Testing low-degree trigonometric polynomials

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Joint work with Klaus Meer

(work supported by DFG, GZ:ME 1424/7-2)

Real number model developed by Blum, Shub and Smale in 1989.

The BSS model focusses on algebraic algorithms.

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Operations:	$+, -, *, :$
Test:	$x \geq 0?$

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Alternative approach: recursive analysis

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Theorem

The problem whether a system of quadratic polynomials has a real common zero (QPS) is $NP_{\mathbb{R}}$-complete

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Example

- $NP_{\mathbb{R}}$ is decidable in single exponential time (Grigoriev & Vorobjov, Renegar, Heintz et al, ...)
- Toda's theorem (Basu & Zell)
- $PCP_{\mathbb{R}}$ theorem (Baartse & Meer 2013)

Theorem (ALMSS 1992, Algebraic proof)

Every $L \in NP$ has a probabilistic verifier that uses $O(\log(n))$ random bits to make $O(1)$ queries to the certificate such that

- *for all $x \in L$ there is a certificate that is accepted and*
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Can the $\mathrm{PCP}_{\mathbb{R}}$ theorem also be proven along the lines of ALMSS?

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Can the $PCP_{\mathbb{R}}$ theorem also be proven along the lines of ALMSS?

To what extend can the coding techniques used by ALMSS be applied in the BSS model? Are there alternatives?

Essential in ALMSS:

Given $g : F^k \to F$ (finite field),

is there a polynomial P with low degree

such that for most $x \in F^k$,

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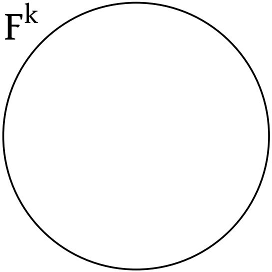

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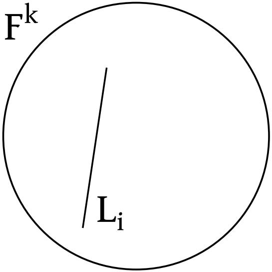

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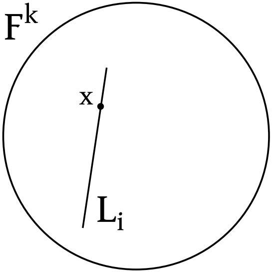

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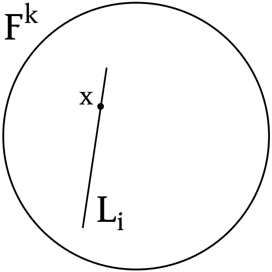
Line test:

Does g agree with p_{L_i} on x?



Theorem (Rubinfeld, Sudan)

If there exist univariate polynomials $p_{L_1}, \ldots, p_{L_m}$ with low degree such that $Pr[p_{L_i}$ agrees with g on $x] \geq 1 - \delta$, then there exists a polynomial $P : F^k \rightarrow F$ with low degree that agrees with g on all but a 2δ fraction of arguments.

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Theorem (Friedl, Hatsagi, Shen)

Let $A \subseteq \mathbb{R}$ be finite. Given $g : A^k \to \mathbb{R}$, performing $O(k)$ line tests establishes that g is close to a low-degree polynomial.

Example:

Let F be the finite field with 17 elements. We look at the differences between considering the "same" function as function from F^2 to F and as function from $\{0, \ldots, 16\}^2 \to \mathbb{R}$.

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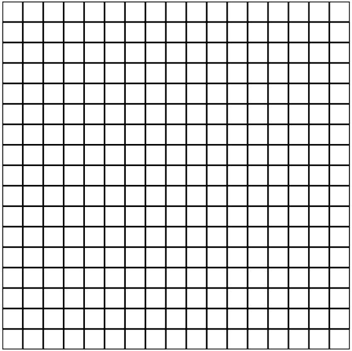

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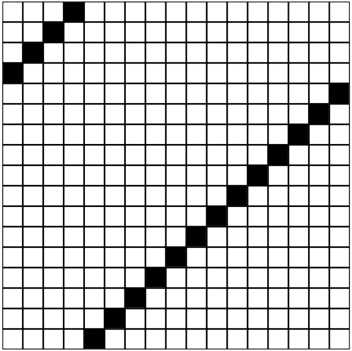

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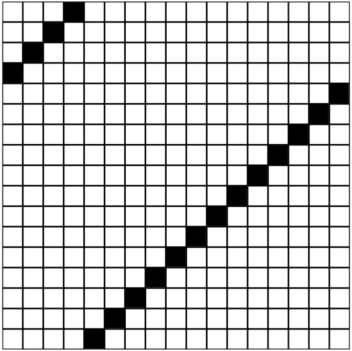

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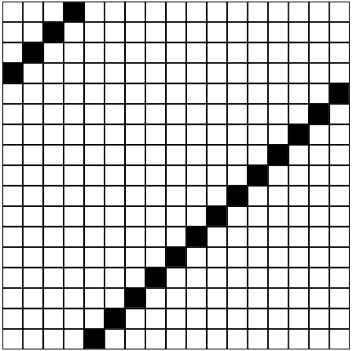

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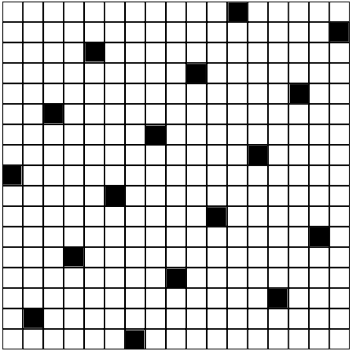

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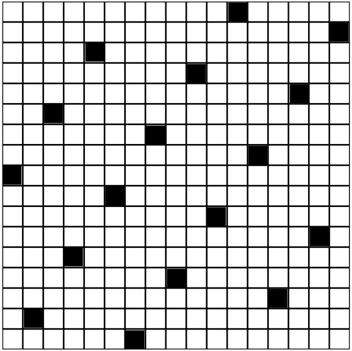

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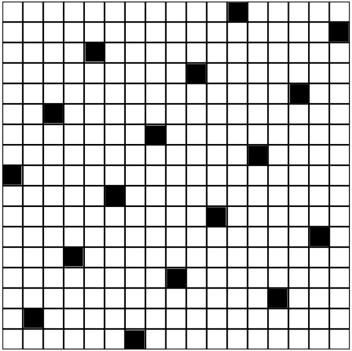

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$$g_L(t) = \begin{cases} (2t+1)(3t+1)^2 & t \leq 5 \\ (2t+1)(3t-16)^2 & 5 < t \leq 7 \\ (2t-16)(3t-16)^2 & 7 < t \leq 10 \\ (2t-16)(3t-33)^2 & 10 < t \end{cases}$$

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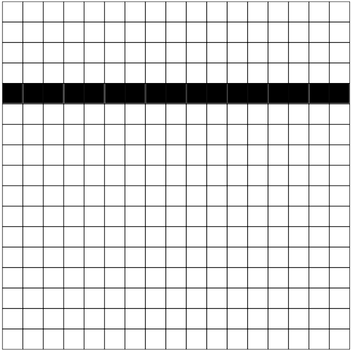

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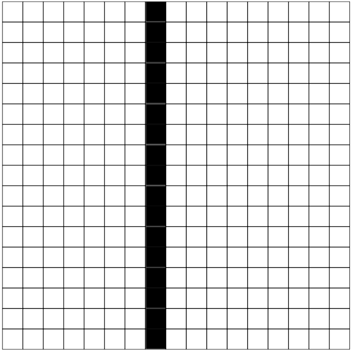

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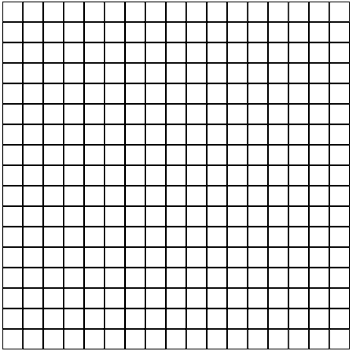

$$
\begin{aligned}
g : \{0,1,\ldots,16\}^k &\to \mathbb{R} \\
x &\mapsto \begin{cases} 0 & x_1 < 8 \\ 1 & x_1 \geq 8 \end{cases}
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Use trigonometric polynomials with appropriate period.

The difference between the multiplication and addition in F and the multiplication and addition in $\mathbb{R}$ becomes irrelevant.

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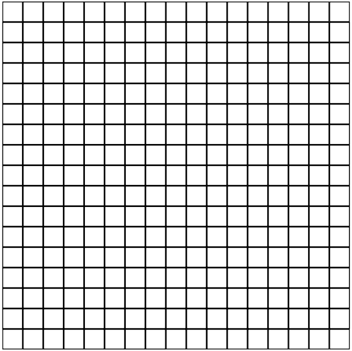

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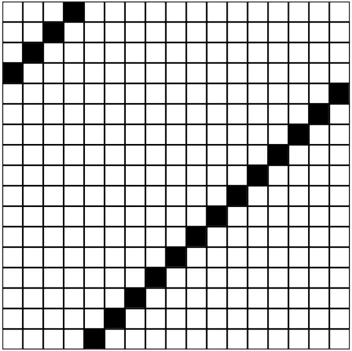

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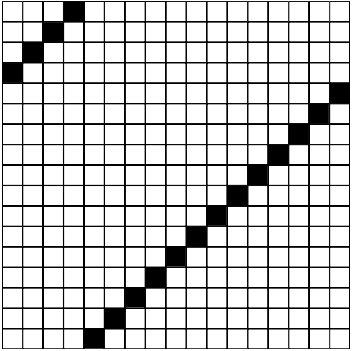

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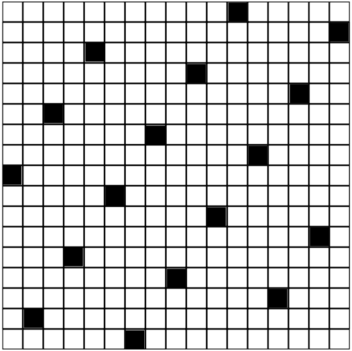

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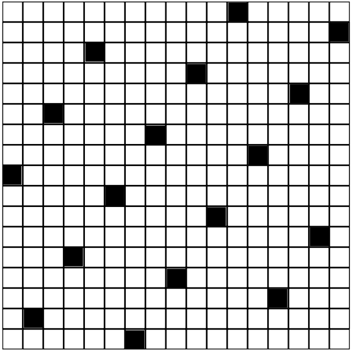

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Outline

- The suitable lines connect F^k well. Let $G = (V, E)$ be the graph with $V = F^k$ and $(x, y) \in E$ if there is a suitable line connecting x and y. The graph G is an expander with expansion parameter $\lambda(G)$ close to 1.

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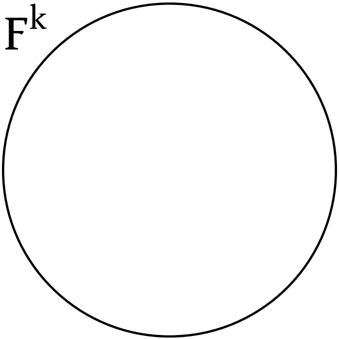
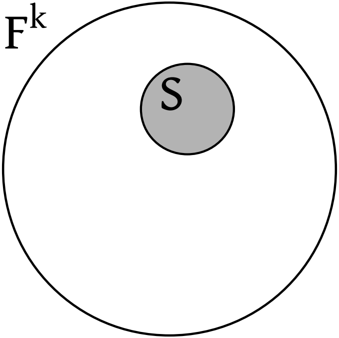
- If $f : F^k \rightarrow \mathbb{R}$ is ϵ-close to a polynomial, then the probability that the line test rejects is about ϵ, but only if ϵ is small.


F^k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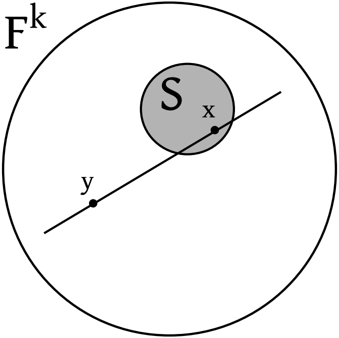
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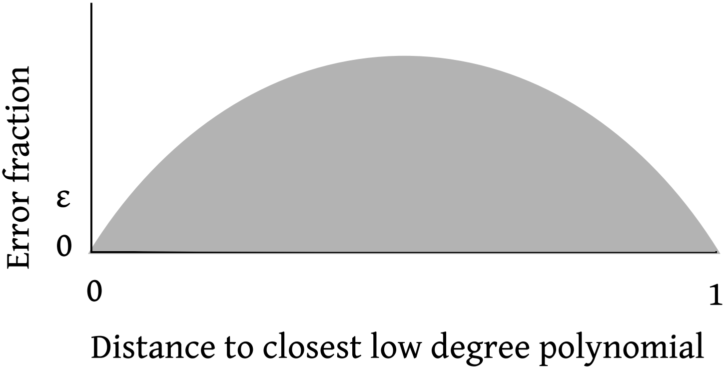
- $f_1 = f$ and f_n is a polynomial
- f_i is very close to f_{i+1}
- for every f_i the probability that the line test rejects is at most two times the probability that the line test rejects f.


Error fraction
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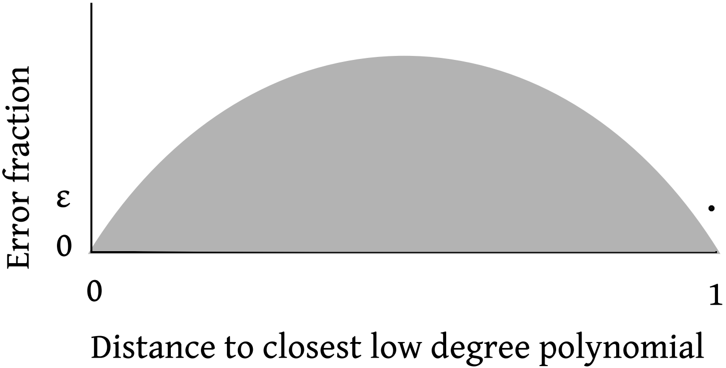
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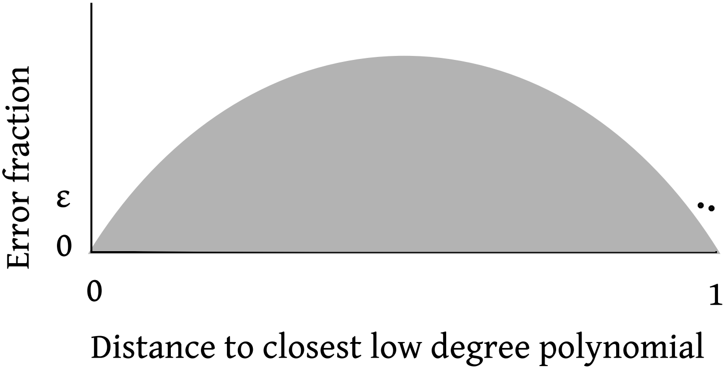
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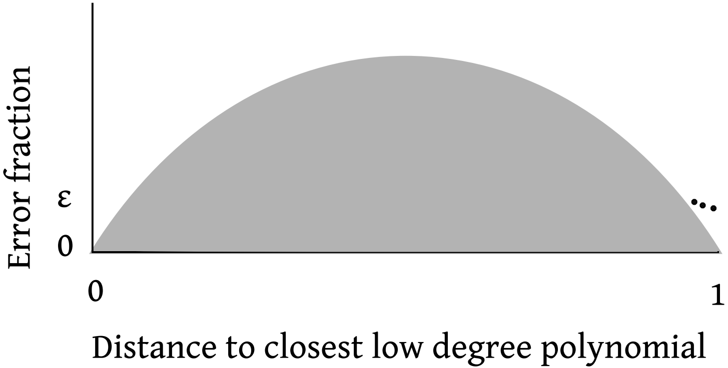
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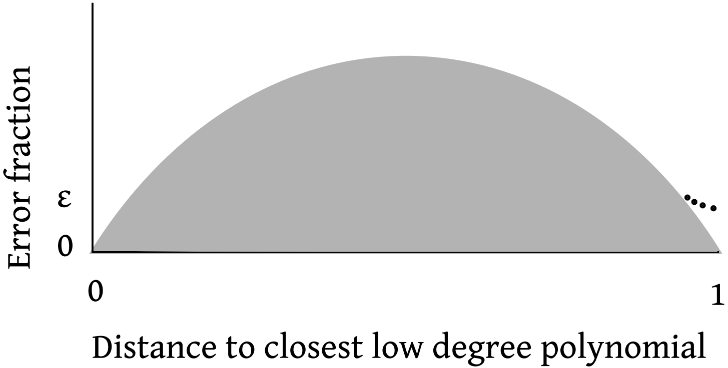
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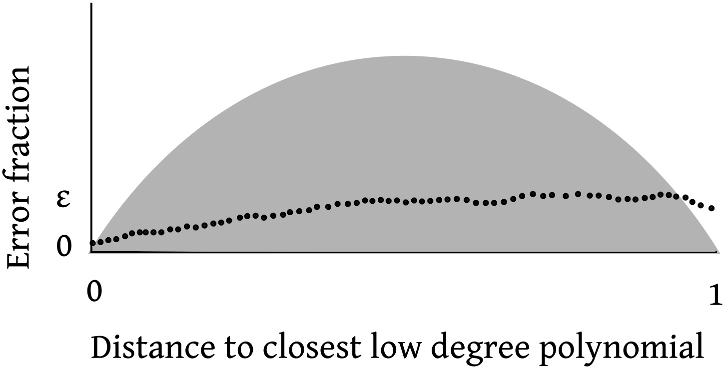
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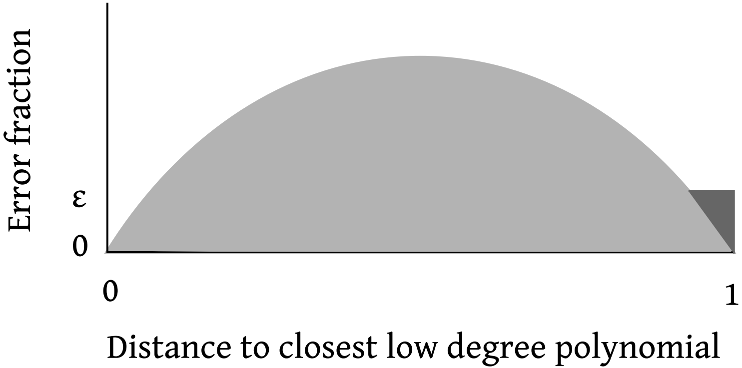
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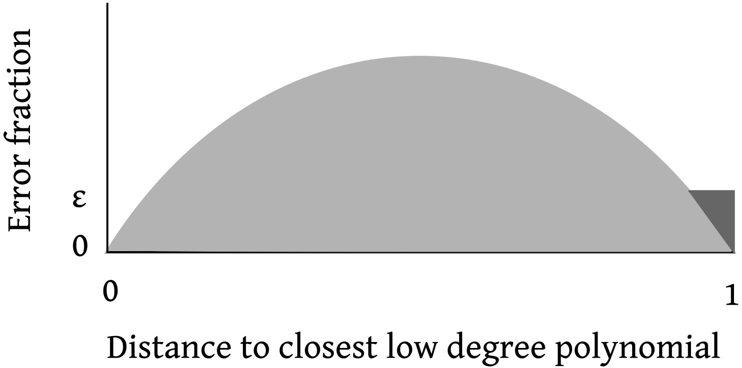

Theorem

If the line test finds an error with probability less than ϵ, then $g : F^k \to \mathbb{R}$ is close to a low degree trigonometric polynomial.

Theorem (All details)

Let F be a finite field with q elements where q is a prime number. Let $d \in \mathbb{N}$, $h := 10^{15}$ and $k > 3h$ such that $q \geq 10^4(2hkd+1)^3$. There exists a probabilistic verification algorithm in the BSS-model of computation over the reals with the following properties:

- *The verifier gets as input a function value table of a multivariate function $f : F^k \to \mathbb{R}$ and a proof string π consisting of at most q^{2k} segments (blocks). Each segment consists of $2hkd + k + 1$ real components. Such a segment is seen as specifying a degree hkd polynomial by its coefficients and claiming that this is the restriction of f to the corresponding line.*
  The verifier first uniformly generates $O(k \cdot \log q)$ random bits; next, it uses the random bits to determine a point $x \in F^k$ together with one segment in the proof string it wants to read. Finally, using the values of $f(x)$ and those of the chosen segment it performs a line test. According to the outcome of the test the verifier either accepts or rejects the input.
  The running time of the verifier is polynomially bounded in the quantity $k \cdot \log q$, i.e., polylogarithmic in the input size $O(k \cdot q^{2k})$.
- *For every function value table representing a trigonometric max-degree d polynomial there exists a proof string such that the verifier accepts with probability 1.*
- *For any $0 < \epsilon < 10^{-19}$ and for every function value table whose distance to a closest max-degree $2hkd$ polynomial is at least 2ϵ the probability that the verifier rejects is at least ϵ, no matter what proof string is given.*

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- What remains to be done in the proof of the $PCP_{\mathbb{R}}$ theorem?
  - segmentation procedure
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- Is there a way to test algebraic polynomials while querying only a constant number of segments?
- Can the number of queries in the $PCP_{\mathbb{R}}$ theorem be reduced to a small number?