Vertex Disjoint Paths in Upward Planar Graphs

Saeed Akhoondian Amiri, Ali Golshani, Stephan Kreutzer, Sebastian Siebertz

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Outline

1. UpPlan-VDPP is NP-complete
2. Linear Time FPT Algorithm for UpPlan-VDPP

Introduction

- The k-vertex disjoint paths problem (VDPP):
  Given: A graph G and terminal pairs $(s_i, t_i), i \in [k]$.
  Question: Are there k pairwise internally vertex disjoint paths linking s_i to t_i.

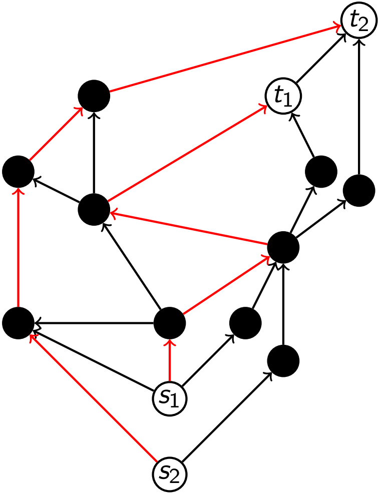
- Undirected graphs: NP-complete [Lynch; 1975], but FPT in k [Robertson, Seymour; 1995].
- Directed graphs: NP-complete for $k = 2$[Fortune, Hopcroft, Wyllie; 1980].
- Directed Planar graphs: FPT [Cygan et al; 2013].

Introduction

Two research directions:

1. Very dense digraphs: In semi-complete digraphs is in $n^{O(k^3)}$ [Chudnovsky, Scott, Seymour; 2012].
2. Sparse digraphs: $2^{2^{O(k^2)}} \cdot P(n)$ in planar digraphs.

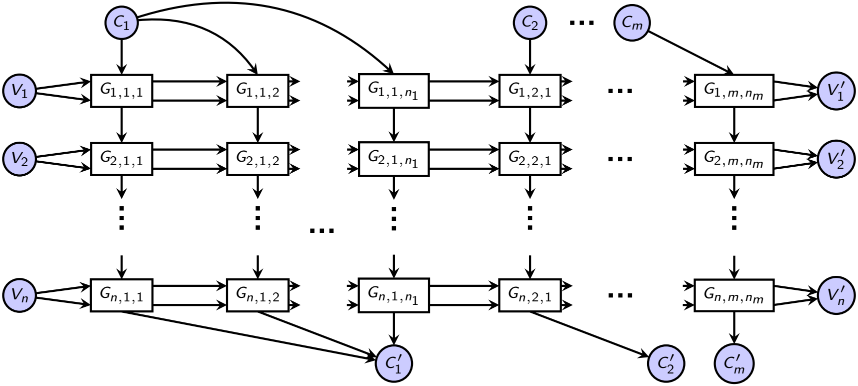
1. Upward planar graphs: A planar digraph which can be drawn in a plane such that each edge is monotonically increasing in y axis.
2. Upward Planar Testing (UPT):NP-complete.
3. UPT is in P if there is a single sink [Garg, Tamassia; 1995].
4. UpPlan-VDPP: VDPP restricted on a graph G with its given upward planar drawing


t_2
t_1
s_1
s_2

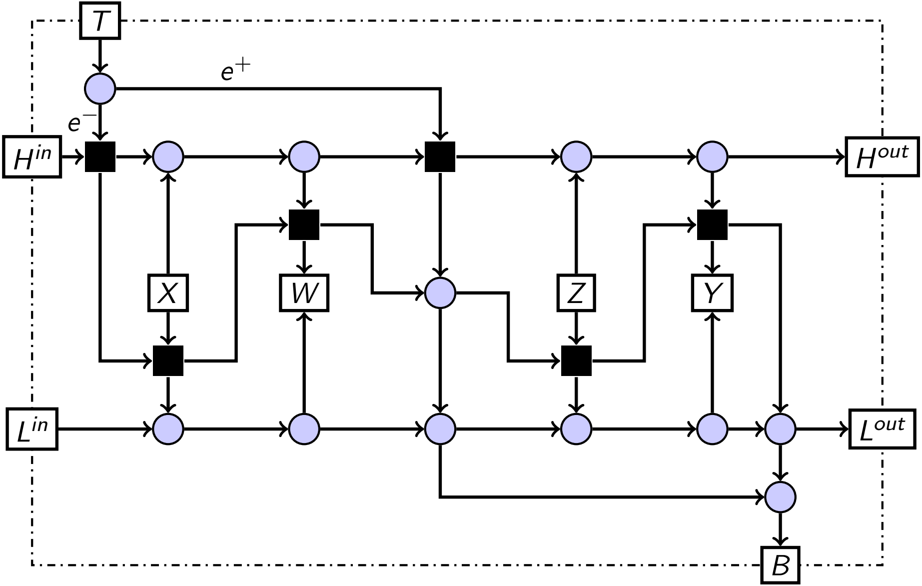
Theorem 1

UpPlan-VDPP *is NP-complete on upward planar graphs.*

Overall View

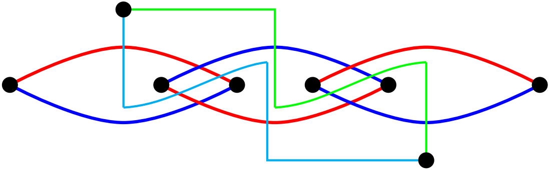


The Crossing Gadget

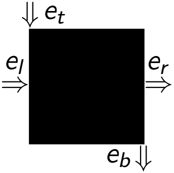


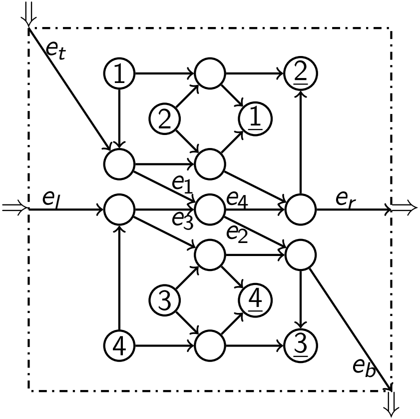
General Idea of Gadgets

1. Subdividing one $V_i \to V'_i$ path to some smaller paths.
2. Forcing routings:



Routing Gadget





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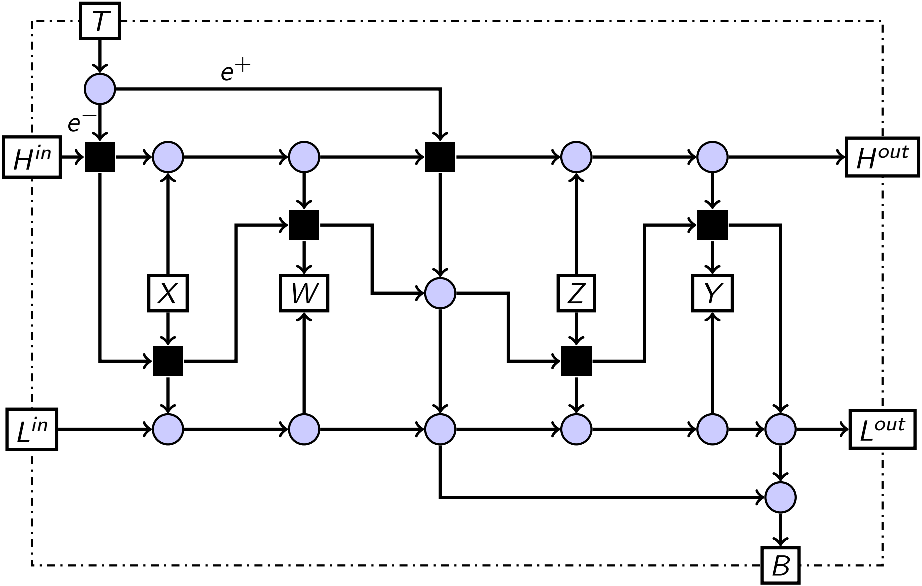

Theorem 2

There is an algorithm for k-UpPlan-VDPP which runs in time $O(k! \cdot n)$.

To The Right Relation

Let P and Q be two internally vertex disjoint paths in an upward planar graph G with a given upward planar drawing D.

1. A point p is to the right of a point q if $p.y = q.y, q.x < p.x$.
2. $Q \prec P$ if there is a point $p \in P$ which is to the right of some point $q \in Q$.
3. $\prec^*$ is the transitive closure of $\prec$.

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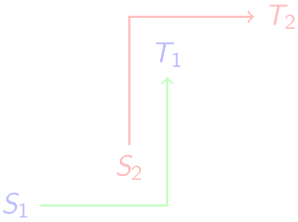

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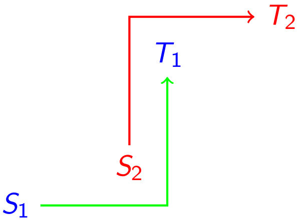
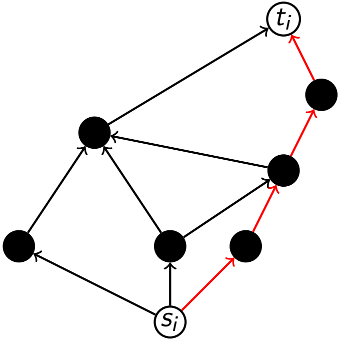
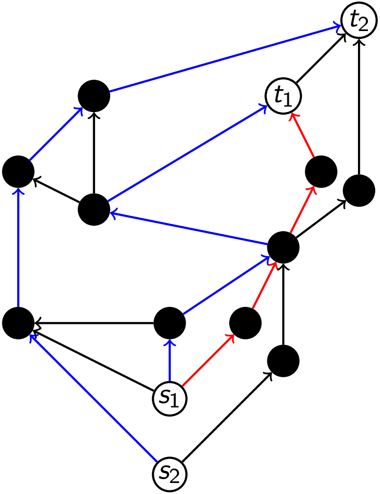


Rightmost Path



General Idea



Algorithm

Let $ST \leftarrow \{(s_i, t_i) \mid i \in [k]\}$.

function UpPlan-VDPP(ST, G, k)
    $Candidates \leftarrow FindCandidates\{C_1, \ldots, C_k\}$
    for $C_i \in Candidates$ **do**
        if UpPlan-VDPP $(ST - (s_i, t_i), G - C_i, k - 1)$ **then**
            return TRUE
    return FALSE

$\prec^*_{\mathcal{P}}$ Is a Partial Order

Let $\mathcal{P}$ be a set of pairwise disjoint paths in upward planar drawing of G.

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Find Candidates in Linear Time

Given two vertices $s, t \in V(G)$ we compute the right-most s-t-path in G, if such a path exists.

(i) DFS starts at s to compute all reachable vertices from s : $Reach(s)$.

(ii) Inverse DFS starts at t to compute all reachable vertices from t : $Reach^{-1}(t)$.

(iii) Select rightmost vertex inductively starting at s from $Reach(s) \cap Reach^{-1}(t)$ untill reach t.

(iv) First two steps are in $O(n)$. The third step takes at most $O(\sum_{i \in [n]} deg(v_i)) = O(n)$.

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Conclusion and Open Problems

1. VDPP on semi-complete digraphs? EDPP is in FPT[Fradkin, Seymour; 2010]; Best known for VDPP is $n^{O(k^3)}$ [Chudnovsky, Scott, Seymour; 2012].
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