THE FIRST BETTI NUMBER OF A COMPACT HYPERBOLIC MANIFOLD AND THE HODGE CONJECTURE FOR COMPACT QUOTIENTS OF THE COMPLEX n BALL

JOHN MILLSON

In 1976 in [3], I showed that for every n the standard arithmetic examples of compact hyperbolic n dimensional manifolds M had nonzero first Betti number by constructing nonseparating totally geodesic hypersurfaces inside them. Now, 36 years later, Nicolas Bergeron, Colette Moeglin and I can show, [1], that if $n \geq 4$ these totally-geodesic hypersurfaces *span* the next-to-top homology of M. There are analogous results for the standard arithmetic quotients $M = \Gamma \backslash X$ of the symmetric spaces X associated to the orthogonal groups $\mathrm{O}(p,q)$.

In very recent work (in progess) Bergeron, Moeglin and I have applied analogous techniques to the standard arithmetic quotients $M = \Gamma \backslash X$ of the symmetric spaces X associated to the unitary groups $\mathrm{U}(p,q)$. For the case in which the unitary group is $\mathrm{U}(n,1)$ the associated symmetric space X is the complex n ball D^n (complex hyperbolic space). We prove that, under the assumption $k < n/3$, the intersection $H^{2k}(M,\mathbb{Q}) \cap H^{k,k}(M,\mathbb{C})$ is spanned by the images in M of totally geodesic $n-k$-balls $D^{n-k} \subset D^n$. Since these image cycles are carried by projective subvarieties, *this proves the Hodge conjecture* in these degrees for the standard arithmetic quotients of the ball.

I will avoid technical details and explain the overall principles: the simple geometric idea behind my 1976 paper and the geometry behind the construction in [2] which uses the Weil representation to construct closed differential forms on the above manifolds which are Poincaré dual to the geodesic cycles.

References

[1] N. Bergeron, J. Millson and C. Moeglin , *Hodge type theorems for arithmetic manifolds asociated to orthogonal groups* , preprint, arXiv:1110.3049.

[2] S.Kudla and J. Millson , *Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables*, Inst. Hautes Études Sci. **71** (1990), 121-172.

[3] J. Millson , *On the first Betti number of a constant negatively curved manifold* , Annals of Math. **104** (1976), 235-247.