(l'_0, \dots, l'_\delta) = N(l''_0, \dots, l''_\delta) \right\} \Rightarrow \mathop{\&}_{\iota=0}^{\delta} (l'_\iota = l''_\iota). \tag{6}$$

This property becomes obvious from the "geometric sense" of the polynomial N: if $l_\iota \le \lambda$, then $l_\delta, \dots, l_0$ are exactly all digits in the expansion of $N(l_0, \dots, l_\delta)$ in the number system with the base $\lambda + 1$. We choose polynomial B such that in the k-ary expansion of the number $B(h_1, \dots, h_\delta, k)$ all non-zero digits are placed rarely just to obtain property (6). Property (6) allows us to give "the geometric interpretation" of identity (5): in the k-ary expansion of the number

$(1 + ak + B(h_1, \dots, h_\delta, k))^\lambda$ the digits are all possible numbers of the form

$$\kappa_{\alpha_0,\dots,\alpha_\delta} a^{\alpha_0} h_1^{\alpha_1} \dots h_\delta^{\alpha_\delta} \tag{7}$$

provided k exceeds each of them.

Without loss of generality we shall assume that the polynomial R is a linear combination of monomials (7):

$$R(a, h_1, \dots, h_\delta)) = \sum_{\alpha_0+\dots+\alpha_\delta \le \lambda} \rho_{\alpha_0,\dots,\alpha_\delta} \kappa_{\alpha_0,\dots,\alpha_\delta} a^{\alpha_0} h_1^{\alpha_1} \dots h_\delta^{\alpha_\delta} \tag{8}$$

Obviously, if $l_1 + \dots + l_\delta \le \lambda$, then

$$N(l_1, \dots, l_\delta) \le \lambda(\lambda + 1)^\delta.$$

Let us denote $\lambda(\lambda + 1)^\delta$ by ν, and the polynomial

$$\sum_{\alpha_0+\dots+\alpha_\delta \le \lambda} \rho_{\alpha_0,\dots,\alpha_\delta} k^{\nu - N(\alpha_0,\dots,\alpha_\delta)}$$

by $V(k)$. One can easily see that the identity of the following type holds:

$$V(k)(1 + ak + B(h_1, \dots, h_\delta, k))^\lambda = \sum_{\iota=0}^{2\nu} T_\iota(a, h_1, \dots, h_\delta) k^\iota, \tag{9}$$

where T_0, ... ,$T_{2\nu}$ are polynomials, whose degrees do not exceed λ.

One can naturally interpret identity (9) if one considers k-ary number system, in which negative digits are allowed, too; for instance, one may require that

$$k > |2T_\iota(a, h_1, \dots, h_\delta)| \qquad (\iota = 0, \dots, 2\nu) \tag{10}$$

and consider the system with digits ranging from $[-(k - 1)/2]$ to $[(k - 1)/2]$.

It is easy to check that (5), (6), (8) and (9) imply the identity

$$T_\nu(a, h_1, \dots, h_\delta) = R(a, h_1, \dots, h_\delta). \tag{11}$$

Thus, if

$$b = B(h_1, \dots, h_\delta, k) \tag{12}$$

and k is large enough so that inequalities (10) are satisfied, then relation (4) holds if and only if the digit on the ν-th place in k-ary expansion of the number $V(k)(1 + ak + b)^\lambda$ is zero. As we shall show below, the latter condition can be easily written by means of one existential quantifier.

Lemma 1. For any a, b, $h_1, \dots, h_\delta$, k satisfying conditions (10) and (12), relation (4) holds if and only if there exists an integer z such that

$$-k^\nu < 2(V(k)(1+ak+b)^\lambda - zk^{\nu+1}) < k^\nu. \tag{13}$$

Necessity. Put

$$z = \sum_{\iota=\nu+1}^{2\nu} T_\iota(a, h_1, \dots, h_\delta)k^{\iota-\nu-1}$$

By (9), (12) and (4),

$$V(k)(1+ak+b)^\lambda - zk^{\nu+1}) = \sum_{\iota=0}^{\nu-1} T_\iota(a, h_1, \dots, h_\delta)k^\iota.$$

We deduce from (10) that

$$\left| 2\sum_{\iota=0}^{\nu-1} T_\iota(a, h_1, \dots, h_\delta)k^\iota \right| \le \tag{14}$$
$$\le \sum_{\iota=0}^{\nu-1} |2T_\iota(a, h_1, \dots, h_\delta)k^\iota| \le$$
$$\le \sum_{\iota=0}^{\nu-1}(k-1)k^\iota = k^\nu - 1 < k^\nu,$$

therefore, inequalities (13) are satisfied.

Sufficiency. It is easy to see that there exists at most one integer y such that

$$-k^\nu < 2(V(k)(1+ak+b)^\lambda - yk^\nu) < k^\nu.$$

On the one hand, accordingly to (13), y equals to zk, on the other hand y equals to

$$\sum_{\iota=\nu}^{2\nu} T_\iota(a, h_1, \dots, h_\delta)k^{\iota-\nu}$$

since , by (9) and (12),

$$V(k)(1+ak+b)^\lambda - \left(\sum_{\iota=\nu}^{2\nu} T_\iota(a, h_1, \dots, h_\delta)k^{\iota-\nu}\right)k^\nu =$$
$$= \sum_{\iota=0}^{\nu-1} T_\iota(a, h_1, \dots, h_\delta)k^\iota$$

and inequalities (14) hold. Thus,

$$\sum_{\iota=\nu}^{2\nu} T_\iota(a, h_1, \dots, h_\delta) k^{\iota-\nu} = zk.$$

Passing from the equality to congruence, we get

$$T_\nu(a, h_1, \dots, h_\delta) \equiv 0 \pmod{k}.$$

This, together with (10), gives us the equality

$$T_\nu(a, h_1, \dots, h_\delta) = 0. \tag{15}$$

Now (4) follows from (11) and (15).

Lemma is proved.

4. Let us transform inequalities (10). Let us denote by γ some positive integer, which exceeds the sum of absolute values of all coefficients of any polynomial $T_1, \dots, T_{2\nu}$ multiplied by 2. Obviously, the following inequalities hold:

$$|2T_\iota(a, h_1, \dots, h_\delta)| < \gamma(\max\{1, a, h_1, \dots, h_\delta\})^\lambda.$$

Thus, if

$$c \geq \max\{h_1, \dots, h_\delta\}$$

and

$$k = K(a, c) = \gamma(2 + a + c)^\lambda,$$

then inequalities (10) are satisfied.

By means of Lemma 1 one can easily show that formula (3) is equivalent to the formula

$$\exists b \exists c[\mathfrak{F}_1 \,\&\, \exists z \mathfrak{F}_2],$$

Here and later on, $\mathfrak{F}_1$ denotes the formula

$$\exists h_1 \dots \exists h_\delta [c \geq \max\{h_1, \dots, h_\delta\} \,\&\, b = H(h_1, \dots, h_\delta, K(a, c))]$$

and $\mathfrak{F}_2$ denotes the formula

$$-(K(a,c))^\nu < 2(V(K(a,c))(1 + aK(a,c) + b)^\lambda - \\ - zK(a,c)^{\nu+1}) < (K(a,c))^\nu.$$

Lemma 2. Formula $\mathfrak{F}_1$ is equivalent to the formula

$$\mathfrak{F}_3 \,\&\, \mathfrak{F}_4 \,\&\, \mathfrak{F}_5, \tag{16}$$

Here and later on $\mathfrak{F}_3$ denotes the formula

$$\exists d[b = d(K(a,c))^{\lambda+1}],$$

$\mathfrak{F}_4$ denotes the formula

$$\mathop{\&}_{\iota=1}^{\delta-1} \exists d \exists e[b = d(K(a,c))^{(\lambda+1)^{\iota+1}} + e \,\&\, e < (c+1)(K(a,c))^{(\lambda+1)^\iota}],$$

and $\mathfrak{F}_5$ denotes the formula

$$b < (c+1)(K(a,c))^{(\lambda+1)^\delta}.$$

The proof of the lemma becomes explicitly obvious, if one notes that both formulas $\mathfrak{F}_1$ and (16) mean that in the expansion of the number b in the number system with the base $K(a,c)$ all non-zero digits can be placed only on $(\lambda+1)$-th, … , $(\lambda+1)^\delta$-th positions and, moreover, these digits do not exceed c.

5. As Lemmas 1 and 2 shows, formula (3) is equivalent to the formula

$$\exists b \exists c[\exists z \mathfrak{F}_2 \,\&\, \mathfrak{F}_3 \,\&\, \mathfrak{F}_4 \,\&\, \mathfrak{F}_5]. \tag{17}$$

Theorem 1. Every recursively enumerable set of nonnegative integers can be represented by the formula of type (1).

Proof. The desired formula can be obtained from the formula (17) by means of easy algebraic transformations.

Formula $\mathfrak{F}_2$ contains variable z, whose possible values are all integers. However, it follows from $\mathfrak{F}_2$ that

$$2z(K(a,c))^{\nu+1} > -(K(a,c))^\nu + 2V(K(a.c))(1+aK(a,c)+b)^\lambda,$$

hence,

$$z \geq V(K(a.c))(1+aK(a,c)+b)^\lambda,$$

Let us denote by $\mathfrak{F}_6$ the formula, which is obtained from $\mathfrak{F}_6$ by substituting the polynomial

$$d + V(K(a.c))(1+aK(a,c)+b)^\lambda$$

instead of z and by distributing terms, which contain and do not contain d, into different sides of the inequalities. Obviously, the formula $\exists z \mathfrak{F}_2$ is equivalent to the formula $\exists d \mathfrak{F}_6$. We transform the formula $\mathfrak{F}_3$ into an equivalent one

$$\exists d[b-1 < (K(a,c))^{\lambda+1}d < b+1]. \tag{18}$$

Equalities in each conjunctive term of the formula $\mathfrak{F}_4$ allow us to express e in terms of a, b, c and d explicitly, and, therefore, allow us to exclude this variable. In addition, we must impose inequality, which corresponds to non-negativity of e. Finally, we obtain the formula

$$\mathop{\&}_{\iota=1}^{\delta-1} \exists d[b-(c+1)(K(a,c))^{(\lambda+1)^\iota} < (K(a,c))^{(\lambda+1)^{\iota-1}}d < b+1. \tag{19}$$

Now we must replace the formula $\mathfrak{F}_5$ by an equivalent one

$$\exists d[b-1 < bd < (c+1)(K(a,c))^{(\lambda+1)^\delta}].$$

Theorem is proved.

6. Now we turn to constructing the formula of type (2), which represents the set $\mathfrak{R}$. For this purpose we first show that for any ϵ there exist polynomials F_ϵ and W_ϵ in 2ϵ and $3\epsilon+1$ variables respectively, such that the following property holds. If numbers $g_1, \dots ,g_\epsilon$, $s_1, \dots ,s_\epsilon$, $t_1, \dots ,t_\epsilon$ satisfy the inequalities

$$0 < g_\iota, \quad t_\iota - s_\iota \le g_\iota \quad (\iota = 1, \dots, \epsilon), \tag{20}$$

then the formula

$$\mathop{\&}_{\iota=1}^{\epsilon} \exists z[s_\iota < zg_\iota < t_\iota] \tag{21}$$

is equivalent to the formula

$$\forall f[f \le F_\epsilon(s_1, \dots, s_\epsilon, t_1, \dots, t_\epsilon) \Rightarrow W_\epsilon(g_1, \dots, g_\epsilon, s_1, \dots, s_\epsilon, t_1, \dots, t_\epsilon, f) > 0].$$

We start with the case $\epsilon = 1$ and find, firstly, polynomials X and Y such that for $g > 0$ then the formula

$$\exists z[s < zg < t] \tag{22}$$

is equivalent to the formula

$$\forall y[-s^2 - t^2 - 2 < y \le s^2 + t^2 + 2 \Rightarrow X(g, s, t, y) > 0 \vee Y(g, s, t, y) > 0]. \tag{23}$$

Formula (23) without existential quantifiers must contain, however, complete information about some integer z that satisfies inequalities

$$s < zg < t. \tag{24}$$

We shall verify equivalence between formulas of type (23) and (22) by means of the following obvious lemma, which may be regarded as a discrete analogue of the Cauchy theorem about vanishing of a continuous function, whose values at the endpoints of an interval have different signs.

Let p and q be integers such that $p < q$, let Φ and Ψ be unary predicates defined for all integers from p to q inclusively. If $\Phi(p) \,\&\, \Psi(q)$ holds and for any w, such that $p < w < q$, $\Phi(w) \vee \Psi(w)$ holds, then there exists an integer r such that $p \le r \le q$ and $\Phi(r) \,\&\, \Psi(r+1)$.

Lemma 3. If

$$g > 0, \tag{25}$$

then formula (22) is equivalent to the formula

$$\forall y[-s^2 - t^2 - 2 < y \le s^2 + t^2 + 2 \Rightarrow (y-1)g - s > 0 \vee t - yg > 0]. \tag{26}$$

Proof. Let g, s, t satisfy conditions (25) and (26). Let us show that they satisfy condition (22), too.

By (25),

$$t-(-s^2-t^2-1)g \geq t+s^2+t^2+1>0,$$
$$(s^2+t^2+1)g-s \geq s^2+t^2+1-s>0.$$

By the discrete analogue of the Cauchy theorem formulated above, we have that there exists z such that

$$t-zg>0 \,\&\, zg-s>0.$$

Thus, condition (22) is satisfied.

Now, let g, s and t satisfy conditions (25) and (22). Let us find z that satisfies inequalities (24). Assume that condition (26) doesn't hold. Let y be a number such that

$$(y-1)g-s \leq 0 \,\&\, t-yg \leq 0. \tag{27}$$

¿From (24) and (27) we obtain that

$$(y-1)g \leq s < zg, \quad zg < t < yg.$$

Consequently

$$y-1<z<y.$$

This contradiction completes the proof of the equivalence of formulas (22) and (26).

Note, that if

$$t-s \leq g, \tag{28}$$

then two inequalities in formula (28) are inconsistent. Moreover, if $(y-1)g-s>0$, then $t-yg<0$, and conversely if $t-yg>0$, then $(y-1)g-s<0$.

This allows us to transform the disjunction of two inequalities into one:

$$(y-1)g-s>0 \vee t-yg>0 \Leftrightarrow ((y-1)g-s>0 \,\&\, t-yg<0)) \vee$$
$$\vee\ (t-yg>0 \,\&\, (y-1)g<0)) \Leftrightarrow ((y-1)g-s)(yg-t)>0.$$

Thus, if inequalities (25) and (28) are satisfied, then formula (22) is equivalent to the formula

$$\forall y[-s^2-t^2-2<y \leq s^2+t^2+2 \Rightarrow Z(g,s,t,y)>0],$$

here and later on $Z(g,s,t,y)$ denotes the polynomial

$$((y-1)g-s)(yg-t).$$

Note, that if $g>0$, then

$$\forall y[y \leq -s^2-t^2-2 \ \vee\ y>s^2+t^2+2 \Rightarrow Z(g,s,t,y)>0]. \tag{29}$$

7. Now consider an arbitrary formula of type (21). If the numbers $g_1, \dots, g_\epsilon, s_1, \dots, s_\epsilon, t_1, \dots, t_\epsilon$ satisfy inequalities (20), then, as shown above, formula (21) is equivalent to the formula

$$(30) \qquad \mathop{\&}_{\iota=1}^{\epsilon} \forall y[-s_\iota^2 - t_\iota^2 - 2 < y \le s_\iota^2 + t_\iota^2 + 2 \Rightarrow Z(g_\iota, s_\iota, t_\iota, y) > 0].$$

Let us introduce the following notation:

$$F_\iota(s_1, \dots, s_\iota, t_1, \dots, t_\iota) = \sum_{\mu=1}^{\iota}(2s_\mu^2 + 2t_\mu^2 + 4) \quad (\iota = 0, \dots, \epsilon),$$

$$Z_\iota(g_\iota, s_1, \dots, s_\iota, t_1, \dots, t_\iota, y) = Z(g_\iota, s_\iota, t_\iota,$$
$$y - F_{\iota-1}(s_1, \dots, s_{\iota-1}, t_1, \dots, t_{\iota-1}) - s_\iota^2 - t_\iota^2 - 2) \quad (\iota = 1, \dots, \epsilon).$$

Obviously, formula (30) is equivalent to the formula

$$(31) \quad \mathop{\&}_{\iota=1}^{\epsilon} \forall y[F_{\iota-1}(s_1, \dots, s_{\iota-1}, t_1, \dots, t_{\iota-1}) < y \le$$
$$\le F_\iota(s_1, \dots, s_\iota, t_1, \dots, t_\iota) \Rightarrow$$
$$\Rightarrow Z_\iota(g_\iota, s_1, \dots, s_\iota, t_1, \dots, t_\iota, y) > 0].$$

Let us denote by $W_\epsilon(g_1 \dots, g_\epsilon, s_1, \dots, s_\epsilon, t_1, \dots, t_\epsilon, y)$ the polynomial

$$\prod_{\iota=1}^{\epsilon} Z_\iota(g_\iota, s_1, \dots, s_\iota, t_1, \dots, t_\iota, y).$$

Lemma 4. If the numbers $g_1 \dots, g_\iota, s_1, \dots, s_\iota, t_1, \dots, t_\iota$ satisfy inequalities (20), then formula (21) is equivalent to the formula

$$\forall f[f \le F_\epsilon(s_1, \dots, s_\epsilon, t_1, \dots, t_\epsilon) \Rightarrow$$
$$\Rightarrow W_\epsilon(g_1 \dots, g_\epsilon, s_1, \dots, s_\epsilon, t_1, \dots, t_\epsilon, f) > 0].$$

One can easily carry out the proof of the lemma using property (29).

Theorem 2. Every recursively enumerable set of non-negative integers can be represented by a formula of type (2).

Proof. Let us transform the formula $\exists z \mathfrak{F}_2 \,\&\, \mathfrak{F}_3 \,\&\, \mathfrak{F}_4 \,\&\, \mathfrak{F}_5$ to the form analogous to (21).

In the formula $\mathfrak{F}_2$ it is sufficient to distribute terms, which contain and do not contain z, into different sides of inequalities. The resulting formula we denote by $\mathfrak{F}_7$.

In the formula $\mathfrak{F}_3$ we replace the variable d, whose admissible values are nonnegative integers, by the variable z, whose admissible values are all integers. Since always

$$b \ge 0, \quad (K(a, c))^{\lambda+1} > 0,$$

the obtained formula is equivalent to the formula $\mathfrak{F}_3$. Let us rewrite the formula we obtained into the form analogous to (18) and denote the new formula by $\mathfrak{F}_8$.

Analogously, in each conjunctive term of the formula $\mathfrak{F}_4$ we replace the variable d by z. Since always

$$b \geq 0, \quad (K(a,c))^{(\lambda+1)^{\iota+1}} > (c+1)(K(a,c))^{(\lambda+1)^{\iota}} > 0,$$

the obtained formula is equivalent to the formula $\mathfrak{F}_4$. Let us make with the obtained formula the same transformations as we made with the formula $\mathfrak{F}_4$ in the proof of Theorem 1. As a result, we obtain the formula $\mathfrak{F}_9$, which is analogous to formula (19).

We replace the formula $\mathfrak{F}_5$ by an equivalent formula

$$\exists z[b-(c+1)(K(a,c))^{(\lambda+1)^{\delta}} < 2(c+1)(K(a,c))^{(\lambda+1)^{\delta}} z < \\ < (c+1)(K(a,c))^{(\lambda+1)^{\delta}} - b],$$

which we denote by $\mathfrak{F}_{10}$.

Formula

$$\exists z \mathfrak{F}_7 \mathbin{\&} \mathfrak{F}_8 \mathbin{\&} \mathfrak{F}_9 \mathbin{\&} \mathfrak{F}_{10} \tag{32}$$

is of the form analogous to (21). The only difference is the following: there are polynomials in the parameters a, b, c instead of the variables g_ι, s_ι, t_ι in (32). It is easy to check that for all values of the parameters, the inequalities analogous to (20) hold. This allows us to find by Lemma 4 the desired polynomials F and W.

Theorem is proved.

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