Constraint Satisfaction with Counting Quantifiers 2

Sequel to Constraint Satisfaction with Counting Quantifiers (CSR 2012)

Barnaby Martin and Juraj Stacho

Foundations of Computing Group, Middlesex University, London

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The Constraint Satisfaction Problem CSP($\mathcal{B}$) takes as input a *primitive positive* ({1}-pp) sentence Φ, i.e. of the form

$$\exists v_1 \ldots v_j \ \phi(v_1, \ldots, v_j),$$

where ϕ is a conjunction of atoms, and asks whether $\mathcal{B} \models \Phi$.

This is equivalent to the Homomorphism Problem – has $\mathcal{A}$ a homomorphism to $\mathcal{B}$?

The structure $\mathcal{B}$ is known as the template.

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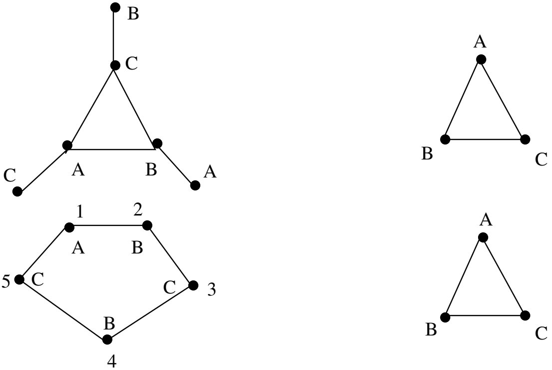
Example homomorphisms.



$$\Phi := \exists v_1, v_2, v_3, v_4, v_5 \quad E(v_1, v_2) \wedge E(v_2, v_1) \wedge E(v_2, v_3) \wedge E(v_3, v_2)$$
$$E(v_3, v_4) \wedge E(v_4, v_3) \wedge E(v_4, v_5)$$
$$E(v_5, v_4) \wedge E(v_5, v_1) \wedge E(v_!, v_5).$$

Finite CSPs occur a lot in nature.

- CSP($\mathcal{K}_m$) is graph m-colourability.
- CSP($\{0,1\}; R_{NAE}$), where B_{NAE} is $\{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$ is not-all-equal 3-satisfiabilty.
- CSP($\{0,1\}; R_{TTT}, R_{TTF}, R_{TFF}, R_{FFF}$) is 3-satisfiabilty.
- CSP($\{0,1\}; \{0\}, \{1\}, \{(0,0),(1,1)\}$) is graph s-t unreachability.

Also vertex cover, clique and hamilton path – but these require non-fixed template.

Infinite CSPs also occur a lot in nature (another story...)

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Feder-Vardi dichotomy conjecture. Each CSP($\mathcal{B}$) is either in P or is NP-complete.

- Compare with Ladner non-dichotomy for NP.

Still open, but known for:

- Structures size 2 (Schaefer 1978).
- Structures size 3 (Bulatov 2002).
- Structures with unary relations (Bulatov 2003).
- Smooth digraphs (Barto, Kozik and Niven 2010).
- Structures size 4 (Marković 2011?).

The Quantified CSP QCSP($\mathcal{B}$) takes as input a *positive Horn* ($\{1, |B|\}$-pp) sentence Φ, i.e. of the form

$$\forall \overline{v}_1 \exists \overline{v}_2 \ldots, Q\overline{v}_j \ \phi(\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_j),$$

where ϕ is a conjunction of atoms, and asks whether $\mathcal{B} \models \Phi$.

QCSP($\mathcal{B}$) is always in Pspace.

Extant classifications

QCSP classifications are harder than CSP classifications.

- Boolean structures. Dichotomy P, Pspace-complete. (Schaefer 1978 + Creignou et al. 2001/ Dalmau 1997.)
- Graphs of permutations. Trichotomy P, NP-complete, Pspace-complete. (Börner et al. 2002.)
- Various digraphs Dichotomies and trichotomies NL, NP-complete, Pspace-complete. (M., Madelaine, Dapić, Marković. 2006, 2011, 2013, 2014)
- Structures with 2-semilattice polymorphism. Dichotomy P, coNP-hard. (Chen 2004.)

The algebraic approach is weaker for QCSPs and the combinatorial method has fewer constructs. Separating NP-hard into NP-complete and Pspace-complete is especially difficult.

For $\mathcal{B}$ with $|B| = n$, let $X \subseteq \{1, \ldots, n\}$. The X-CSP($\mathcal{B}$) has input of the form X-pp

- $\Phi := Q_1 x_1 Q_2 x_2 \ldots Q_m x_m \; \phi(x_1, x_2, \ldots, x_m)$,

where ϕ is a positive conjunction and each Q_i is $\exists^{\geq j}$ for some $j \in X$.

- The yes-instances are those for which $\mathcal{B} \models \Phi$.

Counting quantifiers not studied here before.

- $\exists^{\geq 1}$ is $\exists$ and $\exists^{\geq n}$ is $\forall$.

So,

- $\{1\}$-CSP($\mathcal{B}$) is CSP($\mathcal{B}$), and
- $\{1, |B|\}$-CSP($\mathcal{B}$) is QCSP($\mathcal{B}$).

X-CSP($\mathcal{B}$) is always in Pspace.

Basic results.

(CSR 2012.) Consider X a singleton.

1. $\{1\}$-CSP($\mathcal{B}$) is in NP for all $\mathcal{B}$. For each $n \geq 2$, there exists a template $\mathcal{B}_n$ of size n s.t. $\{1\}$-CSP($\mathcal{B}_n$) is NP-complete.
2. $\{|B|\}$-CSP($\mathcal{B}$) is in L for all $\mathcal{B}$.
3. For each $n \geq 3$, there exists a template $\mathcal{B}_n$ of size n s.t. $\{j\}$-CSP($\mathcal{B}_n$) is Pspace-complete for all $1 < j < n$.

The case of Cycles.

Theorem (CSR 2012)

For $n \geq 3$ and $X \subseteq \{1, \dots, n\}$, the problem X-CSP($\mathcal{C}_n$) is either in L, is NP-complete or is Pspace-complete. Namely:

(i) *X-CSP($\mathcal{C}_n$) $\in \mathrm{L}$ if $n = 4$, or $1 \notin X$, or n is even and $X \cap \{2, \dots, n/2\} = \emptyset$.*

(ii) *X-CSP($\mathcal{C}_n$) is NP-complete if n is odd and $X = \{1\}$.*

(iii) *X-CSP($\mathcal{C}_n$) is Pspace-complete in all other cases.*

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The case of Cliques.

Theorem (CSR 2012)

For $n \in \mathbb{N}$ and $X \subseteq \{1, \ldots, n\}$:

(i) X-CSP($\mathcal{K}_n$) is in L if $n \leq 2$ or $X \cap \{1, \ldots, \lfloor n/2 \rfloor\} = \emptyset$.

(ii) X-CSP($\mathcal{K}_n$) is NP-complete if $n > 2$ and $X = \{1\}$.

(iii) X-CSP($\mathcal{K}_n$) is Pspace-complete if $n > 2$ and either $j \in X$ for $1 < j < n/2$ or $\{1, j\} \subseteq X$ for $j \in \{\lceil n/2 \rceil, \ldots, n\}$.

This is a near trichotomy – where n is even and we have just $\exists^{\geq n/2}$ is open. Clearly $\{1\}$-CSP($\mathcal{K}_2$) is in L.

Theorem (CSR 2014)

(iv-i) $\{2\}$-CSP($\mathcal{K}_4$) is in P.

(iv-ii) $\{j\}$-CSP($\mathcal{K}_{2j}$) is Pspace-complete, for $j \geq 3$.

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Hell and Nešetřil

Theorem (Hell and Nešetřil 1990)

Let $\mathcal{H}$ be a (undirected) graph. Then

- *CSP*$(\mathcal{H}) \in \mathrm{P}$, *if $\mathcal{H}$ is bipartite*
- *CSP*$(\mathcal{H})$ *is NP-complete, otherwise.*

What can we say when we augment $\exists$ with $\exists^{\geq 2}$?

Generalising Hell and Nešetřil

Let $[1^m2^*]$-pp be the fragment of $\{1,2\}$-pp in which m $\exists^{\geq 2}$ quantifiers are followed by nothing but $\exists^{\geq 1} = \exists$.

Theorem (CSR 2012)

Let $\mathcal{H}$ be a graph. Then

- $[2^m1^*]$-*CSP*$(\mathcal{H}) \in \mathrm{P}$ *for all m, if* $\mathcal{H}$ *is a forest or a bipartite graph containing* $\mathcal{C}_4$
- $[2^m1^*]$-*CSP*$(\mathcal{H})$ *is NP-complete for some m, if otherwise.*

Theorem (CSR 2014)

Let $\mathcal{H}$ be a graph. Then

- $\{1,2\}$-*CSP*$(\mathcal{H}) \in \mathrm{P}$, *if* $\mathcal{H}$ *is a forest or a bipartite graph containing* $\mathcal{C}_4$
- $\{1,2\}$-*CSP*$(\mathcal{H})$ *is NP-hard, otherwise.*

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Is that all!?

The sub-case $\{1,2\}$-CSP$(\mathcal{P}_\omega)$ in P is already complicated.

- but seriously...

the contribution seems so slight, but the combinatorics of counting quantifiers is so awkward!

- The CSR submission was 35 pages!

The algebraic method now exists for X-CSP, but it has not proven to be much better.

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$\{2\}$-CSP($\mathcal{K}_4$) is in P

We iteratively construct the following three sets:

$$R^+,\ R^-, \text{ both ternary and } F \supseteq E.$$

X1 If there are $x, y, z \in V(G)$ such that $\{x, y\} < z$ where $xz, yz \in F$, then add xyz into R^-.

X2 If there are vertices $x, y, w, z \in V(G)$ such that $\{x, y, w\} < z$ with $wz \in F$ and $xyz \in R^-$, then add xyw into R^+.

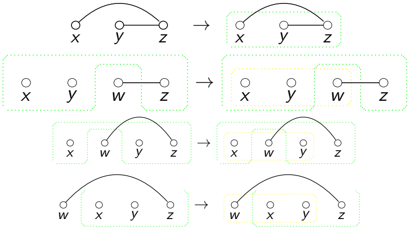

Figure : Illustrating rules (X1) and (X2)

$\{2\}$-CSP($\mathcal{K}_4$) is in P

X3 If there are $x, y, w, z \in V(G)$ s.t. $\{x, y, w\} < z$, $wz \in F$ and $xyz \in R^+$, then if $\{x, y\} < w$, add xyw to R^-, else add xw, yw to F.

X4 If there are vertices $x, y, w, z \in V(G)$ s.t. $\{x, w\} < y < z$ with $xyz \in R^+$ and $wyz \in R^-$, then add xw to F, and add xyw to R^+.

X5 If there are vertices $x, y, w, z \in V(G)$ such that $\{x, y, w\} < z$ where either $xyz, wyz \in R^+$, or $xyz, wyz \in R^-$, then add xyw into R^+.

X6 If there are vertices $x, y, q, w, z \in V(G)$ such that $\{x, y, w\} < q < z$ where either $xyz, wqz \in R^+$, or $xyz, wqz \in R^-$, then add xyw and xyq into R^+.

X7 If there are vertices $x, y, q, w, z \in V(G)$ such that $\{x, y, w\} < q < z$ where either $xyz \in R^+$ and $wqz \in R^-$, or $xyz \in R^-$ and $wqz \in R^+$, then add xyq into R^-, and if $\{x, y\} < w$, also add xyw into R^-, else add xw and yw into F.

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{2}-CSP($\mathcal{K}_4$) is in P

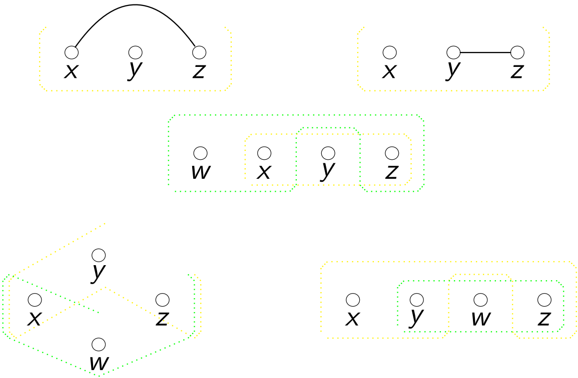

Figure : Five forbidden configurations of {2}-CSP($\mathcal{K}_4$)

Further interesting results

Theorem

If $\mathcal{H}$ is a bipartite graph, then either $\{1,2\}$-CSP($\mathcal{H}$) is in P, or $\{1,2\}$-CSP($\mathcal{H}$) is Pspace-complete.

Theorem

Let H be a (partially reflexive) graph on at most three vertices, then either $\{1,2\}$-CSP(H) is in P or it is Pspace-complete.

The smallest graph $\mathcal{H}$ so that $\{1,2\}$-CSP($\mathcal{H}$) is in NP-complete is size 4.

Theorem

Let $\mathcal{H}$ be a graph with reflexive dominating vertex, then $\{1,2\}$-CSP($\mathcal{H}$) is either in P or is NP-complete.

Theorem

Let $\mathcal{H} := \mathcal{K}_{a_1,\dots,a_n}$ be a complete multipartite graph with respective parts of size $a_1,\dots,a_n$.

- (i) *If $n = 2$, then $\{1,2\}$-CSP($\mathcal{H}$) is in L.*
- (ii) *If $n > 2$ and the multiset $\{\{a_1,\dots,a_n\}\}$ contains at most 1, then $\{1,2\}$-CSP($\mathcal{H}$) is NP-complete.*
- (iii) *$\{1,2\}$-CSP($\mathcal{H}$) is Pspace-complete all other cases.*

Conjecture

Let $\mathcal{H}$ be a graph. Either $\{1,2\}$-CSP($\mathcal{H}$) is in P, or it is NP-complete, or it is Pspace-complete.

Combinatorics to a Galois theory.

- A homomorphism from $\mathcal{B}^k$ to $\mathcal{B}$ is termed a k-ary polymorphism.

Let $\mathrm{Pol}(\mathcal{B}), \mathrm{sPol}(\mathcal{B})$ be the polymorphisms, surjective pols of $\mathcal{B}$.

- $\mathrm{Inv}(\mathrm{Pol}(\mathcal{B})) = \langle \mathcal{B} \rangle_{\{1\}\text{-pp}}$.
- $\mathrm{Inv}(\mathrm{sPol}(\mathcal{B})) = \langle \mathcal{B} \rangle_{\{1,|B|\}\text{-pp}}$.

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Call a function $f : B^k \to B$ expanding if,

- for each m, $|X_1|, \ldots, |X_k| = m$ implies $|f(X_1, \ldots, X_k)| \geq m$.

Let $\mathrm{ePol}(\mathcal{B})$ be the expanding polymorphisms of $\mathcal{B}$.

Theorem (Bulatov and Hedayaty 2012)

For finite $\mathcal{B}$, $\mathrm{Inv}(\mathrm{ePol}(\mathcal{B})) = \langle \mathcal{B} \rangle_{\{1,\ldots,|B|\}\text{-pp}}$.

There is some hope this can help in the Mal'tsev case.

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Conjecture.

- For $j > 1$, $\{1, j\}$-CSP($\mathcal{B}$) is either in P, NP-complete or Pspace-complete.
- $\{1, \ldots, |B|\}$-CSP($\mathcal{B}$) is either in P or Pspace-complete.