Block Products and Nesting Negations in FO^2

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Main result

Theorem: Let $L \subseteq A^*$ and $m \geq 1$. The following are equivalent:

1. L is definable in $\Sigma_m^2[<]$.
2. The ordered syntactic monoid $\mathrm{Synt}(L)$ of L is in $\mathbf{W}_m$.
3. $\mathrm{Synt}(L)$ is in **DA** and satisfies $U_m \leq V_m$.

Logic

- **Syntax** FO:

$$\varphi \;=\; \exists x\big(\, a(x) \wedge \forall y(\neg b(y) \,\vee\, x < y) \wedge \exists z(x < z \,\wedge\, b(z))\,\big)$$

- **Syntactic properties / resources**:
  - 3 variables: FO^3
  - quantifier depth 2
  - 1 quantifier alternation: Σ_2

  $\rightarrow \Sigma_2^3$

- **Semantics** FO:
  - $L(\varphi) = \{a, c\}^* a \,\{a, b, c\}^* \, b \,\{a, b, c\}^*$

- **Alternative**:

$$\psi \;=\; \exists x(b(x)) \,\wedge\, \forall x \exists y\big(b(x) \,\rightarrow\, (y < x \,\wedge\, a(y))\big)$$

Understanding a logic fragment $\mathcal{F}$: (e.g. $\mathcal{F} = \mathsf{FO}$)

- Complexity of computational problems for $\mathcal{F}$?
  (satisfiability for FO is non-elementary)
- Which languages can be defined in $\mathcal{F}$? (FO = star-free)
- How can we decide whether a given regular language L is definable in $\mathcal{F}$? (FO = aperiodic)
- Closure properties of the $\mathcal{F}$-definable languages?
  (FO is closed under inverse homomorphisms)
- Which other fragment also defines the $\mathcal{F}$-definable languages?
  ($\mathsf{FO} = \mathsf{LTL} = \mathsf{FO}^3$)
- Is separation by $\mathcal{F}$-definable languages decidable? (yes)
  Computation of Separators? (yes)

$\Rightarrow$ descriptive complexity theory within the regular languages

The role of algebra:

- Many effective characterizations of definability in $\mathcal{F}$ rely on algebra.
- **Outline:** $L \subseteq A^*$ is $\mathcal{F}$-definable $\Leftrightarrow$ $\mathrm{Synt}(L) \in \mathbf{V}$ for some class $\mathbf{V}$ of finite monoids
- membership in $\mathbf{V}$ decidable $\Rightarrow$ definability in $\mathcal{F}$ decidable
- Sometimes algebra can only be found below the surface, e.g. FO = counter-free.
- Usually, classes of finite monoids and operations on finite monoids are easier to handle than in the case of automata.
- *"Good"* algebraic characterizations can often be translated to automata.
- Many closure properties come for free!

ω-terms

- M finite monoid, $u \in M$ is idempotent if $u^2 = u$
- there exists $\omega(M) \geq 1$ such that $\forall u \in M$: $u^{\omega(M)}$ is idempotent
- idea behind ω-terms:
  Use one formal symbol ω which works for all finite monoids
- ω-terms: $s ::= x \mid ss \mid s^{\omega}$ for variable $x \in \Omega$
- a mapping $h : \Omega \to M$ extends to homomorphism
  $h : \{\omega\text{-terms over } \Omega\} \to M$ by setting $h(s^{\omega}) = h(s)^{\omega(M)}$
- M satisfies an identity $s = t$ if $h(s) = h(t)$ for all $h : \Omega \to M$
- Example 1: $xy = yx$ defines the finite commutative monoids
- Example 2: $(xy)^{\omega}x(xy)^{\omega} = (xy)^{\omega}$ defines **DA**
- Example 3: $(xy)^{\omega}x(ts)^{\omega} = (xy)^{\omega}s(ts)^{\omega}$ defines **J**
- identities $s \leq t$ for ordered monoids
- "distance" from ω-terms to logic is rather large

Block products

- For a homomorphism $h : A^* \to N$ let $A_N = N \times A \times N$ and let $\sigma_h : A^* \to A_N^*,\ a_1 \cdots a_n \mapsto b_1 \cdots b_n$ with $b_i = \big(h(a_1 \cdots a_{i-1}), a_i, h(a_{i+1} \cdots a_n)\big)$.
- $L \subseteq A^*$ is recognized by monoid in $\mathbf{V} ** \mathbf{W}$ if there exists a homomorphism $h : A^* \to N \in \mathbf{W}$ such that L is union of languages $\sigma^{-1}(K) \cap L_h$ with $K \subseteq A_N^*$ being recognized by monoid in $\mathbf{V}$ and $L_h \subseteq A^*$ being recognized by h.
- Equivalent construction using monoids only (no homomorphisms, no recognition) is called the block product.
- Block products do not automatically give decidability.
- “distance” between block products and logic is rather small

Main result

- $\Sigma_m^2[<]$: two variables, m blocks $=$ $m-1$ nested negations
- Example: $A^* a_1 A^* \cdots a_k A^*$ is definable in $\Sigma_1^2[<]$.
- $\mathbf{W}_1 = [\![\, x \le 1 \,]\!]$, $\mathbf{W}_{m+1} = \mathbf{W}_m ** \mathbf{J}$
- $U_1 = z$, $V_1 = 1$,
  $U_{m+1} = (U_m x_m)^\omega U_m (y_m U_m)^\omega$,
  $V_{m+1} = (U_m x_m)^\omega V_m (y_m U_m)^\omega$

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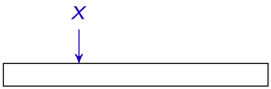

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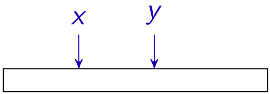

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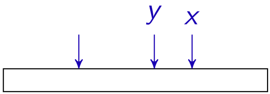

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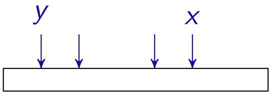

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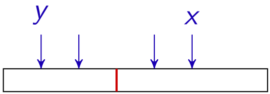

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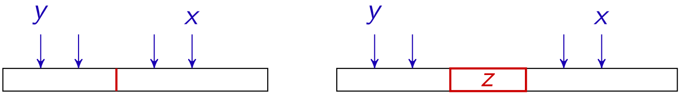

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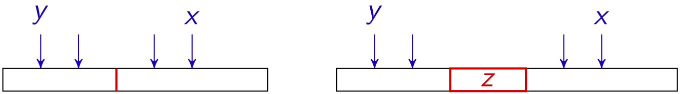

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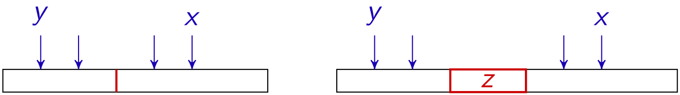

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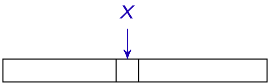

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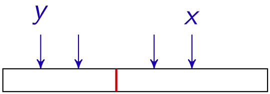

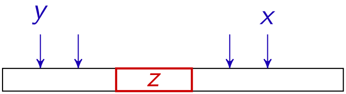

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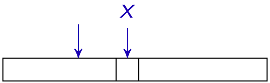

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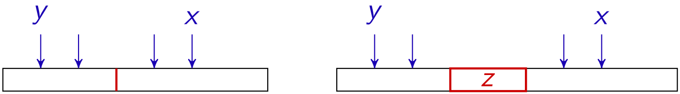

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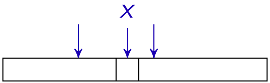

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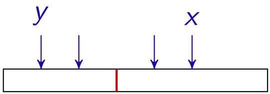

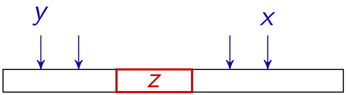

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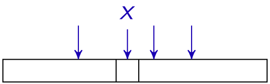

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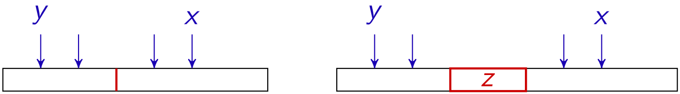

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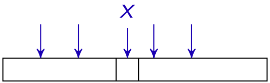

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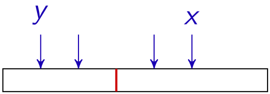

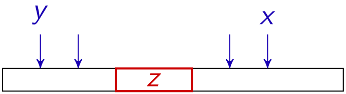

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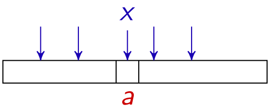

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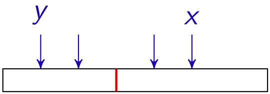

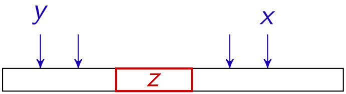

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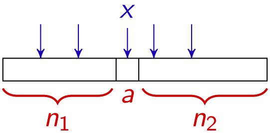

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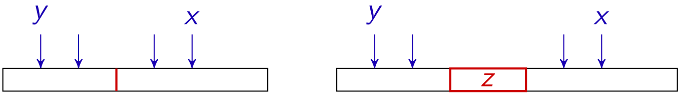

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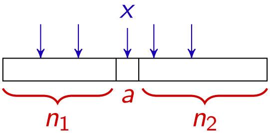

$n_1, n_2 \in N$ for $N \in \mathbf{J}$ [Simon's Theorem]

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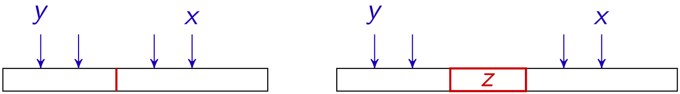

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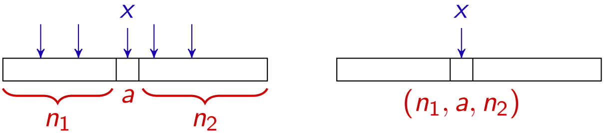

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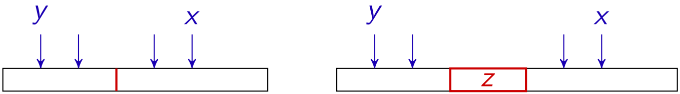

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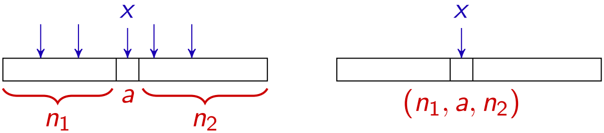

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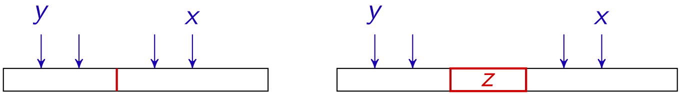

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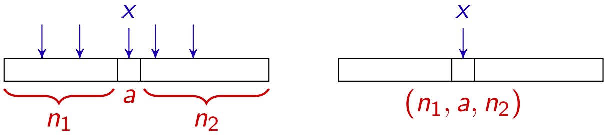

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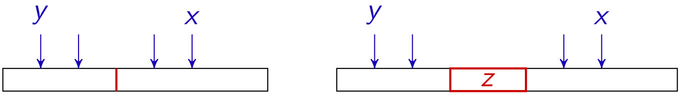

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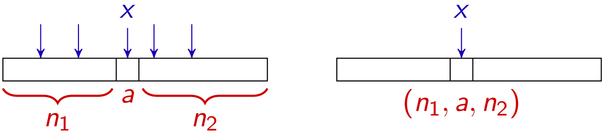

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- "equivalent" formula $\varphi' \in \Sigma_{m-1}^2[<]$ over alphabet $N \times A \times N$
- Induction: $\varphi' \in \mathbf{W}_{m-1}$
- Thus $\varphi \in \mathbf{W}_m = \mathbf{W}_{m-1} ** \mathbf{J}$. $\square$

Remarks

- Step from $\mathbf{W}_m$ to $U_m \leq V_m$ is also easy.
- Difficult part (as usual): from $U_m \leq V_m$ to $\Sigma^2_m[<]$
- Related results: Effective characterizations of $\mathrm{FO}^2_m[<]$
  [K., Weil 2012], [Krebs, Straubing 2012]
- No immediate connection between $\mathrm{FO}^2_m[<]$ and $\Sigma^2_m[<]$

Thank you!