Dynamic Complexity of Planar 3-connected Graph Isomorphism

Jenish C. Mehta

Dynamic Complexity

Fixed Problem

Input

Computed Solution

slight change

Complexity of updating the solution?

Dynamic Complexity

<u>Fixed Problem</u>

Input
A Relation filled with tuples

slight change
Insertion/Deletion of a tuple

Computed Solution
A set of Relations

Complexity of updating the solution?
Complexity Class in which the Relations can be updated?

Definition. For any static complexity class C, we define its dynamic version, DynC as follows: Let $\rho = \langle R_1^{a_1}, ..., R_s^{a_s}, c_1, ..., c_t \rangle$, be any vocabulary and $S \subseteq STRUC(\rho)$ be any problem. Let $R_{n,\rho} = \{ins(i, a'),\ del(i, a'),\ set(j, a) \mid 1 \leq i \leq s,\ a' \in \{0, ..., n-1\}^{a_i},\ 1 \leq j \leq t\}$ be the request to insert/delete tuple a' into/from the relation R_i, or set constant c_j to a.

Let $eval_{n,\rho} : R^*_{n,\rho} \rightarrow STRUC(\rho)$ be the evaluation of a sequence or stream of requests. Define $S \in \mathsf{DynC}$ iff there exists another problem $T \subset STRUC(\tau)$ (over some vocabulary τ) such that $T \in \mathsf{C}$ and there exist maps f and g:

$$f : R^*_{n,\rho} \rightarrow STRUC(\tau),\ g : STRUC(\tau) \times R_{n,\rho} \rightarrow STRUC(\tau)$$

satisfying the following properties:

1. ***(Correctness)*** For all $r' \in R^*_{n,\rho}$, $(eval_{n,\rho}(r') \in S) \Leftrightarrow (f(r') \in T)$
2. ***(Update)*** For all $s \in R_{n,\rho}$, and $r' \in R^*_{n,\rho}$, $f(r's) = g(f(r'), s)$
3. ***(Bounded Universe)*** $||f(r')|| = ||eval_{n,\rho}(r')||^{O(1)}$
4. ***(Initialization)*** The functions g and the initial structure $f(\emptyset)$ are computable in C as functions of n.

Definition. For any static complexity class $\mathcal{C}$, we define its dynamic version, Dyn$\mathcal{C}$ as follows: Let $\mu = \langle R_1^{a_1}, \ldots, R_s^{a_s}, c_1, \ldots, c_t \rangle$ be any vocabulary and $S \subseteq STRUC(\mu)$ be any problem. Let $R_{n,\mu} = \{ins(i, \bar{a}'), del(i, \bar{a}'), set(j, a) \mid 1 \le i \le s, \bar{a}' \in \{0, \ldots, n-1\}^{a_i}, 1 \le j \le t\}$ be the request to insert/delete tuple $\bar{a}'$ into/from the relation R_i, or set constant c_j to a.

Let $eval_{n,\mu} : R_{n,\mu}^* \to STRUC(\mu)$ be the evaluation of a sequence or stream of requests. Define $S \in$ Dyn$\mathcal{C}$ iff there exists another problem $T \subseteq STRUC(\tau)$ (over some vocabulary τ) such that $T \in \mathcal{C}$ and there exist maps f and g:

$$f : R_{n,\mu}^* \to STRUC(\tau),\ g : STRUC(\tau) \times R_{n,\mu} \to STRUC(\tau)$$

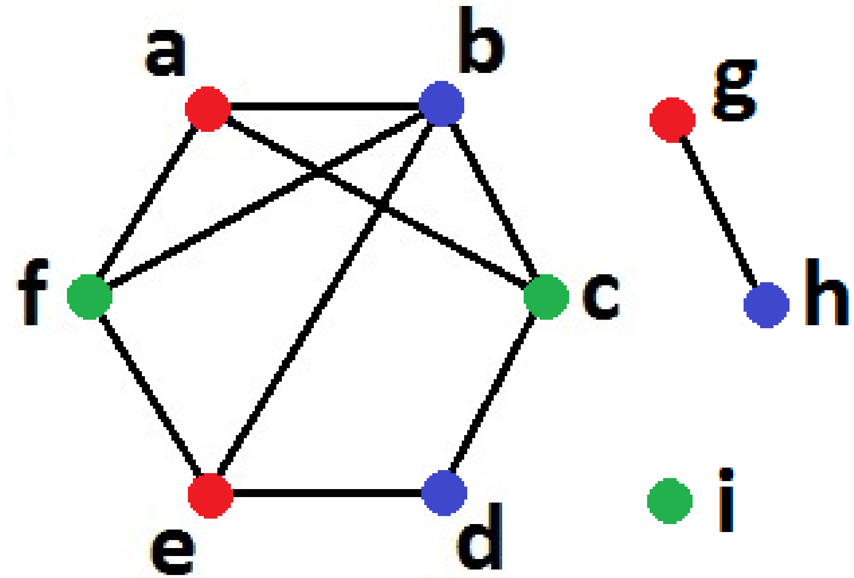
satisfying the following properties:

1. *(Correctness)* For all $\bar{r} \in R_{n,\mu}^*$, $(eval_{n,\mu}(\bar{r}) \in S) \Leftrightarrow (f(\bar{r}) \in T)$
2. *(Update)* For all $s \in R_{n,\mu}$, and $\bar{r} \in R_{n,\mu}^*$, $f(\bar{r}s) = g(f(\bar{r}), s)$
3. *(Bounded Universe)* $\|f(\bar{r})\| = \|eval_{n,\mu}(\bar{r})\|^{O(1)}$
4. *(Initialization)* The functions g and the initial structure $f(\emptyset)$ are computable in $\mathcal{C}$ as functions of n.

Dynamic Complexity

Problem: Vertex-colouring a graph using 3 colours?

Input: Relation (graph) *G(x,y)*
(a,b), (b,c), (c,d), (d,e), (e,f),
(a,c), (b,e), (b,f), (c,f), (g,h)



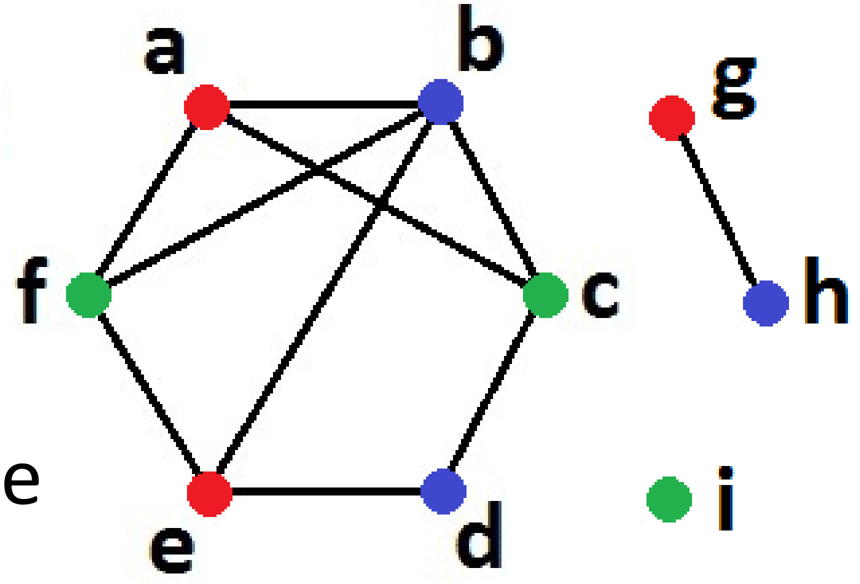
Solution:
R = a,e,g B = b,d,h G = c,f,i

Change: Insertion/Deletion of an edge, or tuple in *G*

Dynamic Complexity

Problem: Vertex-colouring a graph using 3 colours?

Relations Maintained:
A(x,y), B(x,y,z,w), R(p,q,r), D(a,b,c,d,e), C(s,r)



Dynamic Complexity:
Complexity class *C*, to update the relations A,B,C,D,R and find the solution from them after insertion/deletion

Problem is in *DynC*

Dynamic Complexity

Problem: Parity of the String?

Input: Relation (string) *S(p,b)*

1 0 1 1 1 0 * 0 * 1
0 1 2 3 4 5 6 7 8 9

Relations:

B(z) = To find the parity of the string.
The only tuple in the relation will be the parity of the string.

Simple *DynP, DynL* solution

Dynamic Complexity

Problem: Parity of the String?

Input: Relation (string) *S(p,b)*

1 0 1 1 1 0 * 0 * 1
0 1 2 3 4 5 6 7 8 9

Relations:

A(x,y) = To store the old string

B(z) = To find the parity of the string.
The only tuple in the relation will be the parity of the string.

Dynamic Complexity

Problem: Parity of the String?

1 0 1 1 1 0 * 0 * 1

0 1 2 3 4 5 6 7 8 9

S(p,b) =

(0,1), (1,0), (2,1), (3,1), (4,1), (5,0), (7,0), (9,1)

A(x,y) =

(0,1), (1,0), (2,1), (3,1), (4,1), (5,0), (7,0), (9,1)

B(z) = (1)

Dynamic Complexity

Problem: Parity of the String?

User: *insert(p,b)*

1	**0**	**1**	**1**	**1**	**0**	*****	**0**	*****	**1**
0	1	2	3	4	5	6	7	8	9

A'(x,y) = A(x,y) **OR**
x=p **AND** y=b

Dynamic Complexity

User: *insert(p,b)*
[assume insert(6,1)]

1 0 1 1 1 0 * 0 * 1
0 1 2 3 4 5 6 7 8 9

B'(z) = A(p,b) **AND** B(z) **OR**
!A(p,b) **AND**
b=0 **AND** B(z) **OR**
b=1 **AND**
z=1 **AND** B(0) **OR**

z=0 **AND** B(1)

$R_2(v,x) = BFSEdge(v,a,b) \wedge Path(v,v,x,\{a,b\})$
$R_1(v,y) = \neg R_2(v,y)$
$PR(v,s,t) = R_1(v,s) \wedge R_2(v,t) \wedge Edge(s,t)$ {All edges connecting R_1 and R_2}
$l_{min}(v,w) \leftarrow \min\{level_v(s) + 1 + level_t(w) : \; PR(v,s,t)\}$ {Length of the new shortest path from v to w}
$PR_{min}(v,w,s,t) = R_2(v,w) \wedge PR(v,s,t) \wedge (level_v(s) + 1 + level_t(w) = l_{min}(v,w))$ {Set of edges that lead to the shortest path}
$PR_{lex,min}(v,w,s,t) = PR_{min}(v,w,s,t) \wedge (s \le t) \wedge (\forall p,q, \; PR_{min}(v,w,p,q) \Rightarrow (s < p) \vee ((s = p) \wedge (t \le q)))$
{Choosing the lexicographically smallest edge. $PR_{lex,min}$ is the set of new edges that will be added. The queries are now exactly similar to insertion of edges}

{$|P_2| < |P_1|$ or $|P_1| = |P_2| \wedge P_2 <_c P_1$, and $\{x,y,z\}$ are on $|P_2|$}
$(l_{old} > l_{new}) \vee (l_{old} = l_{new} \wedge n_1 > n_2)$ and
$(CPath(v,v_e,v,\alpha,\{x,y,z\}) \wedge CPath(v,v_e,x,y,z))$ {All on the path from v to α}
$\vee(CPath(\beta,\beta_e,w,\{x,y,z\}) \wedge CPath(\beta,\beta_e,x,y,z))$ {All on the path from β to w}
$\vee(CPath(v,v_e,v,\alpha,\{x\}) \wedge CPath(\beta,\beta_e,\beta,w,\{y,z\}) \wedge CPath(\beta,\beta_e,\beta,y,z))$ {x on $path_{v,v_e}(v,\alpha)$ and y,z on $path_{\beta,\beta_e}(\beta,w)$}
$\vee(CPath(v,v_e,v,\alpha,\{x,z\}) \wedge CPath(v,v_e,v,z,x) \wedge CPath(\beta,\beta_e,\beta,w,y))$ {x,z on $path_{v,v_e}(v,\alpha)$ and y on $path_{\beta,\beta_e}(\beta,w)$}
{$EmbPar(v,v_e,x,n_p)$ denotes that the embedding number of x's parent in $[v,v_e]$ is n_p}
$EmbPar(v,v_e,x,n_p) = \exists x_p, Parent(v,v_e,x_p,x) \wedge Emb(x,x_p,n_p)$
$Emb_p(v,v_e,t,x,n_x) = Edge(x,t) \wedge \exists n_p, d_t, n_{old}, Deg(t,d_t) \wedge EmbPar(v,v_e,t,n_p) \wedge Emb(t,x,n_{old})$
$\wedge(n_{old} \ge n_p \Rightarrow n_x = n_{old} - n_p) \wedge (n_{old} < n_p \Rightarrow n_x = n_{old} + d_x - n_p)$
$Emb_f(v,x,n_x) = \exists n_{old}, d_v, Emb(v,x,n_{old}) \wedge Deg(v,d_v) \wedge (n_x = d_v - 1 - n_{old})$

$R_2(v,x) = BFSEdge(v,a,b) \wedge Path(v,v,x,\{a,b\})$
$R_1(v,y) = \neg R_2(v,y)$
$PR(v,s,t) = R_1(v,s) \wedge R_2(v,t) \wedge Edge(s,t)$ {All edges connecting R_1 and R_2}
$l_{min}(v,w) \leftarrow \min\{level_v(s) + 1 + level_t(w) :\ PR(v,s,t)\}$ {Length of the new shortest path from v to w}
$PR_{min}(v,w,s,t) = R_2(v,w) \wedge PR(v,s,t) \wedge (level_v(s) + 1 + level_t(w) = l_{min}(v,w))$ {Set of edges that lead to the shortest path}
$PR_{lex,min}(v,w,s,t) = PR_{min}(v,w,s,t) \wedge (s \leq t) \wedge (\forall p,q.\ PR_{min}(v,w,p,q) \Rightarrow (s < p) \vee ((s = p) \wedge (t \leq q)))$
{Choosing the lexicographically smallest edge. $PR_{lex,min}$ is the set of new edges that will be added. The queries are now exactly similar to insertion of edges}

{$|P_2| < |P_1|$ or $|P_1| = |P_2| \wedge P_2 <_c P_1$, and $\{x,y,z\}$ are on $|P_2|$}
$(l_{old} > l_{new}) \vee (l_{old} = l_{new} \wedge n_1 > n_2)$ and
$(CPath(v,v_e,v,\alpha,\{x,y,z\}) \wedge CPath(v,v_e,x,y,z))$ {All on the path from v to α}
$\vee(CPath(\beta,\beta_e,w,\{x,y,z\}) \wedge CPath(\beta,\beta_e,x,y,z))$ {All on the path from β to w}
$\vee(CPath(v,v_e,v,\alpha,\{x\}) \wedge CPath(\beta,\beta_e,\beta,w,\{y,z\}) \wedge CPath(\beta,\beta_e,\beta,y,z))$ {x on $path_{v,v_e}(v,\alpha)$ and y,z on $path_{\beta,\beta_e}(\beta,w)$}
$\vee(CPath(v,v_e,v,\alpha,\{x,z\}) \wedge CPath(v,v_e,v,z,x) \wedge CPath(\beta,\beta_e,\beta,w,y))$ {x,z on $path_{v,v_e}(v,\alpha)$ and y on $path_{\beta,\beta_e}(\beta,w)$}

{$EmbPar(v,v_e,x,n_p)$ denotes that the embedding number of x's parent in $[v,v_e]$ is n_p}
$EmbPar(v,v_e,x,n_p) = \exists x_p.\ Parent(v,v_e,x_p,x) \wedge Emb(x,x_p,n_p)$
$Emb_p(v,v_e,t,x,n_x) = Edge(x,t) \wedge \exists n_p,d_t,n_{old}.\ Deg(t,d_t) \wedge EmbPar(v,v_e,t,n_p) \wedge Emb(t,x,n_{old})$
$\wedge(n_{old} \geq n_p \Rightarrow n_x = n_{old} - n_p) \wedge (n_{old} < n_p \Rightarrow n_x = n_{old} + d_x - n_p)$
$Emb_f(v,x,n_x) = \exists n_{old},d_v.\ Emb(v,x,n_{old}) \wedge Deg(v,d_v) \wedge (n_x = d_v - 1 - n_{old})$

Dynamic Complexity

Parity is NOT in *FO* (uniform AC^0)

Parity is in *DynFO*!

Undirected Reachability is in *DynFO*!

Dynamic Complexity

DST ('93) – FOIES, Acyclic Reach

IP ('97) – Dynamic Complexity, Undirected Reach

Hesse ('01) – Reach in $DynTC^0$

HI ('02) – Complete problems for DynC

DHK ('14) – Triangulated PlanarReach in DynFO

Schwentick ('13) – Perspectives

Isomorphism in PlanarLand

	Trees	3-connected planar graphs	Planar Graphs
Quadratic/ Linear time	Elementary	Weinberg (‘66); Hopcroft, Tarjan (‘73)	Hopcroft, Wong (‘74)
Logspace	Lindell (‘92)	Datta, Limaye, Nimbhorkar (‘08)	Datta, Limaye, Nimbhorkar, Thierauf, Wagner (‘09)
DynFO	Etessami (‘98)	**This work**	?

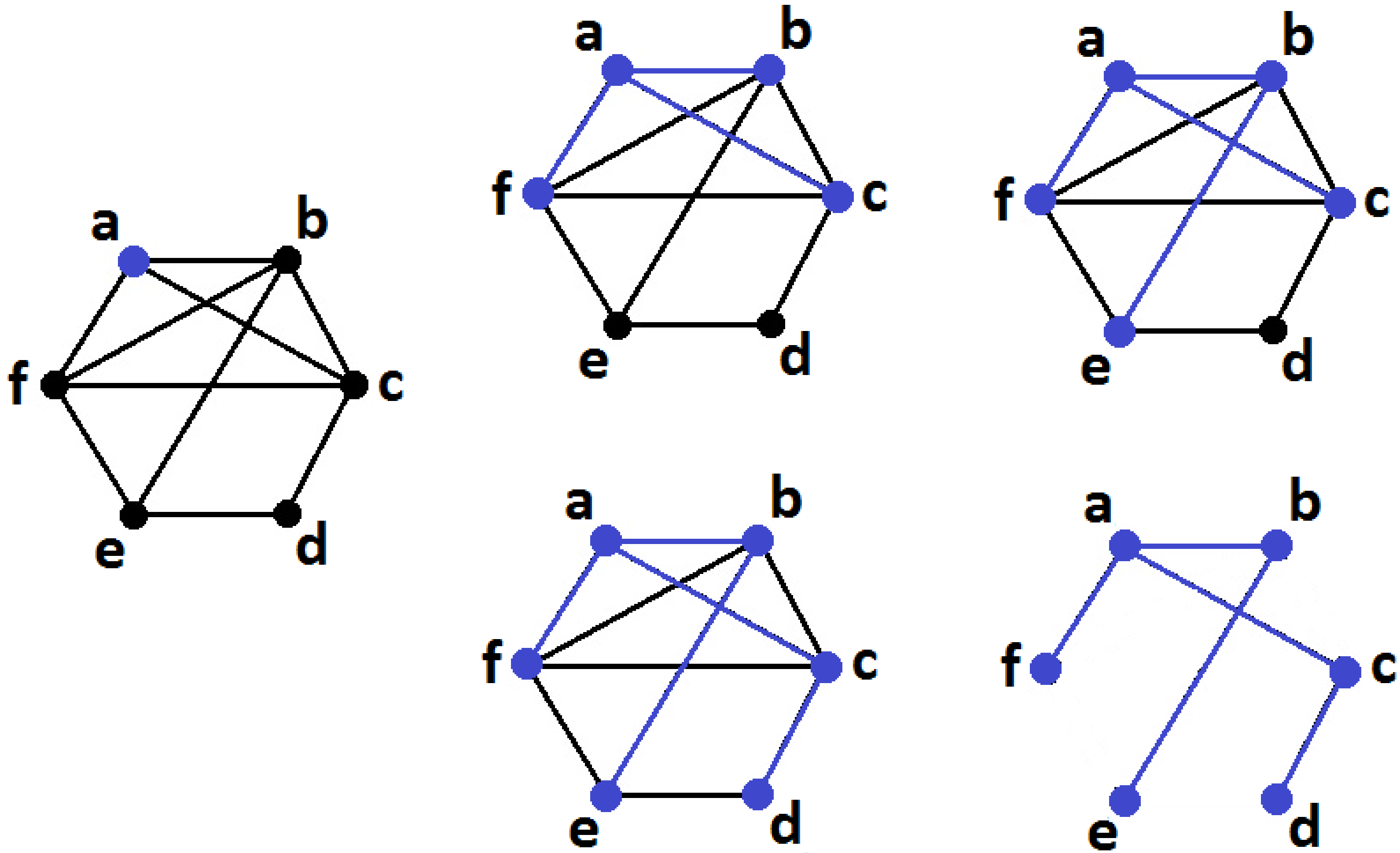
This work

Main Results:

1. Breadth-First Search for general undirected graphs is in *DynFO*

2. Isomorphism for Planar 3-connected graphs is in *DynFO*+

Breadth-First Search in DynFO
(general undirected graphs)

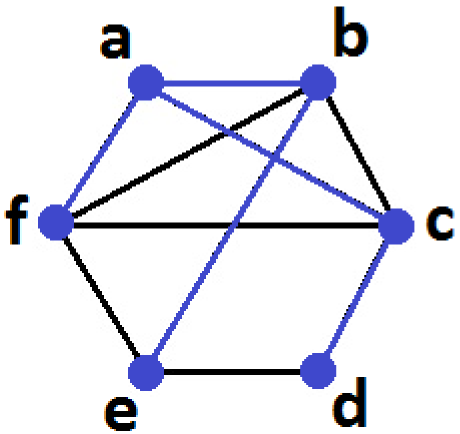


Breadth-First Search in DynFO
(general undirected graphs)

Main Idea:

Maintain BFS-tree from ***every*** vertex in the graph

Breadth-First Search in DynFO
(general undirected graphs)

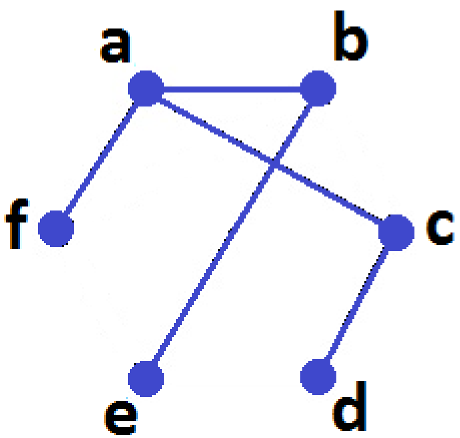


Edge (x, y)

(a,b), (b,a),
(b,c), (c,b), …

Level (v, x, l)

(a, b, 1),
(a, d, 2), …



BFSEdge (v, x, y)

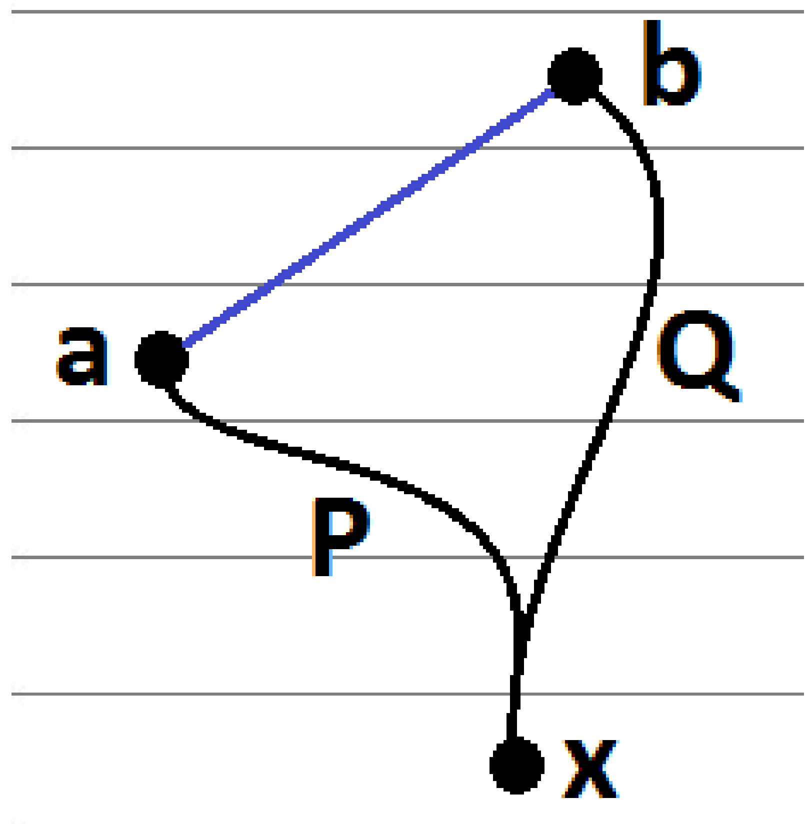
(a, a, b),
(a, b, e), …

Path (v, x, y, z)

(a, e, d, b),
(a, a, d, c), …

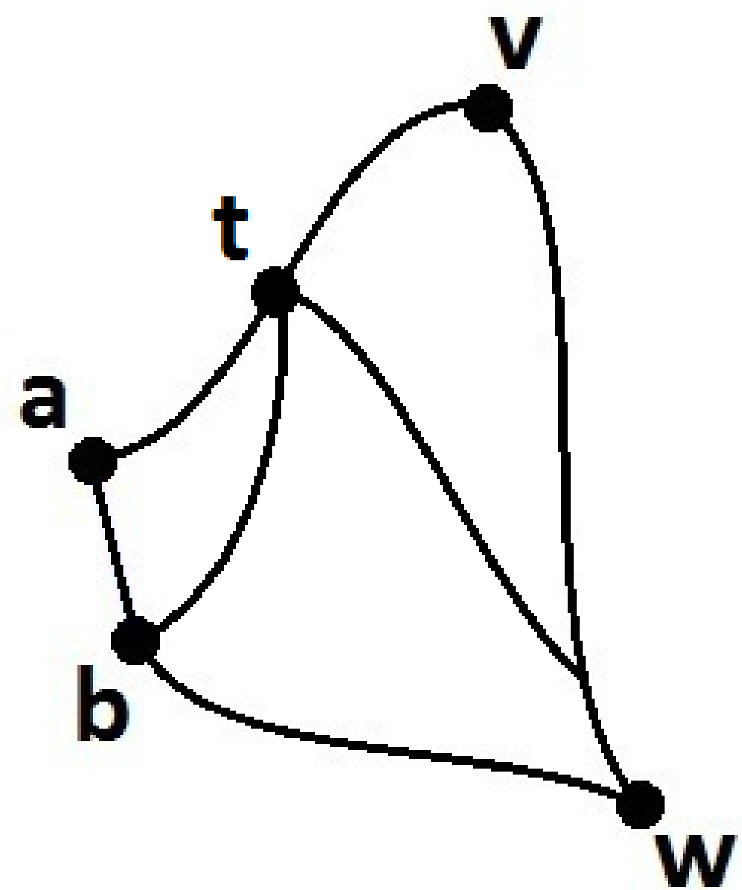
Breadth-First Search in DynFO
(general undirected graphs)

Lemma 1:
After the insertion of edge $\{a,b\}$, the level of a vertex x cannot change both in the BFS trees of a and b.



Breadth-First Search in DynFO
(general undirected graphs)

Lemma 2:
If any vertex *t* lies on *path(b,b,w)* and on *path(v,v,a),* then the shortest path from *v* to *x* does not change after the insertion of *(a,b)*

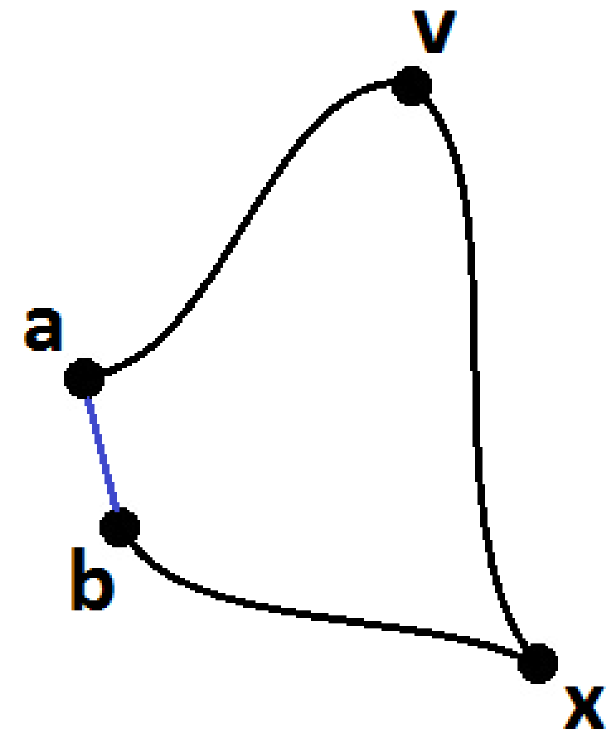


Breadth-First Search in DynFO
(general undirected graphs)

insert (a,b)

- Find the shorter path: *path(a,a,x)* or *path(b,b,x)* [Lemma 1]

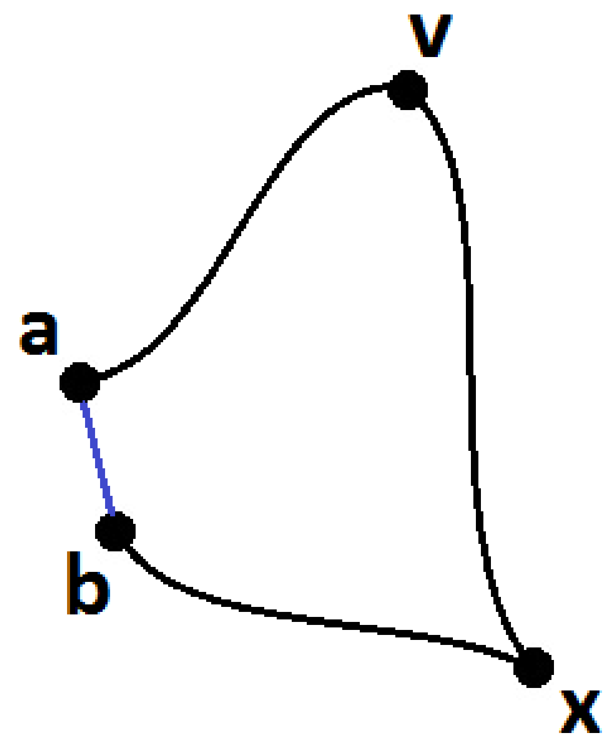
- Only New path to consider: *path(v,v,a)* + *(a,b)* + *path(b,b,x)*



Breadth-First Search in DynFO
(general undirected graphs)

insert (a,b)

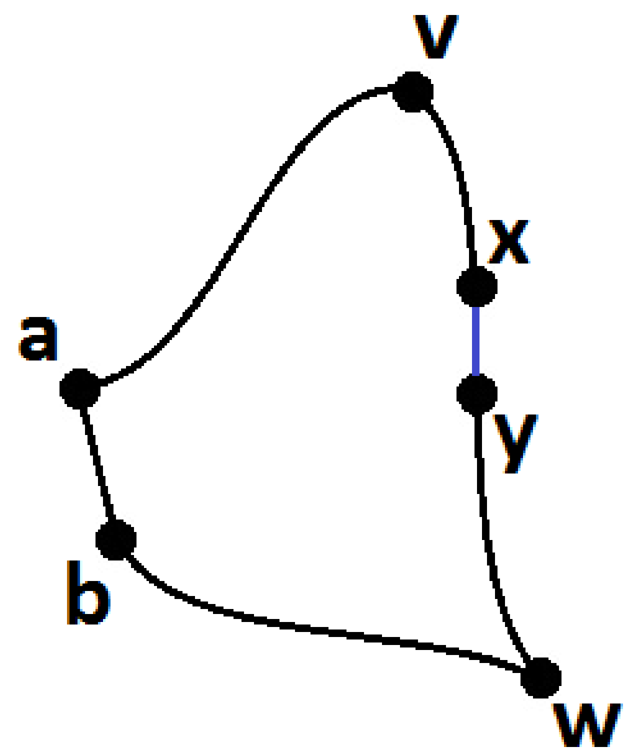
- Find the shorter path:
  path(v,v,x) or
  path(v,v,a) + *(a,b)* + *path(b,b,x)*
  [Lemma 2]
- Update the relations if new path is shorter



Breadth-First Search in DynFO
(general undirected graphs)

BFSEdge(v,x,y):

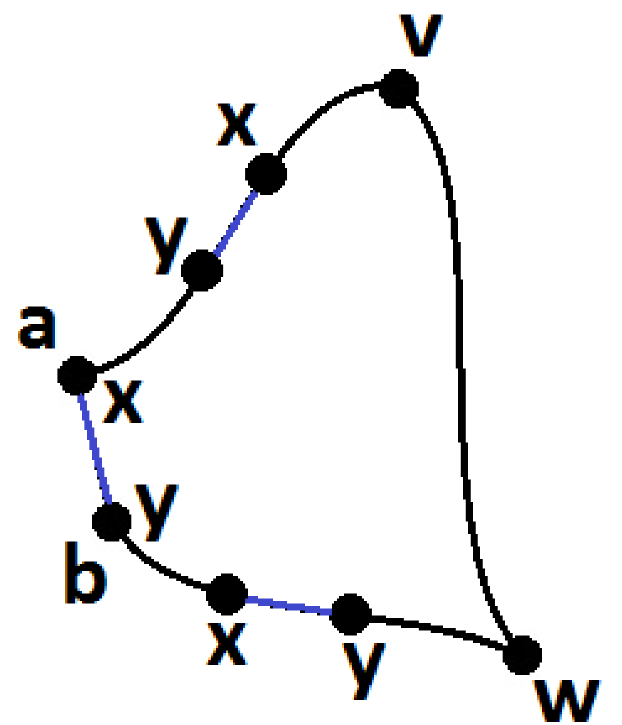
Edge (*x*,*y*) belongs to the BFS tree of vertex *v*, if: There exists a vertex *w* in BFS tree of *v* whose level has <u>not changed</u> AND (*x*,*y*) lies on the path from *v* to *w* OR …



Breadth-First Search in DynFO
(general undirected graphs)

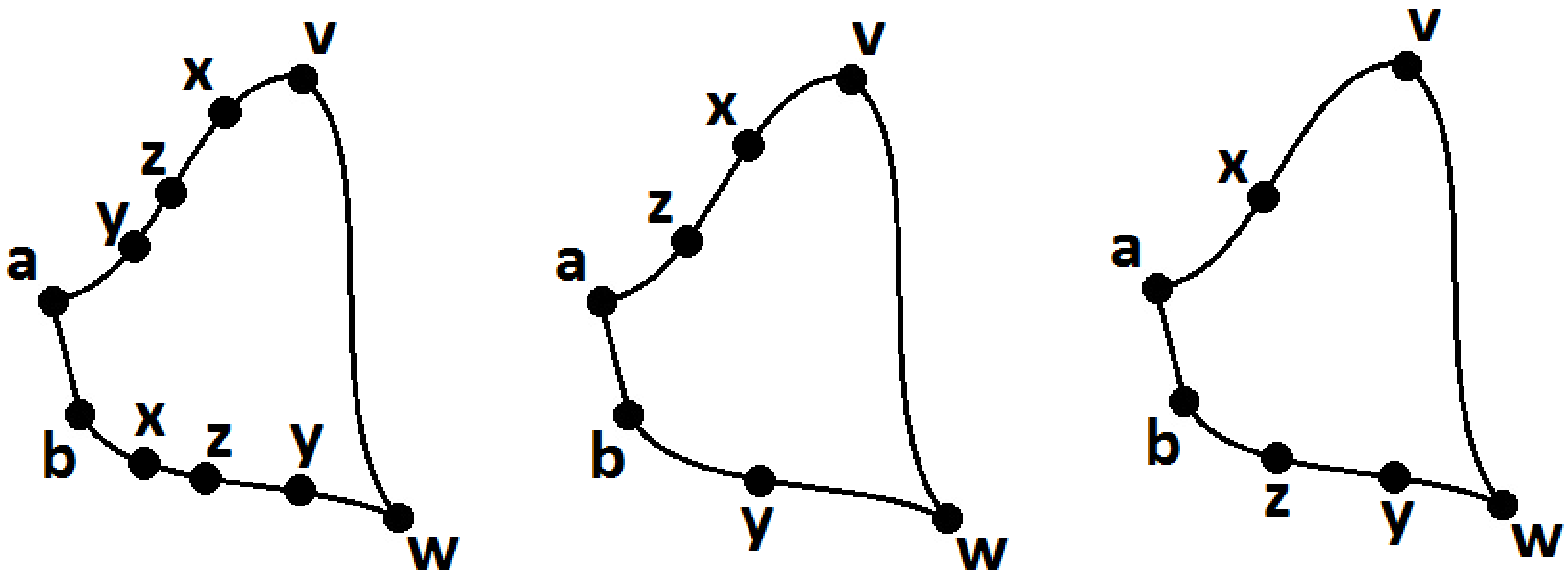
BFSEdge(v,x,y):

… OR
There exists a vertex *w* in BFS tree of *v* whose level has <u>changed</u> AND (*x*,*y*) lies on the path from *v* to *a* OR the path from *b* to *w* OR is (*a*,*b*).



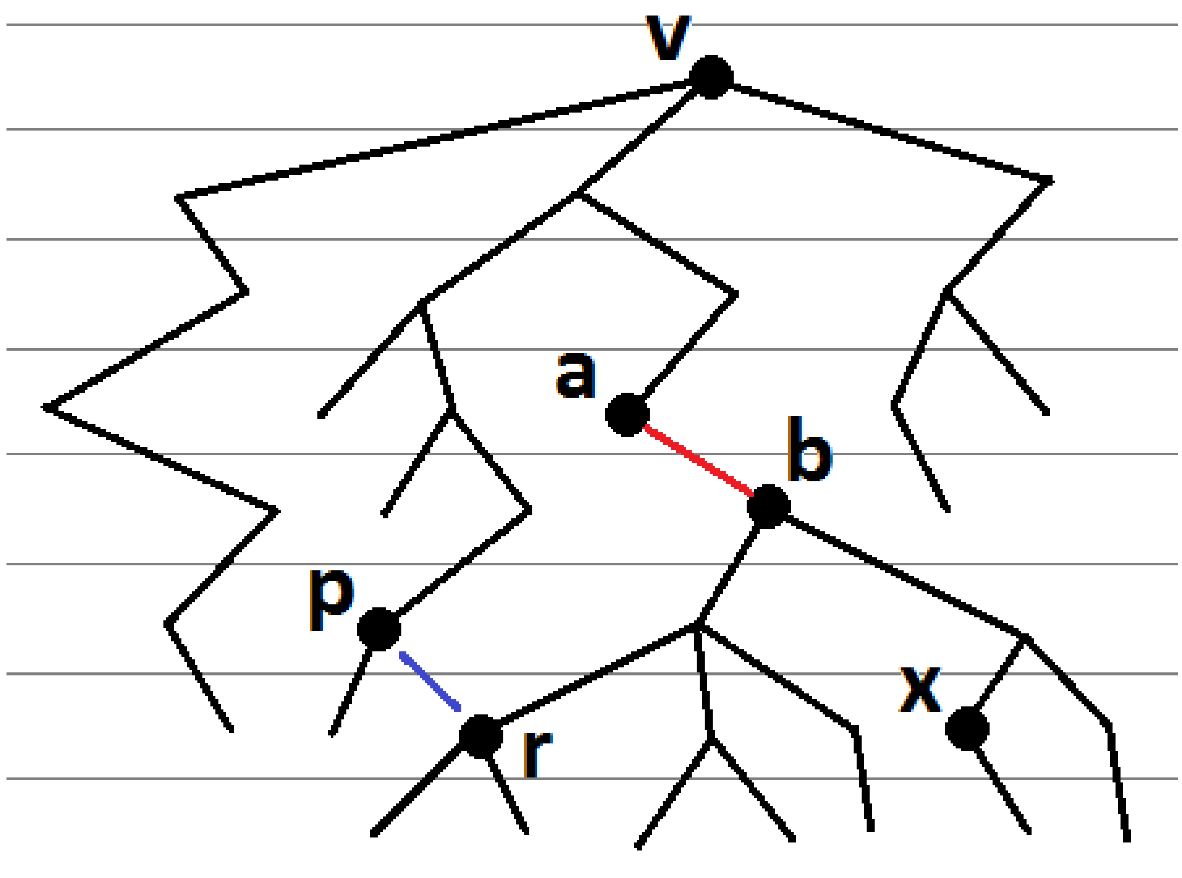
Breadth-First Search in DynFO
(general undirected graphs)

Path(v,x,y,z):



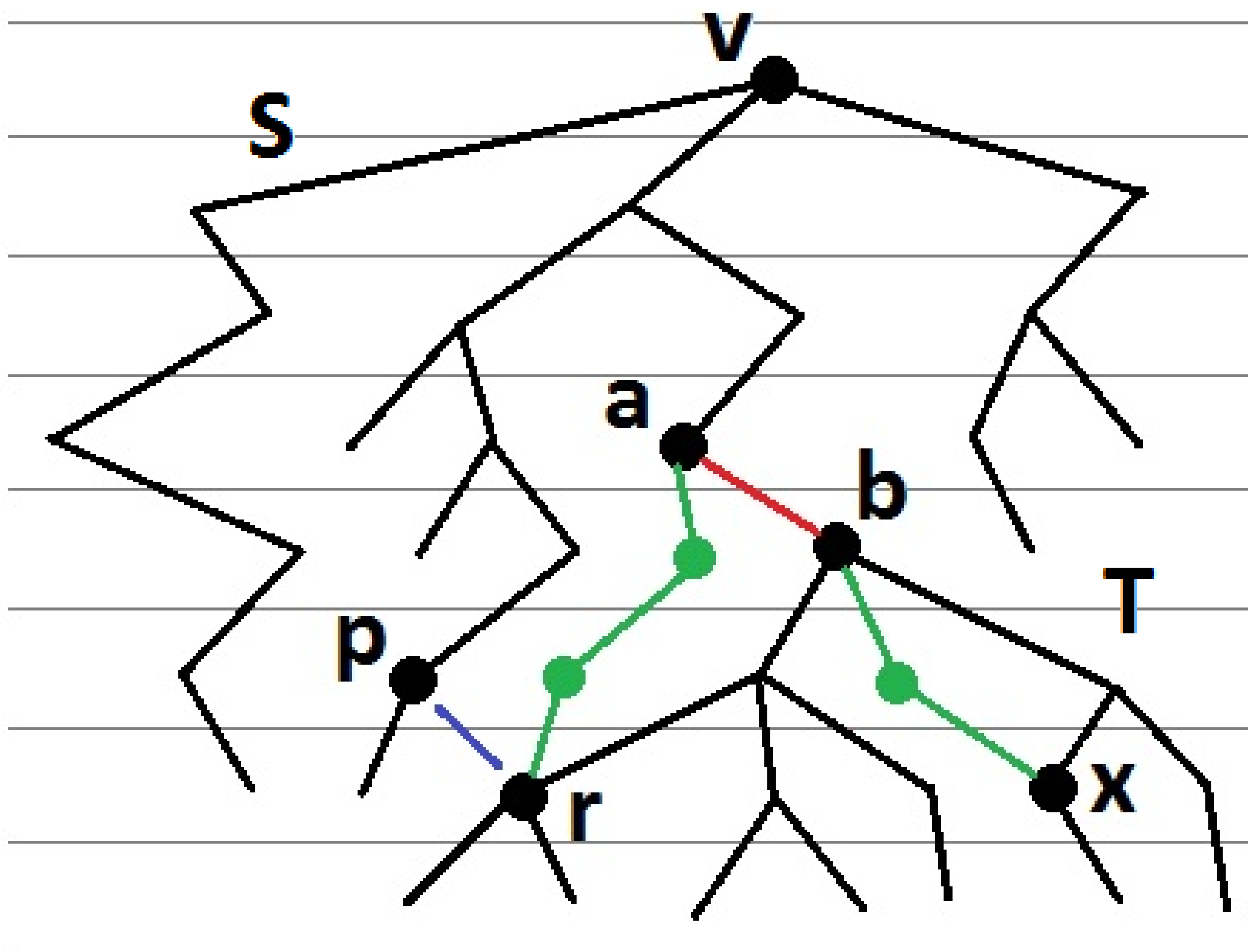
Breadth-First Search in DynFO
(general undirected graphs)

delete(a,b):



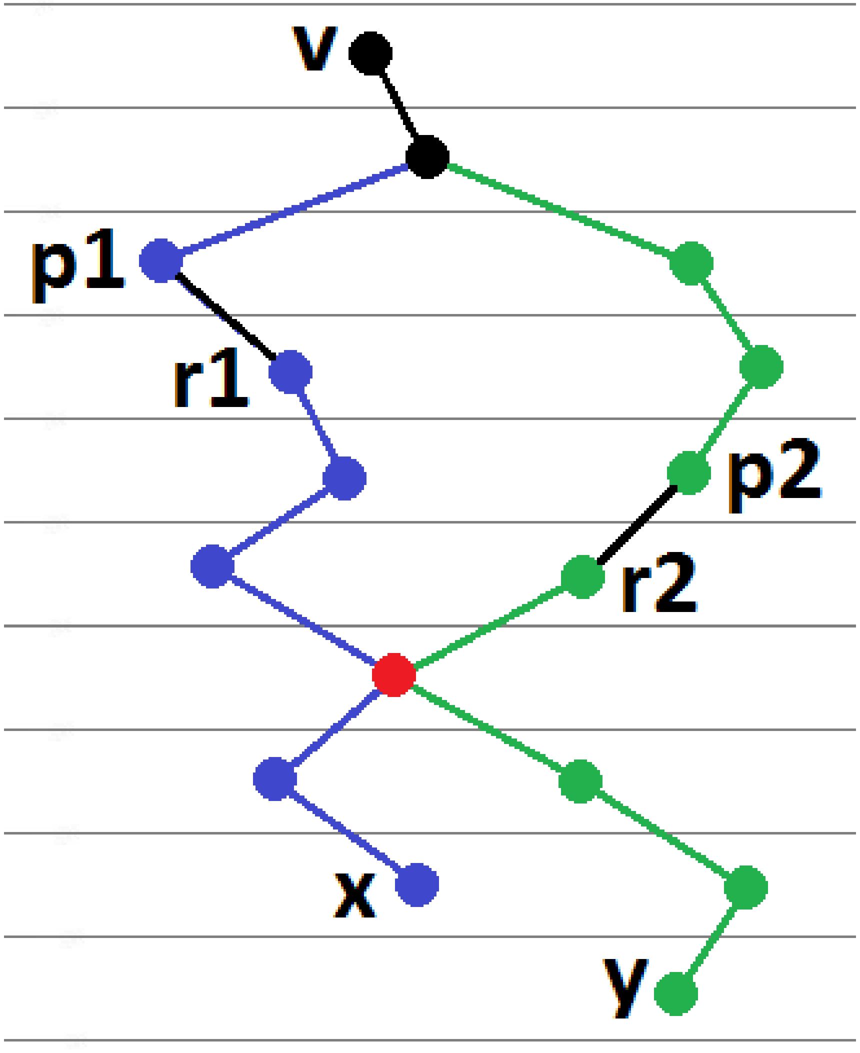
Breadth-First Search in DynFO
(general undirected graphs)

Lemma 3: When an edge (*a*,*b*) separates a set of vertices *T* from the BFS tree of *v*, and *r* and *x* are vertices belonging to *T*, then *path(r,r,x)* cannot pass through (*a*,*b*)



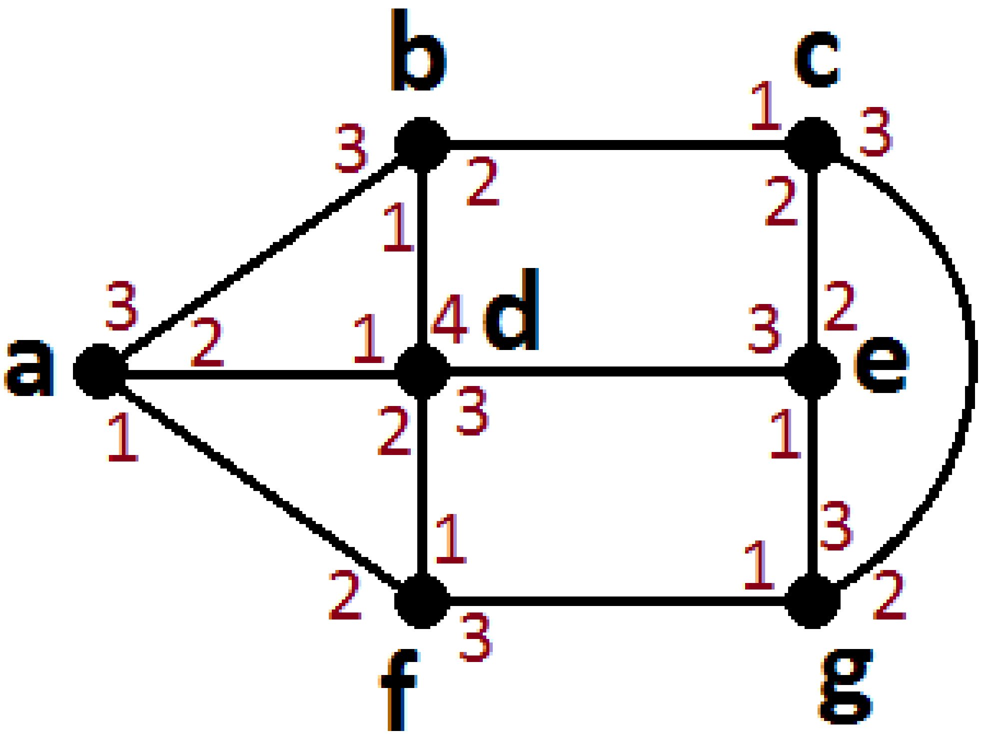
Breadth-First Search in DynFO
(general undirected graphs)

Consistency?



A Theorem of Whitney

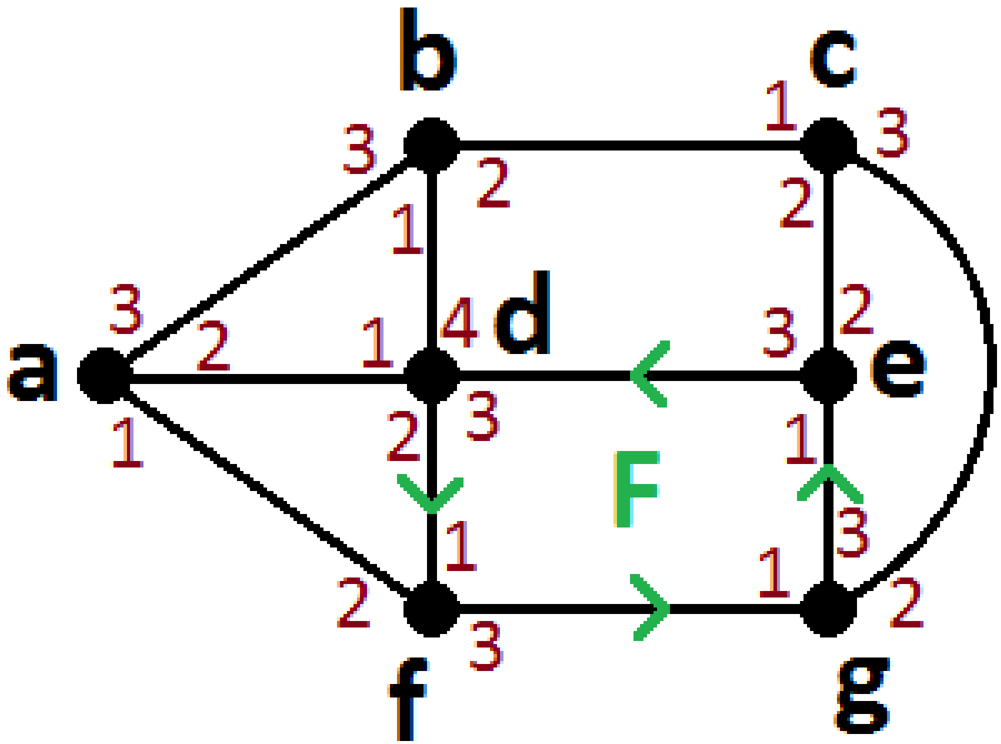
Theorem (Whitney, 1933):
A planar 3-connected graph has a unique embedding on the sphere



Anti/clockwise from *d*:
e b a f e

Impossible to re-draw such that ordering is:
e a f b e

Embedding a planar 3-connected graph



Emb (v, x, n):

(d, a, 1),

(g, e, 3), …

Face (f, x, y, z):

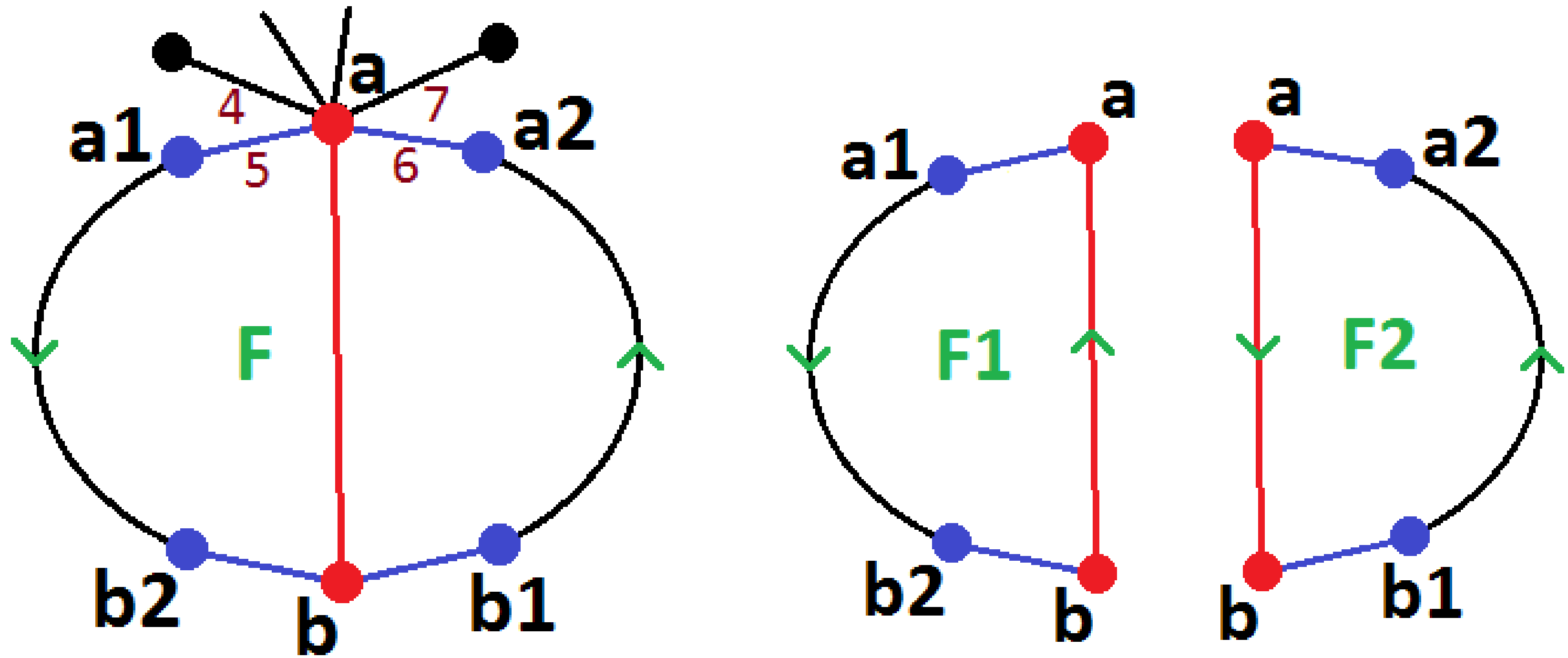
(F, e, g, f),

(F, d, f, d),

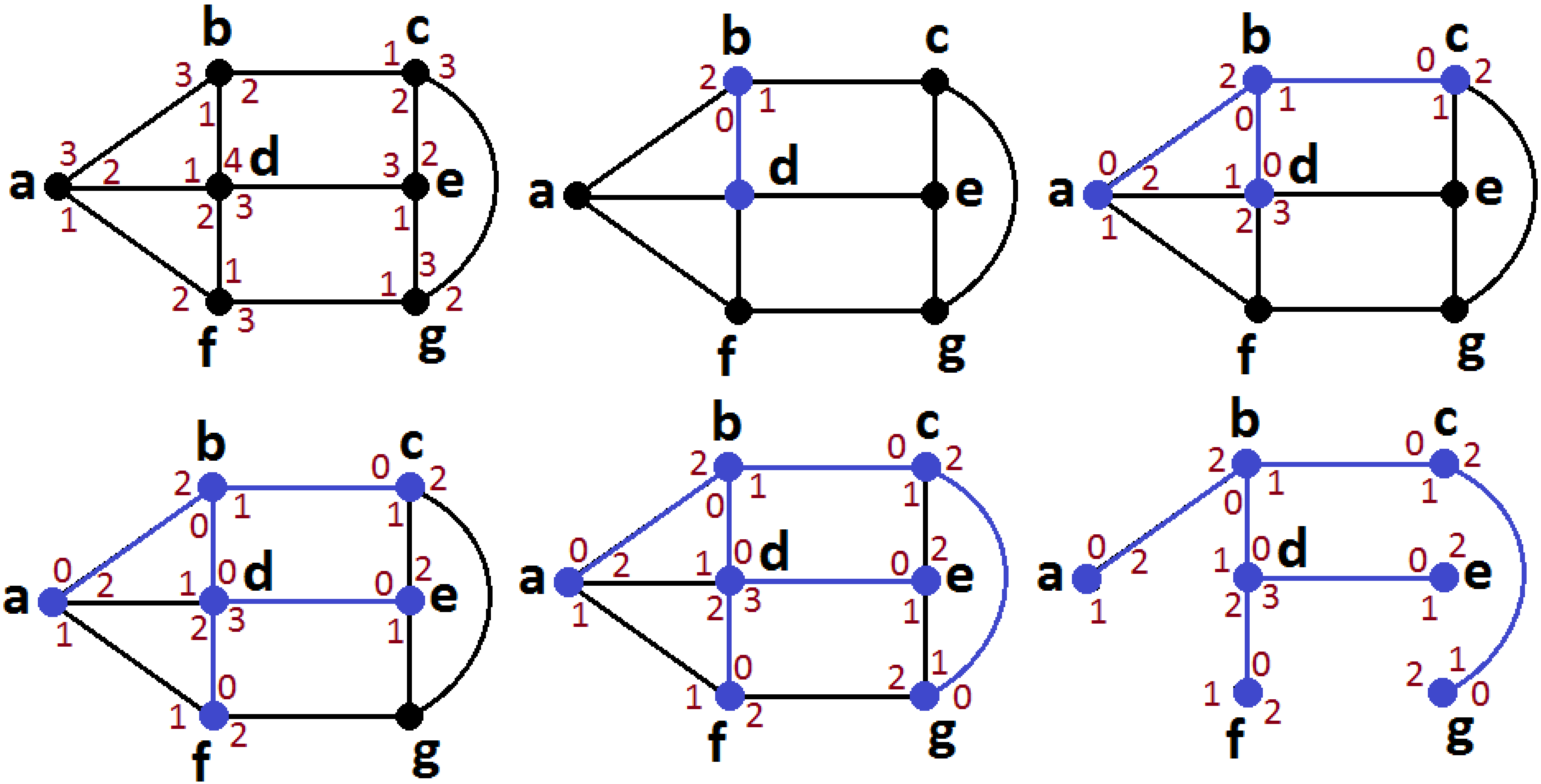
(F, g, d, e), …

Embedding a planar 3-connected graph

Lemma:
Two distinct vertices lie on at most *one* face in a 3-connected planar graph



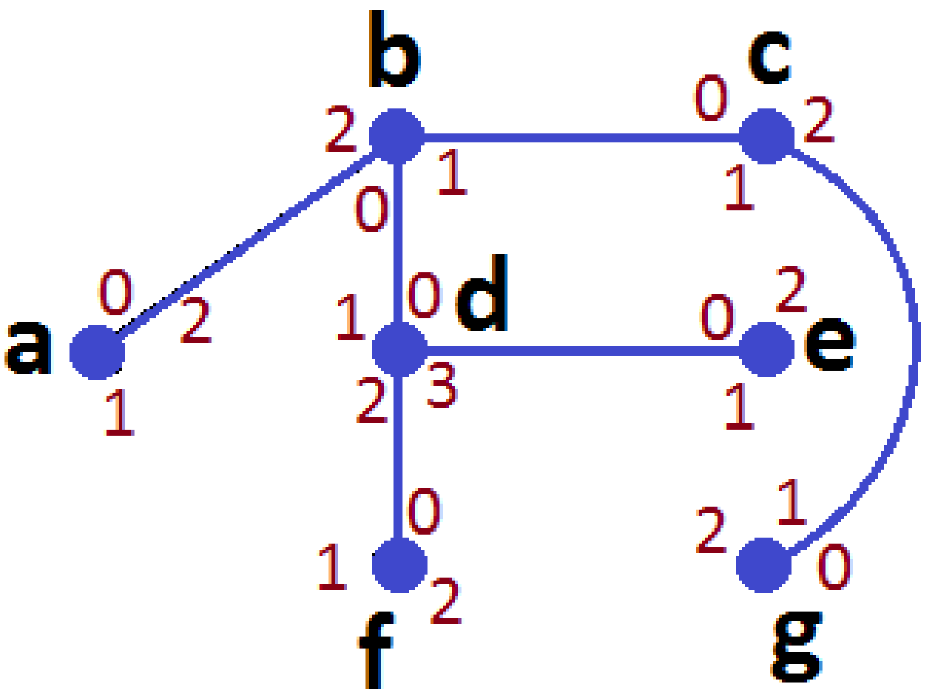
Canonical Breadth-First Search
(Thierauf, Wagner, 2007)



Canonical Breadth-First Search in DynFO+
(planar 3-connected graphs)

Key Idea:
Maintain CBFS-trees from *every* vertex, for *every* edge taken as the starting embedding edge

Canonical Breadth-First Search in DynFO+
(planar 3-connected graphs)



Edge (x, y), Level (v, x, l)

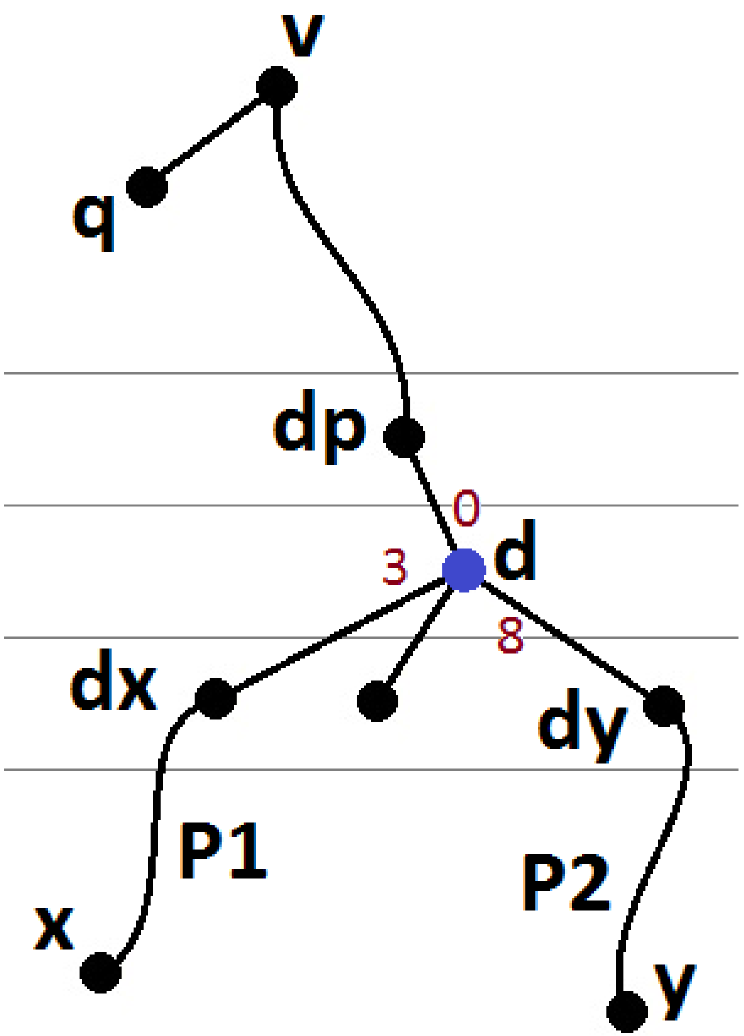
CBFSEdge (v, q, x, y):
(b, d, c, g),
(b, d, b, a), …

CPath (v, q, x, y, z):
(b, d, f, g, c),
(b, d, e, f, d), …

Canonical Breadth-First Search in DynFO+
(planar 3-connected graphs)

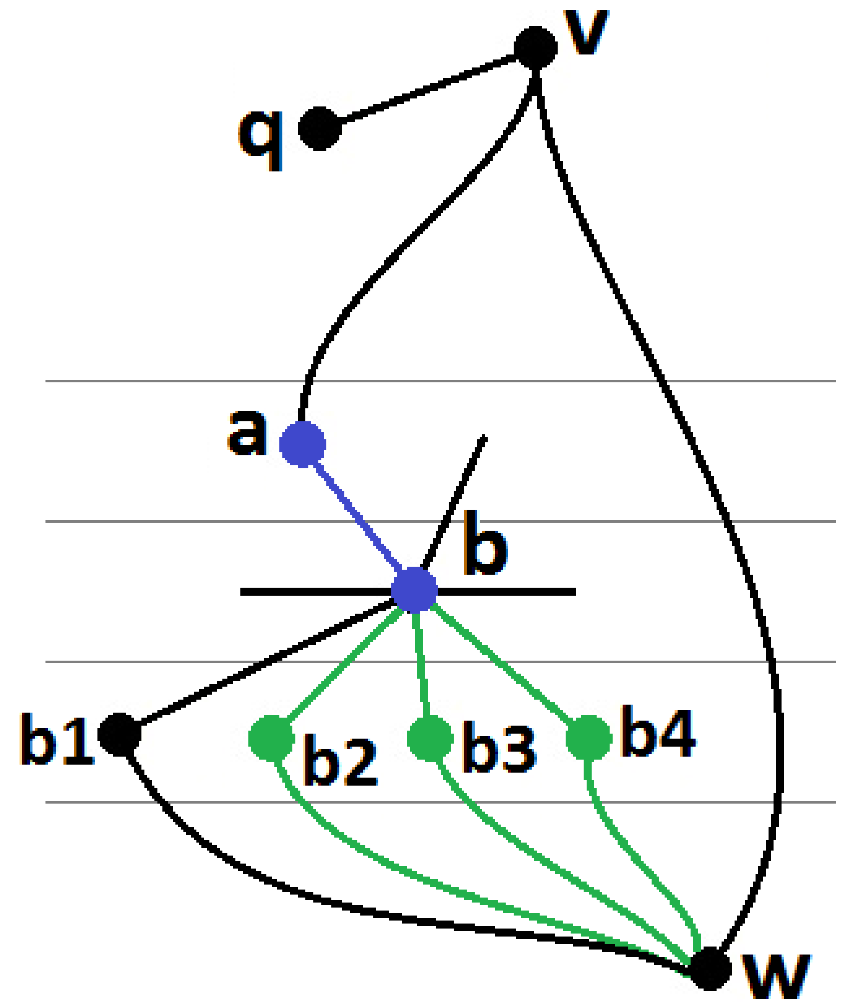
Canonical Ordering on Paths: $P1 <_c P2$ if

- $|P1| < |P2|$ OR
- $|P1| = |P2|$
  AND
  d = lca(x,y),
  emb(d,dp) = 0,
  emb(d,dx) < emb(d,dy)



Canonical Breadth-First Search in DynFO+
(planar 3-connected graphs)

insert (a,b)



Canonical Breadth-First Search in DynFO+

(planar 3-connected graphs)

delete(a,b):

Use $<_c$ relation to find the edge *(p,r)*

Canon from a CBFS tree

Canon(v,q,x) =

{

(l, m) :

for some ancestor w of x,

let pw be the parent of w,

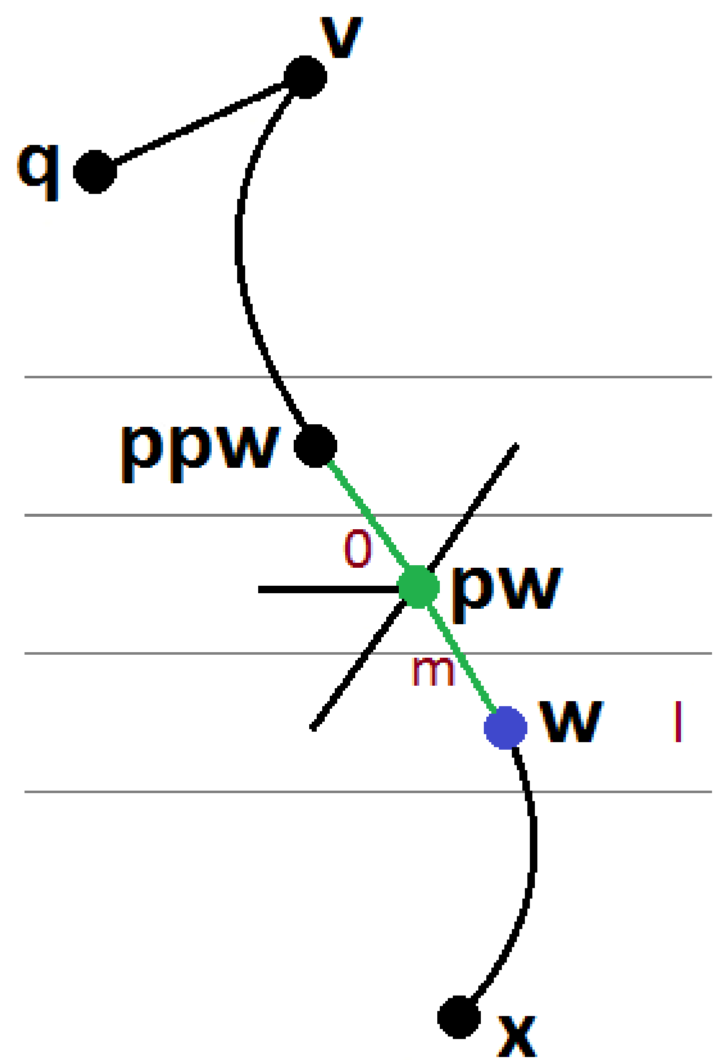
ppw be the parent of pw,

emb (v, q, pw, ppw) = 0,

l = level (v, w) AND

m = emb (v, q, pw, w)

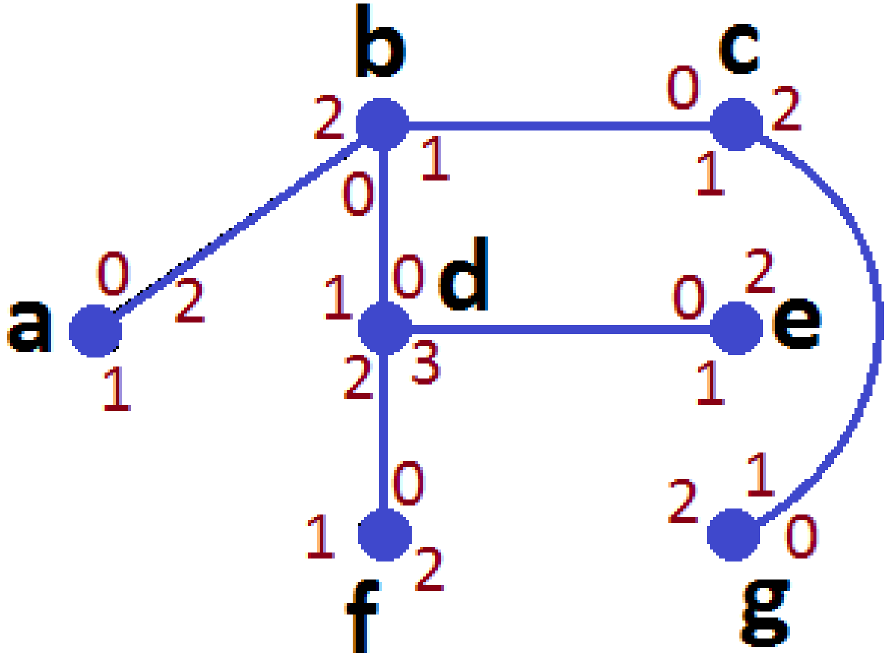
}



Canon from a CBFS tree

Starting vertex: *b*

Starting edge: *(b,d)*



Pre-canon:

(a)= { (b,0), (a,2) }

(b)= { (b,0) }

(c)= { (b,0), (c,1) }

(d)= { (b,0), (d,0) }

(e)= { (b,0), (d,0), (e,3) }

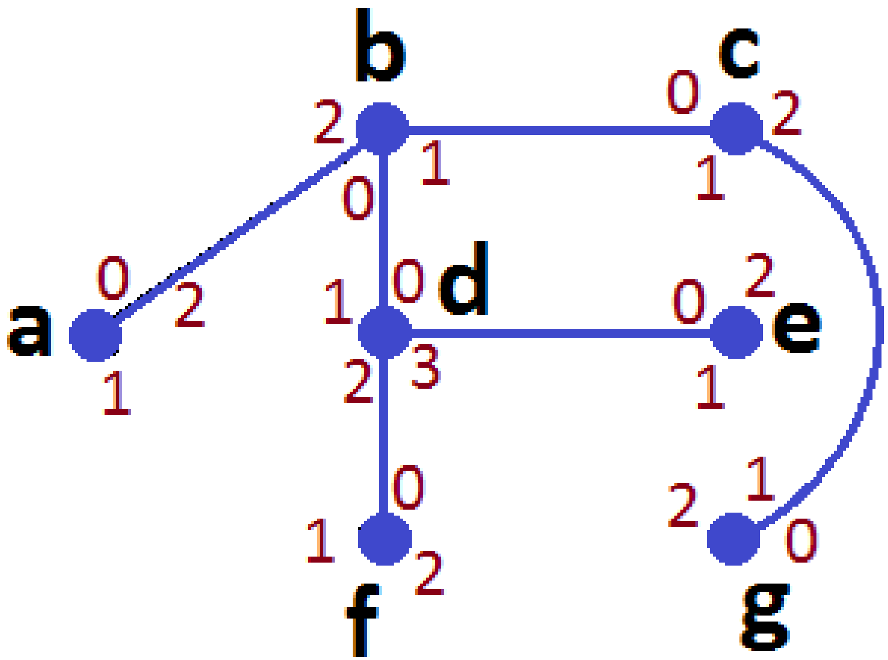
(f)= { (b,0), (d,0), (f,2) }

(g)= { (b,0), (c,1), (g,2) }

Canon from a CBFS tree

Starting vertex: *b*

Starting edge: *(b,d)*



Canon:

(a)= { (0,0), (1,2) }

(b)= { (0,0) }

(c)= { (0,0), (1,1) }

(d)= { (0,0), (1,0) }

(e)= { (0,0), (1,0), (2,3) }

(f)= { (0,0), (1,0), (2,2) }

(g)= { (0,0), (1,1), (2,2) }

Canon from a CBFS tree

Canon:

(a)= { (0,0), (1,2) }

(b)= { (0,0) }

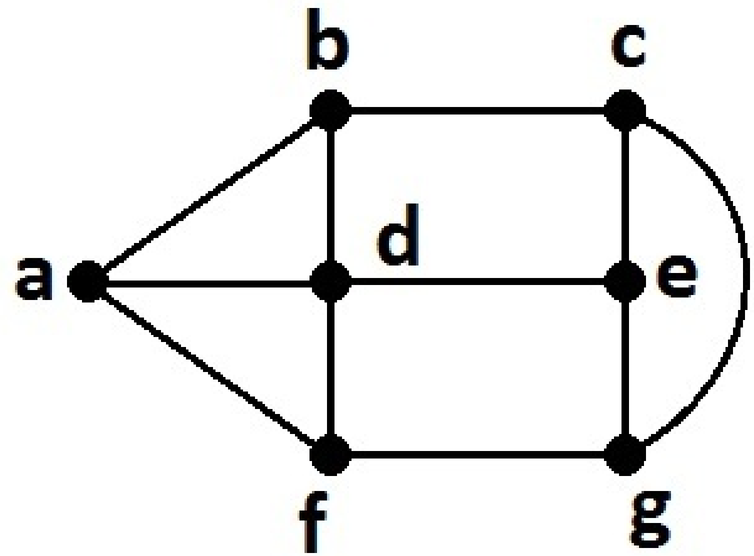
(c)= { (0,0), (1,1) }

(d)= { (0,0), (1,0) }

(e)= { (0,0), (1,0), (2,3) }

(f)= { (0,0), (1,0), (2,2) }

(g)= { (0,0), (1,1), (2,2) }



Canon for the Graph:

Canon(G,b,d) =

{ ({ (0,0), (1,2) },
{ (0,0), (1,0) }), ... }

Isomorphism

Testing for isomorphism between G and H:
Graphs *G* and *H* are isomorphic if and only if:
For some starting vertex/edge pair *(v,q)* in *G*,
There exists a vertex/edge pair *(w,r)* in *H*,
Such that, *Canon(G,v,q) = Canon(H,w,r)*

Open Problems

Is Planar Graph Isomorphism decidable in *DynFO*?

Yes

Does the dynamic version of every language in *L* belong to *DynFO*?

No

(Static Complexity) Upper Bound for *DynFO*?

?

arxiv.org/abs/1312.2141

Thank You