Notions of metric dimension of corona products: combinatorial and computational results

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Overview

- Graph products and graph parameters
- Combinatorial results
- Complexity results
- Conclusions

Graph products and graph parameters

- A recurring theme in graph combinatorics:
- Bound parameters of a product graph by parameters of its constituents!
- The results (proofs) are often of a computational nature,
- but of little practical algorithmic use.

Here: Good (exact) bounds could yield computational insights.

Corona product (Frucht, Harary 1970) a lesser known asymmetric graph product

G, H: graphs of order n_G, n_H.

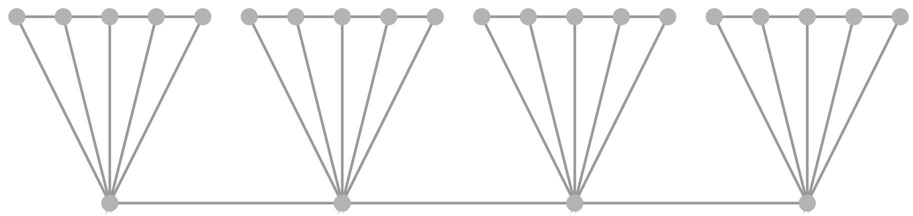
The *corona product (graph)* $G \odot H$ is obtained by

— taking one copy of G and n_G copies of H and

— introducing an edge between

*** each vertex from the i^{th} copy of H and

*** the i^{th} vertex of G.

$P_4 \odot P_5$: 

An abstract detour (with applications): Let (X, d) be a metric space.
The *diameter* of a point set $S \subseteq X$ is $\mathrm{diam}(S) := \sup\{d(x, y) : x, y \in S\}$.
A point $z \in X$ is said to *distinguish* two points x and y of X if $d(z, x) \neq d(z, y)$.
A *generator* of (X, d) is a set $S \subseteq X$ such that any pair of points of X is distinguished by some point of S.
If the only distances are $0, 1, \ldots, k$, then x, y are *neighbors* if $d(x, y) = 1$.
A *local generator* of (X, d) is a set $S \subseteq X$ such that any pair of neighbored points of X is distinguished by some point of S.

A possible application: a traveler lost in some metric space can locate himself by knowing his distance to all generator points.
$\leadsto$ Navigation applications of "locating sets".
Local generators help with local disorientation.

Graphs, metrices and derived parameters

Let $G = (V, E)$ be a connected graph.
$d_G(x, y)$: the length of a shortest path between vertices u and v.
Clearly, (V, d_G) is a metric space. The *diameter* of a graph is thus understood.
$S \subseteq V$ is a *metric generator* for G if it is a generator of (V, d_G).
A minimum metric generator is known as a *metric basis*, and its cardinality is the *metric dimension* of G, denoted by $\mathrm{dim}(G)$.
see: Slater 1975; Harary, Melter 1976; for applications: Johnson 1993/1998
Derived notions: *local metric generator*, giving rise to the *local metric dimension* of G, denoted by $\mathrm{dim}_l(G)$; see Okamoto 2010.

Alternative myopic metrization of V: $d_{G,2}(x, y) = \min\{d_G(x, y), 2\}$. 
Can only differentiate neighbors from non-neighbors.
Derived notions: *(local) adjacency generator*, leading to the *(local) adjacency dimension* of G, denoted by $\mathrm{dim}_A(G)$ or $\mathrm{dim}_{A,l}(G)$; see Saputro 2013; very much related to that of a 1-locating dominating set Charon, Hudry, Lobstein 2003.

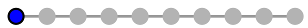
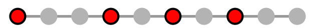
Simple facts

By definition, the following inequalities hold for any graph G:

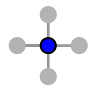
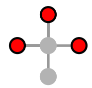
- $\dim(G) \leq \dim_A(G)$; if $\text{diam}(G) \leq 2$, then $\dim(G) = \dim_A(G)$;
- $\dim_l(G) \leq \dim_{A,l}(G)$;
- $\dim_l(G) \leq \dim(G)$;
- $\dim_{A,l}(G) \leq \dim_A(G)$;
- $\gamma(G) \leq \dim_A(G) + 1$ (if S is an adjacency generator, then at most one vertex is not dominated by S);
- $\dim_{A,l}(G) \leq \beta(G)$ (each vertex cover is a local adjacency generator).

Concrete facts for paths and stars

1. $\dim_l(P_n) = \dim(P_n) = 1 \leq \left\lfloor \frac{n}{4} \right\rfloor \leq \dim_{A,l}(P_n) \leq \left\lceil \frac{n}{4} \right\rceil \leq \left\lfloor \frac{2n+2}{5} \right\rfloor = \dim_A(P_n)$, $n \geq 7$;

   $n = 10$:  metric versus adjacency 

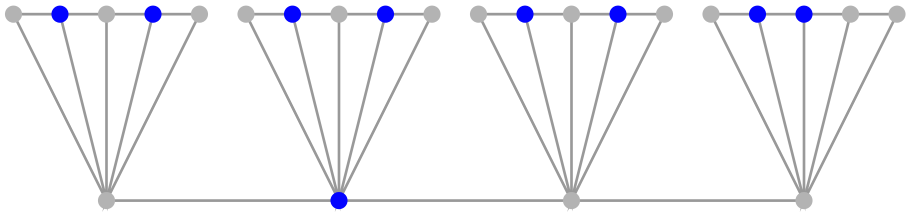
2. $\dim_l(K_{1,n}) = \dim_{A,l}(K_{1,n}) = 1 \leq n - 1 = \dim(K_{1,n}) = \dim_A(K_{1,n})$, $n \geq 2$;

   $n = 4$:  local versus global 

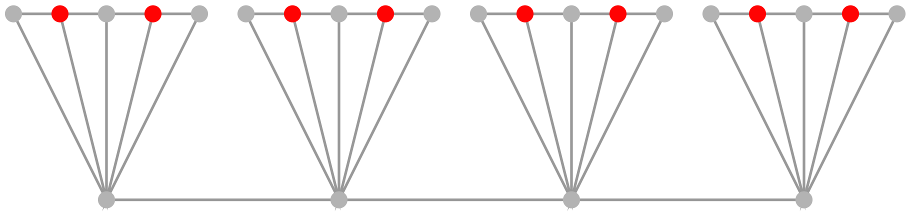
3. $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil \leq \left\lfloor \frac{2n+2}{5} \right\rfloor = \dim_A(P_n)$, $n \geq 7$;
4. $\left\lfloor \frac{n}{4} \right\rfloor \leq \dim_{A,l}(P_n) \leq \left\lceil \frac{n}{4} \right\rceil \leq \left\lfloor \frac{n}{2} \right\rfloor = \beta(P_n)$, $n \geq 2$.

A small example for the corona product

The blue vertices forms an adjacency basis for $P_4 \odot P_5$ but not a dominating set.



The metric basis is smaller in this example:



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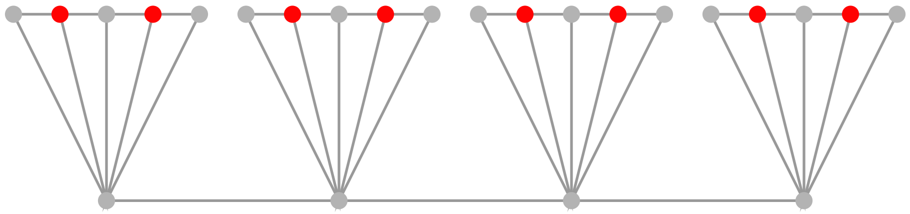
The main combinatorial result

Theorem 1 *For any connected graph G of order $n_G \geq 2$ and for any non-trivial graph H,* $\dim(G \odot H) = n_G \cdot \dim_A(H)$.

Hence, $\dim(P_4 \odot P_5) = 4 \cdot \dim_A(P_5) = 4 \cdot \left\lfloor \frac{2 \cdot 5 + 2}{5} \right\rfloor = 4 \cdot 2 = 8$:

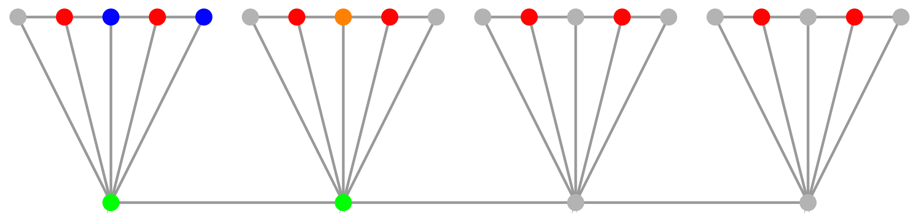


If G is a connected graph with $n_G \geq 2$ and H is non-trivial, $\dim(G \odot H) = n_G \cdot \dim_A(H)$.

Claim 1: If S is an adjacency generator of H, then n_G copies of S form a metric generator of $G \odot H$.

– $\{v\} \times H_v \subseteq G \odot H$ has $\text{diam}(\{v\} \times H_v) = 2$. $\rightsquigarrow \forall x, y \in H_v \; \exists z \in S_v \; d(z, x) \neq d(z, y)$.
– Consider $u, v \in G$, $u \neq v$. Pick $z \in S_v$ so that $d(z, u) > d(z, v)$.
– Let $x \in H_v$ and $y \in H_u$. Then, for $z \in S_v$, $d(z, x) \neq d(z, y)$.
– For $v \in G$ and $x \in H_v$, choose $z \in S_u$ s.t. $d(z, x) > d(z, v)$.
– For $v \in G$ and $x \in H_u$, choose $z \in S_v$ s.t. $d(z, x) > d(z, v)$.
The last three items are due to the following Fact: G forms a separator in $G \odot H$.

This "bottleneck argument" also yields: Claim 2: The restriction of any metric generator of $G \odot H$ to some copy H_v is an adjacency generator of H_v.



More combinatorial results: Going into some technical details

Theorem 2 *Let G be a connected graph of order $n_G \geq 2$ and let H be a non-trivial graph. If there exists an adjacency basis for H which is also a dominating set and if, for any adjacency basis S for H, there exists some $v \in V(H) - S$ such that $S \subseteq N_H(v)$, then*

$$\dim_A(G \odot H) = n_G \cdot \dim_A(H) + \gamma(G).$$

Corollary 3 *Let $r \geq 2$. Let G be a connected graph of order $n_G \geq 2$. Then,*

$$\dim_A(G \odot K_r) = n_G(r-1) + \gamma(G).$$

More combinatorial results: Assume G is connected and H is non-trivial.

Theorem 4 *Let $n_G \geq 2$. The following statements are equivalent:*

1. *There exists an adjacency basis S for H, which is also a dominating set, such that for every $v \in V(H) - S$ it is satisfied that $S \not\subseteq N_H(v)$.*
2. $\dim_A(G \odot H) = n_G \cdot \dim_A(H)$.
3. $\dim_A(G \odot H) = \dim(G \odot H)$.

Theorem 5 *Let $n \geq 3$. The following statements are equivalent:*

1. *No adjacency basis for H is a dominating set.*
2. $\dim_A(G \odot H) = n_G \cdot \dim_A(H) + n_G - 1$.
3. $\dim_A(G \odot H) = \dim(G \odot H) + n_G - 1$.

More combinatorial results: Going local

Theorem 6 *For any connected graph G of order $n_G \geq 2$ and any non-trivial graph H,* $\dim_l(G \odot H) = n_G \cdot \dim_{A,l}(H)$.

Under the same conditions, we can obtain:

Theorem 7 *The following assertions are equivalent.*

1. *There exists a local adjacency basis S for H such that $\forall v \in V(H) - S : S \not\subseteq N_H(v)$.*
2. $\dim_{A,l}(G \odot H) = n_G \cdot \dim_{A,l}(H)$.
3. $\dim_l(G \odot H) = \dim_{A,l}(G \odot H)$.

Theorem 8 *The following assertions are equivalent.*

1. *For any local adjacency basis S for H, there exists some $v \in V(H) - S$ with $S \subseteq N_H(v)$.*
2. $\dim_{A,l}(G \odot H) = n_G \cdot \dim_{A,l}(H) + \gamma(G)$.
3. $\dim_l(G \odot H) = \dim_{A,l}(G \odot H) - \gamma(G)$.

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Decidability problems

DIM: Given: G and k, decide if $\mathtt{dim}(G) \leq k$ or not.
LOCDIM: Given G and k, decide if $\mathtt{dim}_l(G) \leq k$ or not.
ADJDIM: Given G and k, decide if $\mathtt{dim}_A(G) \leq k$ or not.
LOCADJDIM: Given G and k, decide if $\mathtt{dim}_{A,l}(G) \leq k$ or not.

VC: Given G and k, decide if $\beta(G) \leq k$ or not.
DOM: Given G and k, decide if $\gamma(G) \leq k$ or not.

Using combinatorial results for $\mathcal{NP}$-hardness proofs I

Theorem 9 ADJDIM *is $\mathcal{NP}$-complete.*

For the hardness, recall Cor. 3: $\dim_A(G \odot K_2) = n_G + \gamma(G)$.
If $\dim_A(G \odot K_2)$ could be determined in poly-time, so could $\gamma(G)$.

Theorem 10 *(other reductions known)* DIM *is $\mathcal{NP}$-complete.*

For the hardness, recall that Thm. 1 yields: $\dim(K_2 \odot H) = 2 \cdot \dim_A(H)$.
If $\dim(K_2 \odot H)$ could be determined in poly-time, so could $\dim_A(H)$.

Using combinatorial results for $\mathcal{NP}$-hardness proofs II

Theorem 11 LocAdjDim *is $\mathcal{NP}$-complete.*

For the hardness, check out the conditions of Thm. 8. Hence,

$$\dim_{A,l}(G \odot K_2) = n_G \cdot \dim_{A,l}(K_2) + \gamma(G) = n_G + \gamma(G).$$

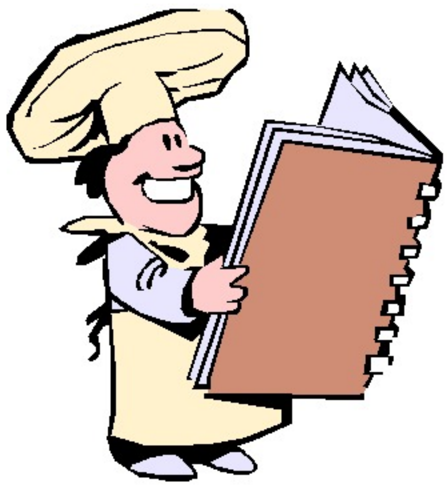
If $\dim_{A,l}(G \odot K_2)$ could be determined in poly-time, so could $\gamma(G)$.

Theorem 12 LocDim *is $\mathcal{NP}$-complete.*

By Thm. 6, $\dim_l(K_2 \odot H) = 2 \cdot \dim_{A,l}(H)$.
If $\dim_l(K_2 \odot H)$ could be determined in poly-time, so could $\dim_{A,l}(H)$.

Conclusions

- Precise combinatorial results (not "only" bounds) that relate different graph parameters are very useful for complexity results.
  $\rightsquigarrow$ Reduction cooks, look up comb. recipes!
  Mathematicians, produce characterizations!
- Our reductions also yield non-existence of sub-exponential $\mathcal{O}^*(2^{o(n)})$ algorithms for our problems, assuming ETH.
- Picture is less clear for approximability or parameterized complexity.



Thanks for your attention !

Theoretische

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