This is a preprint of a paper accepted for publication in:

Journal of Graph Theory © (2003) John Wiley & Sons, Inc.

Some Probabilistic Restatements of the Four Color Conjecture

Yuri Matiyasevich

STEKLOV INSTITUTE OF MATHEMATICS AT SAINT-PETERSBURG

27 FONTANKA, SAINT-PETERSBURG, 191023, RUSSIA

URL: `http://logic.pdmi.ras.ru/~yumat`

ABSTRACT

With every triangulation of sphere we associate in a natural way a probabilistic space and define several random events. The Four Color Conjecture turns out to be equivalent to different statements about positive correlation among some pairs of these events.

1. INTRODUCTION AND THE RESULTS

The famous Four Color Conjecture (4CC for short) has, as many outstanding mathematical problems have, numerous equivalent reformulations (see, for example, [5, 6, 7, 8, 9, 10, 11, 12, 15, 21, 20] and further references in these publications). Sometimes such reformulations are given in terms very remote from maps, graphs and colorings.

In this paper several new restatements of the 4CC will be given, formally as assertions about correlations of some random events. Of course now that we have proofs of the Four Color Theorem given by K. Appel, W. Haken and J. Koch [2, 3, 4] and, more recently, by N. Robertson, D. Sanders, P. Seymour and R. Thomas [19], these restatements become corollaries of the Four Color Theorem.

It is well-known that it is sufficient to prove the 4CC for an arbitrary maximal planar graph G. Let $G = \langle V, E\rangle$ be such a graph having $3n$ edges, i.e., $E = \{e_1, \ldots, e_{3n}\}$.

Graph G, being drawn on a sphere, defines its triangulation (without loss of generality we assume that $n > 0$). Let us cut the sphere along the edges of the graph G into triangular facets. This results in a graph H_G consisting of $2n$ copies of the full graph K_3 which will be called the *cut graph* of graph G.

By a coloring of this graph H_G we shall always mean a coloring of its edges in three colors, 0, 1, and 2. Clearly, there are 6^{2n} such colorings.

We are to introduce two notions of similarity of colorings of the graph H_G; the 4CC will be shown to be equivalent to various statements about correlations between these notions.

First, the graph H_G inherits from the graph G an additional structure, namely, a cyclic order of edges in every connected component. We shall say that such a component is colored *positively* if colors 0, 1 and 2 follow in the "clock-wise order", and *negatively* otherwise. Further, we shall say that a coloring of H_G is *even* or *odd* depending on whether the number of positively (or, equivalently, negatively) colored components is even or odd. At last, we shall say that two colorings of the graph H_G have the same *parity* if both of them are either even or odd.

Second, every coloring of the graph H_G *induces* an assignment of the colors to the edges of the graph G in the following way: to get the color of some edge e of the graph G we add modulo 3 the colors of the two edges of the graph H_G into which the edge e was split during the generation of the graph H_G from the graph G. We shall say that two colorings of graph H_G are *equivalent* if they induce the same assignments of colors to the edges of graph G.

Now, let two colorings of the graph H be selected at random independently one from another, that is, the probability of the elementary event of selecting any particular pair of colorings is equal to 6^{-4n}. Let A_G be the random event "the selected colorings have the same parity", and let B_G be the random event "the selected colorings are equivalent".

It turns out that the 4CC is equivalent to the existence of correlation between the events A_G and B_G.

Theorem 1. For every maximal planar graph G with $3n$ edges

$$\mathrm{Prob}\{B_G \mid A_G\} - \mathrm{Prob}\{B_G\} \;=\; \frac{1}{4 \cdot 48^n}\chi_G(4) \tag{1}$$

where $\chi_G(4)$ is the number of colorings of vertices of the graph G in 4 colors.

Corollary 1.1. For every maximal planar graph G with $3n$ edges

$$\mathrm{Prob}\{B_G \mid A_G\} - \mathrm{Prob}\{B_G \mid \overline{A_G}\} \;=\; \frac{1}{2 \cdot 48^n}\chi_G(4). \tag{2}$$

Corollary 1.2. The Four Color Conjecture is equivalent to the assertion that for every maximal planar graph G the events A_G and B_G are (positively) correlated.

Corollary 1.3. The Four Color Conjecture is equivalent to the assertion that for every maximal planar graph G the probability for two random edge colorings of its cut graph H_G to be equivalent under the condition that the colorings have equal parity is different from (greater than) the similar probability under the condition that the colorings have opposite parities.

The event A_G can be split into two events, A_G^{even}, "both selected colorings are even", and A_G^{odd}, "both selected colorings are odd". It follows from Theorem 1 that at least one of these two events should positively correlate with the event B_G. In fact, both of them correlate positively, but to a different extent.

Theorem 2. For every maximal planar graph G with $3n$ edges

$$\mathrm{Prob}\{B_G \mid A_G^{\mathrm{even}}\} - \mathrm{Prob}\{B_G \mid A_G^{\mathrm{odd}}\} \;=\; \frac{1}{144^n}\chi_G(4). \tag{3}$$

Corollary 2.1. For every maximal planar graph G with $3n$ edges

$$\mathrm{Prob}\{B_G \mid A_G^{\mathrm{even}}\} - \mathrm{Prob}\{B_G\} = \left(\frac{1}{4\cdot 48^n} + \frac{1}{2\cdot 144^n}\right)\chi_G(4), \tag{4}$$

$$\mathrm{Prob}\{B_G \mid A_G^{\mathrm{odd}}\} - \mathrm{Prob}\{B_G\} = \left(\frac{1}{4\cdot 48^n} - \frac{1}{2\cdot 144^n}\right)\chi_G(4), \tag{5}$$

$$\mathrm{Prob}\{B_G \mid A_G^{\mathrm{even}}\} - \mathrm{Prob}\{B_G \mid \overline{A_G}\} = \left(\frac{1}{2\cdot 48^n} + \frac{1}{2\cdot 144^n}\right)\chi_G(4), \tag{6}$$

$$\mathrm{Prob}\{B_G \mid A_G^{\mathrm{odd}}\} - \mathrm{Prob}\{B_G \mid \overline{A_G}\} = \left(\frac{1}{2\cdot 48^n} - \frac{1}{2\cdot 144^n}\right)\chi_G(4). \tag{7}$$

Corollary 2.2. The Four Color Conjecture is equivalent to the assertion that for every maximal planar graph G the events A_G^{even} and B_G are (positively) correlated.

Corollary 2.3. The Four Color Conjecture is equivalent to the assertion that for every maximal planar graph G the events A_G^{odd} and B_G are (positively) correlated.

Corollary 2.4. The Four Color Conjecture is equivalent to the assertion that for every maximal planar graph G the probability for two random edge colorings of its cut graph H_G to be equivalent under the condition that both colorings are even is different from (greater than) the similar probability under the condition that both colorings are odd.

Corollary 2.5. The Four Color Conjecture is equivalent to the assertion that for every maximal planar graph G the probability for two random edge colorings of its cut graph H_G to be equivalent under the condition that both colorings are even is different from (greater than) the similar probability under the condition that the colorings have opposite parities.

Corollary 2.6. The Four Color Conjecture is equivalent to the assertion that for every maximal planar graph G the probability for two random edge colorings of its cut graph H_G to be equivalent under the condition that both colorings are odd is different from (greater than) the similar probability under the condition that the colorings have opposite parities.

The proofs are based on expressing $\chi_G(4)$ via the coefficients of so called *graph polynomial* of the *line graph* of graph G. Such expressions were presented in [13, 14] (for proofs see [15]) and, for the case of Theorem 1, in [1, Theorem 1.4]. The only but essential novelty of the present paper are the probabilistic interpretations of these expressions. In order to make this paper selfcontained, full proofs are given here.

The author found a number of similar theorems giving different probabilistic restatements of the 4CC. Among them, the restatements selected for the present publication seem to the author the most elegant. Another probabilistic restatement of the 4CC can be found in [17].

Theorem 1 can be generalized to triangulations of arbitrary surfaces (in the case of a non-orientable surface, two colorings are said to have the same parity if the cyclic order of colors is different on even number of triangles). Theorem 2 seems to depend essentially on the planarity.

2. PROOF OF THEOREM 1

A maximal planar graph G can, in a natural way, be represented by the set

$$T = \{\langle e_{i_1}, e_{j_1}, e_{k_1}\rangle, \dots, \langle e_{i_{2n}}, e_{j_{2n}}, e_{k_{2n}}\rangle\} \tag{8}$$

of triples of edges belonging to the same triangular face; we will assume that the edges are listed in the “clock-wise order”.

A coloring μ of the graph H_G can be viewed as a map from T into the set consisting of the six triples

$$\langle 0,1,2\rangle, \langle 1,2,0\rangle, \langle 2,0,1\rangle, \langle 2,1,0\rangle, \langle 1,0,2\rangle, \langle 0,2,1\rangle. \tag{9}$$

Let μ_1 and μ_2 be the two randomly selected colorings of graph H_G. We shall use a kind of generating function to represent all possible choices of μ_1 and μ_2, these functions will be polynomials in formal variables $x_1, \dots, x_{3n}$.

The assignment of colors 0, 1 or 2 to an edge e_p will be represented by the monomials

$$1, \quad x_p, \quad x_p^2. \tag{10}$$

Respectively, the six possible colorings (9) of a triangle $\langle e_p, e_q, e_r\rangle$ from T will be represented by the six monomials

$$x_q x_r^2, \quad x_p x_q^2, \quad x_p^2 x_r, \quad x_p^2 x_q, \quad x_p x_r^2, \quad x_q^2 x_r. \tag{11}$$

The "positiveness" or "negativeness" of the coloring of the triangle $\langle e_p, e_q, e_r\rangle$ will be reflected by choice of sign $+$ or $-$ in the formal sum

$$L_1(x_p, x_q, x_r) = x_q x_r^2 + x_p x_q^2 + x_p^2 x_r - x_p^2 x_q - x_p x_r^2 - x_q^2 x_r. \tag{12}$$

The product

$$M_1(x_1, \dots, x_{3n}) = \prod_{l=1}^{2n} L_1(x_{i_l}, x_{j_l}, x_{k_l}) \tag{13}$$

can be formally expanded into the sum of 6^{2n} monomials which are in a natural one-to-one correspondence with 6^{2n} possible choices of the coloring μ_1. The signs of these monomials correspond to the parity of the colorings.

To represent induced assignments of colors to the edges of the graph G we introduce an operator $\mathcal{R}$ which replaces the exponent of each variable x_p by its value modulo 3. Equivalent choices of coloring μ_1 correspond to equal (up to the sign) monomials in $\mathcal{R}M_1(x_1, \dots, x_{3n})$.

Possible choices for μ_2 will be represented in a similar way with the following modification: coloring an edge e_p in colors 0, 1, or 2 will be represented by the monomial

$$1, \quad x_p^2, \quad x_p \tag{14}$$

respectively. These monomials will be called *complimentary* to the monomials (10) and this notion naturally extends to products of several variables (formally, two monomials N_1 and N_2 are complimentary if $\mathcal{R}N_1N_2$ is a number). So the six possible colorings (9) of a triangle $\langle e_p, e_q, e_r\rangle$ from T are now represented by the six monomials

$$x_q^2 x_r, \quad x_p^2 x_q, \quad x_p x_r^2, \quad x_p x_q^2, \quad x_p^2 x_r, \quad x_q x_r^2 \tag{15}$$

complimentary to the monomials (11).

Respectively, the 6^{2n} choices for μ_2 are in one-to-one correspondence with 6^{2n} summands of the (expanded) product

$$M_2(x_1,\ldots,x_{3n}) = \prod_{l=1}^{2n} L_2(x_{i_l}, x_{j_l}, x_{k_l}) \tag{16}$$

where

$$\begin{aligned} L_2(x_p,x_q,x_r) &= x_q^2x_r + x_p^2x_q + x_px_r^2 - x_px_q^2 - x_p^2x_r - x_qx_r^2 && (17)\\ &= \mathcal{R}L_1(x_p^2,x_q^2,x_r^2) && (18)\\ &= -L_1(x_p,x_q,x_r). && (19)\end{aligned}$$

The summands of the polynomial $\mathcal{R}M_2(x_1,\ldots,x_{3n})$ correspond to the induced assignments of colors to the edges of graph G under the complimentary representation (14).

Two summands, one taken from $\mathcal{R}M_1(x_1,\ldots,x_{3n})$ and the other taken from $\mathcal{R}M_2(x_1,\ldots,x_{3n})$, correspond to a choice of equivalent colorings μ_1 and μ_2 if and only if these summands are complimentary. So if we apply the operator $\mathcal{R}$ to the (expanded) product $\mathcal{R}M_1(x_1,\ldots,x_{3n})\mathcal{R}M_2(x_1,\ldots,x_{3n})$, then pairs of equivalent colorings, $\langle\mu_1,\mu_2\rangle$, and only them, would contribute to the constant term of the polynomial

$$\begin{aligned} R(x_1,\ldots,x_{3n}) &= \mathcal{R}(\mathcal{R}M_1(x_1,\ldots,x_{3n})\mathcal{R}M_2(x_1,\ldots,x_{3n})) && (20)\\ &= \mathcal{R}(M_1(x_1,\ldots,x_{3n})M_2(x_1,\ldots,x_{3n})). && (21)\end{aligned}$$

Let us calculate this constant term, $R_0 = R(0,\ldots,0)$, in two ways.

The first way is connected with the left-hand side of (1). By definition, the conditional probability $\mathrm{Prob}\{B_G \mid A_G\}$ is equal to $\frac{\mathrm{Prob}\{A_G\cap B_G\}}{\mathrm{Prob}\{A_G\}}$. Clearly, $\mathrm{Prob}\{A_G\} = \frac{1}{2}$ so

$$\mathrm{Prob}\{B_G \mid A_G\} - \mathrm{Prob}\{B_G\} = 2\mathrm{Prob}\{A_G\cap B_G\} - \mathrm{Prob}\{B_G\}. \tag{22}$$

We have:

- If μ_1 and μ_2 are equivalent colorings of the same parity, then, on the one hand, they contribute 1 to R_0. On the other hand, they contribute 6^{-4n} to both $\mathrm{Prob}\{A_G \cap B_G\}$ and $\mathrm{Prob}\{B_G\}$, hence their contribution to (22) is also 6^{-4n}.
- If μ_1 and μ_2 are equivalent colorings of the opposite parities, then, on the one hand, they contribute -1 to R_0. On the other hand, they contribute 0 to $\mathrm{Prob}\{A_G \cap B_G\}$ and 6^{-4n} to $\mathrm{Prob}\{B_G\}$, hence their contribution to (22) is now equal to -6^{-4n}.

Thus, we get:

$$\mathrm{Prob}\{B_G \mid A_G\} - \mathrm{Prob}\{B_G\} = 6^{-4n} R_0. \tag{23}$$

It is well-known that there is a natural correspondence between colorings of the vertices of our graph G in 4 colors and colorings of edges of its dual graph in 3 colors (so called "Tait colorings", see numerous papers and books about the 4CC, for example, [7, 18, 21]). Instead of considering this dual graph, we prefer to extend the notion of Tait colorings to assignments of 3 colors to the edges of graph G: such an assignment will be called a *Tait coloring* if 3 edges bounding the same triangular facet are colored in 3 different colors. (Note that the assignment of colors to the edges of the graph G induced by a coloring of its cut graph H_G need not be a Tait coloring at all.)

The correspondence between vertex 4-colorings of graph G and its Tait colorings is not one-to-one, in fact, every single Tait coloring corresponds to 4 different vertex 4-colorings so the number of the Tait colorings is equal to $\frac{1}{4}\chi_G(4)$.

The polynomial R has degree at most 2 in each of its $3n$ variables, so it could be determined by its values taken for 3^{3n} suitable choices of the values of the variables. We select this values by allowing each of the variables to take 3 values

$$1, \quad \omega, \quad \omega^2 \tag{24}$$

where $\omega = \frac{-1+\sqrt{-3}}{2}$ is a primitive cubic root of 1 and hence the values (24) are the 3 cubic roots of unity.

By the Interpolation Theorem we have:

$$R(x_1, \dots, x_{3n}) = \sum_{\lambda} R(\omega^{\lambda(e_1)}, \dots, \omega^{\lambda(e_{3n})}) P_\lambda(x_1, \dots, x_{3n}) \tag{25}$$

where the summation is taken over all 3^{3n} maps

$$\lambda : E \to \{0,1,2\} \tag{26}$$

and

$$P_\lambda(x_1,\ldots,x_{3n}) = \prod_{p=1}^{3n} S(x_p,\lambda(e_p)), \tag{27}$$

$$S(x,q) = \prod_{\substack{0\le l\le 2\\ l\ne q}} \frac{x-\omega^l}{\omega^q-\omega^l}. \tag{28}$$

Thanks to the choice of the cubic roots of unity (24) as the coordinates of the interpolation points, we have according to (21):

$$R(\omega^{\lambda(v_1)},\ldots,\omega^{\lambda(v_{3n})}) = M_1(\omega^{\lambda(v_1)},\ldots,\omega^{\lambda(v_{3n})})M_2(\omega^{\lambda(v_1)},\ldots,\omega^{\lambda(v_{3n})}). \tag{29}$$

Polynomial L_1, occurring in the definition (13) of polynomial M_1, can be factored:

$$\begin{aligned} L_1(x_p,x_q,x_r) &= x_qx_r^2+x_px_q^2+x_p^2x_r-x_p^2x_q-x_px_r^2-x_q^2x_r \\ &= (x_p-x_r)(x_r-x_q)(x_q-x_p). \end{aligned} \tag{30}$$

Thus $M_1(\omega^{\lambda(e_1)},\ldots,\omega^{\lambda(e_{3n})})$ is equal to 0 as soon as the map λ is not a Tait coloring of graph G and hence the summation in (25) can be restricted to λ's ranging over Tait colorings of graph G.

Let us now determine the value of (29) when λ is a Tait coloring of graph G. In such a case λ can be viewed also as a coloring of the graph H_G and we can use the terminology introduced above (positive and negative colorings, the parity).

If $\langle e_p, e_q, e_r\rangle \in T$, then all 3 colors 0, 1, and 2, are used to color the edges e_p, e_q, e_r and hence

$$\begin{aligned} L_1(\lambda(e_p),\lambda(e_q),\lambda(e_r)) &= \pm(1-\omega)(\omega-\omega^2)(\omega^2-1) \\ &= \pm 3\sqrt{-3}. \end{aligned} \tag{31}$$

Respectively,

$$M_1(\omega^{\lambda(e_1)},\dots,\omega^{\lambda(e_{3n})}) = \pm 3^{3n}. \tag{32}$$

According to (19),

$$M_2(\omega^{\lambda(e_1)},\dots,\omega^{\lambda(e_{3n})}) = M_1(\omega^{\lambda(e_1)},\dots,\omega^{\lambda(e_{3n})}) \tag{33}$$

and so the value of (29) is equal to 3^{6n}. Substituting it into (25) we get

$$R(x_1,\dots,x_{3n}) = 3^{6n}\sum_{\lambda} P_\lambda(x_1,\dots,x_{3n}) \tag{34}$$

where the summation is taken over all $\frac{1}{4}\chi_G(4)$ maps (26) which are Tait colorings.

In order to calculate R_0, we are to substitute $x_1 = \ldots = x_{3n} = 0$ in (34). According to (27)–(28),

$$P_\lambda(0,\dots,0) = \prod_{p=1}^{3n} S(0,\lambda(e_p)). \tag{35}$$

It is easy to check that

$$S(0,0) = S(0,1) = S(0,2) = \tfrac{1}{3}. \tag{36}$$

Substituting these values into (35), we see that $P_\lambda(0,\dots,0) = 3^{-3n}$, hence, according to (34) $R_0 = \frac{1}{4}3^{3n}\chi_G(4)$ which together with (23) gives the required equality (1).

3. PROOF OF THEOREM 2

The proof is similar to the proof of Theorem 1, so we concentrate only on the new ideas.

In order to distinguish the events A_G^{even} and A_G^{odd}, we introduce one more indeterminant, J. The six possible colorings (9) will now be represented by the polynomial

$$L_1''(x_p,x_q,x_r,J) = x_q x_r^2 + x_p x_q^2 + x_p^2 x_r + J x_p^2 x_q + J x_p x_r^2 + J x_q^2 x_r \tag{37}$$

in the case of μ_1, and by the polynomial

$$L_2''(x_p,x_q,x_r,J) \;=\; x_q^2 x_r + x_p^2 x_q + x_p x_r^2 - J x_p x_q^2 - J x_p^2 x_r - J x_q x_r^2 \tag{38}$$

in the case of μ_2.

Further, the 6^{2n} choices of μ_1 and μ_2 are now represented by the 6^{2n} summands, respectively, in the (expanded) polynomials

$$M_1''(x_1, \ldots, x_{3n}, J) = \prod_{l=1}^{2n} L_1''(x_{i_l}, x_{j_l}, x_{k_l}, J) \tag{39}$$

and in

$$M_2''(x_1, \ldots, x_{3n}, J) = \prod_{l=1}^{2n} L_2''(x_{i_l}, x_{j_l}, x_{k_l}, J). \tag{40}$$

The parity of a coloring μ_1 is now represented by the parity of the exponent of J in the corresponding monomial; the parity of a coloring μ_2 is represented both by the parity of the exponent of J and by the sign of corresponding monomial .

We extend the action of the operator $\mathcal{R}$ on J in the following way:

$$\mathcal{R}J^{2k} = 1, \qquad \mathcal{R}J^{2k+1} = J. \tag{41}$$

Now we calculate the constant term of the polynomial

$$\begin{aligned} R''(x_1, \ldots, x_{3n}, J) &= \mathcal{R}(\mathcal{R}M_1''(x_1, \ldots, x_{3n}, J)\mathcal{R}M_2''(x_1, \ldots, x_{3n}, J)) \\ &= \mathcal{R}(M_1''(x_1, \ldots, x_{3n}, J)M_2''(x_1, \ldots, x_{3n}, J)). \end{aligned} \tag{42}$$

Clearly, $\mathrm{Prob}\{A_G^{\mathrm{even}}\} = \mathrm{Prob}\{A_G^{\mathrm{odd}}\} = \frac{1}{4}$ so the left-hand side of (3) is equal to

$$4\mathrm{Prob}\{A_G^{\mathrm{even}} \cap B_G\} - 4\mathrm{Prob}\{A_G^{\mathrm{odd}} \cap B_G\}. \tag{43}$$

Two colorings, μ_1 and μ_2, contribute to $R_0'' = R''(0, \ldots, 0)$ if and only if they are equivalent and have the same parity. Moreover:

- If these colorings are both even, then, on the one hand, they contribute 1 to R_0''. On the other hand, they contribute 6^{-4n} to $\mathrm{Prob}\{A_G^{\mathrm{even}} \cap B_G\}$ and 0 to $\mathrm{Prob}\{A_G^{\mathrm{odd}} \cap B_G\}$, hence their contribution to (43) is equal to $4 \cdot 6^{-4n}$.

- If these colorings are both odd, then, on the one hand, then they contribute -1 to R''_0. On the other hand, they contribute 0 to $\mathrm{Prob}\{A_G^{\mathrm{even}} \cap B_G\}$ and 6^{-4n} to $\mathrm{Prob}\{A_G^{\mathrm{odd}} \cap B_G\}$, hence their contribution to (43) is equal now equal to $-4 \cdot 6^{-4n}$.

Finally, we have:

$$\mathrm{Prob}\{B_G \mid A_G^{\mathrm{even}}\} - \mathrm{Prob}\{B_G \mid A_G^{\mathrm{odd}}\} = 4 \cdot 6^{-4n} R''_0. \tag{44}$$

Applying the Interpolation Theorem we use, in agreement with (41), values $J = 1$ and $J = -1$:

$$\begin{aligned} &R''(x_1, \ldots, x_{3n}, J) = \\ &\textstyle\sum_\lambda R''(\omega^{\lambda(v_1)}, \ldots, \omega^{\lambda(v_{3n})}, 1) P_\lambda(x_1, \ldots, x_{3n}) \frac{1+J}{2} + \\ &\textstyle\sum_\lambda R''(\omega^{\lambda(v_1)}, \ldots, \omega^{\lambda(v_{3n})}, -1) P_\lambda(x_1, \ldots, x_{3n}) \frac{1-J}{2}. \end{aligned} \tag{45}$$

It is easy to see that

$$M''_1(x_1, \ldots, x_{3n}, -1) = M_1(x_1, \ldots, x_{3n}), \tag{46}$$

$$M''_2(x_1, \ldots, x_{3n}, 1) = M_2(x_1, \ldots, x_{3n}), \tag{47}$$

so $R''(\omega^{\lambda(e_1)}, \ldots, \omega^{\lambda(e_{3n})}, \pm 1)$ is equal to 0 as soon as the map λ is not a Tait coloring of graph G and hence the summations in (45) again can be restricted to λ's ranging over Tait colorings of graph G.

Let us check that the sign in (32) is in fact always "+". Indeed, the sign in (31) depends on whether the map λ colors the triangle $\langle e_p, e_q, e_r \rangle$ positively or negatively. It turns out that the parity of the number of positively or negatively colored triangles coincides with the parity of n. This fact should be well-known but the author did not find a proper reference. To keep the paper self-contained, a proof is given.

We proceed by induction. The case $n = 1$ is trivial. If not all triangles are colored in the same way, then there are two neighboring triangles colored differently. We can remove the common edge and glue together edges colored in the same color (it is easy to show that no loop would arise). The resulting maximal planar graph has $3n - 3$ edges and by the induction hypothesis the parity of positive and negative triangles coincides with the parity of the number of edges. We have eliminated one positive and one negative triangle, so the same coincidence of the parities takes place for the original graph as well.

Now suppose that all triangles are colored in the same way, say, positively. It is easy to see that in such a case the degree of each vertex is divisible by 3. On the other hand, it is well-known that a planar graph has a vertex of degree not greater than 5. Thus our graph has a vertex of degree 3. Removing it we get again a maximal planar graph with $3n-3$ edges satisfying the induction hypothesis. Now we have eliminated three positive triangles and got a new one colored negatively, again keeping the required coincidence of the parities.

Thus, we refined (32) to $M_1(\omega^{\lambda(e_1)},\dots,\omega^{\lambda(e_{3n})}) = 3^{3n}$ and according to (33) the same value has $M_2(\omega^{\lambda(e_1)},\dots,\omega^{\lambda(e_{3n})})$.

Let us now determine what are the value of $M_1''(\omega^{\lambda(v_1)},\dots,\omega^{\lambda(v_{3n})},1)$ and the value of $M_2''(\omega^{\lambda(v_1)},\dots,\omega^{\lambda(v_{3n})},-1)$. If $\langle e_p, e_q, e_r\rangle \in T$, then all 3 colors 0, 1, and 2, are used to color the edges e_p, e_q, e_r and hence

$$L_1''(\lambda(e_p),\lambda(e_q),\lambda(e_r),1) = L_1''(1,\omega,\omega^2,1) = -3, \tag{48}$$

$$L_2''(\lambda(e_p),\lambda(e_q),\lambda(e_r),-1) = L_2''(1,\omega,\omega^2,-1) = -3 \tag{49}$$

and hence

$$M_1''(\omega^{\lambda(e_1)},\dots,\omega^{\lambda(e_{3n})},1) = M_2''(\omega^{\lambda(e_1)},\dots,\omega^{\lambda(e_{3n})},-1) = 3^{2n}. \tag{50}$$

Respectively,

$$R_1''(\omega^{\lambda(e_1)},\dots,\omega^{\lambda(e_{3n})},\pm 1) = 3^{5n}. \tag{51}$$

Substituting the latter value into (45) we get

$$R''(x_1,\dots,x_{3n},J) = 3^{5n}\sum_{\lambda} P_\lambda(x_1,\dots,x_{3n}) \tag{52}$$

where the summation is taken over all $\frac{1}{4}\chi_G(4)$ maps (26) which are Tait colorings.

It remains to substitute here $x_1 = \dots = x_{3n} = 0$ and $J = 0$ and get that $R_0'' = \frac{1}{4}3^{2n}\chi_G(4)$ which together with (44) gives the required equality (3).

References

[1] N. Alon and M. Tarsi, A note on graph colorings and graph polynomials, J Combin Theory B 70 (1997), 197–201.

[2] K. Appel and W. Haken, Every planar map is four colorable. Part I. Discharging, Illinois J Mathematics, 21 (1977), 429–490.

[3] K. Appel, W. Haken and J. Koch, Every planar map is four colorable. Part II. Reducibility, Illinois J Mathematics, 21 (1977), 491–567.

[4] K. Appel and W. Haken, Every planar map is four colorable, AMS Contemporary Mathematics, 98, Amer. Math. Soc., Providence, RI, 1989.

[5] Yves Colin de Verdiere, On a new graph invariant and a criterion for planarity, AMS Contemporary Mathematics, 147, Amer. Math. Soc., Providence, RI, 1993.

[6] T. Hoffman, J. Mitchem and E. Schmeichel, On edge-coloring graphs, Ars Combinatoria 33 (1992), 119–128.

[7] T. R. Jensen and B. Toft, Graph Coloring Problems, Wiley & Sons, New York, 1995.

[8] P. C. Kainen, Is the four color theorem true?, Geombinatorics 3 (1993), 41–56.

[9] L. Kauffman, Map coloring and the vector cross product, J Combin Theory B 48 (1990), 145–154.

[10] L. H. Kauffman, Combinatorial recoupling theory and 3-manifold invariants, Low-dimensional topology and quantum field theory (Cambridge, 1992), (NATO Adv. Sci. Inst. Ser. B Phys., 315), Plenum, New York, 1993, pp. 1–17.

[11] L. H. Kauffman, Spin networks, topology and discrete physics, Adv Ser Math Phys, 17, World Sci. Publishing, River Edge, NJ, 1994, 234–274.

[12] L. H. Kauffman and H. Saleur, An algebraic approach to the planar coloring problem, Communications in Mathematical Physics 152 (1993), 565–590.

[13] Yu. Matiyasevich, Problem 21 (in Russian), Combinatorial and Asymptotical Analysis, issue 2, G. P. Egorychev (Editor), Krasnoyarskii State University, Krasnoyarsk, 1977, 178–179; MR 58 #27504, ZBl 431.00010.

[14] Yu. Matiyasevich, A Polynomial related to colourings of triangulation of sphere, URL: `http://logic.pdmi.ras.ru/~yumat/Journal/Triangular/triang.htm`, mirrored at `http://www.informatik.uni-stuttgart.de/ifi/ti/personen/Matiyasevich/Journal/Triangular/triang.htm`.

[15] Yu. Matiyasevich, Some arithmetical restatements of the four color conjecture, Theoretical Computer Science, 257 (2001), 167–183; a version of this paper is available via URL: `http://logic.pdmi.ras.ru/~yumat/Journal/jcontord.htm`, mirrored at `http://www.informatik.uni-stuttgart.de/ifi/ti/personen/Matiyasevich/Journal/jcontord.htm`.

[16] Yu. V. Matiyasevich, Some algebraic methods for calculation of the number of colorings of a graph (in Russian), Zapiski Nauchnykh Seminarov POMI, 283 (2001), 193–205; available via URL `http://www.pdmi.ras.ru/znsl/2001/v283.html`.

[17] Yu. V. Matiyasevich, A probabilistic equivalent to the Four Color Conjecture (in Russian), Teoriya Veroyatnostei i ee Primeneniya, 48(2003), 411–416.

[18] O. Ore, The four-color problem, New York-London, Academic Press, 1967.

[19] N. Robertson, D. Sanders, P. Seymour and R. Thomas, The Four-colour Theorem, J Combin Theory B 70 (1997), 2-44.

[20] R. Thomas, An update on the Four-Colour Theorem, Notices of the AMS, 45 (1998), 848–859; available via URLs: `http://www.ams.org/noticies/199807/thomas.{ps,pdf}`.

[21] T. L. Saaty, Thirteen colorful variations on Guthrie's four-color conjecture, American Mathematical Monthly, 79 (1972), 2–43.

[22] T. L. Saaty and P. C. Kainen, The four-color problem, Düsseldorf etc., McGraw-Hill, 1977.

Received (received)

Revised (revised)

Accepted (accepted)