POLYNOMIAL EVALUATION AND INTERPOLATION BY MATRIX METHODS

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SOME EARLIER WORKS ON STRUCTURED MATRICES

SIAM REVIEW, 1992 (BY P.)

POLYNOMIAL AND MATRIX COMPUTATIONS (415 pages), Birkhäuser, 1994 (by D. Bini and P.)

STRUCTURED MATRICES AND POLYNOMIAL (278 pages), Birkhäuser/Springer, 2001 (by P.)

$n \times n$ STRUCTURED MATRIX S

a) IS REPRESENTED WITH $O(n)$ PARAMETERS (vs. n^2)

b) IS LINKED TO POLYNOMIALS AND RATIONAL FUNCTIONS

c) ITS INVERSE (IF IT EXISTS) KEEPS THE STRUCTURE

d) WE NEED $\tilde{O}(n)$ OPS TO COMPUTE $S\mathbf{V}$ (vs. $2n^2 - n$)

e) WE NEED $\tilde{O}(n)$ OPS TO SOLVE $S\mathbf{x} = \mathbf{b}$ (vs. $\approx \frac{2}{3}n^3$).

WE IGNORE LOGARITHMIC FACTORS:

$\tilde{O}(n)$ MEANS $O(n \log^d(n))$ FOR A CONSTANT d

Table : FOUR CLASSES OF STRUCTURED MATRICES

TOEPLITZ $(t_{i-j})_{i,j=0}^{n-1}$ $\begin{pmatrix} t_0 & t_{-1} & \cdots & t_{1-n} \\ t_1 & t_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & \cdots & t_1 & t_0 \end{pmatrix}$	HANKEL $(h_{i+j})_{i,j=0}^{n-1}$ $\begin{pmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ h_1 & h_2 & \iddots & h_n \\ \vdots & \iddots & \iddots & \vdots \\ h_{n-1} & h_n & \cdots & h_{2n-2} \end{pmatrix}$
VANDERMONDE $\left(s_i^j\right)_{i,j=0}^{n-1}$ $\begin{pmatrix} 1 & s_0 & \cdots & s_0^{n-1} \\ 1 & s_1 & \cdots & s_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & s_{n-1} & \cdots & s_{n-1}^{n-1} \end{pmatrix}$	CAUCHY $\left(\frac{1}{s_i - t_j}\right)_{i,j=0}^{n-1}$ $\begin{pmatrix} \frac{1}{s_0 - t_0} & \cdots & \frac{1}{s_0 - t_{n-1}} \\ \frac{1}{s_1 - t_0} & \cdots & \frac{1}{s_1 - t_{n-1}} \\ \vdots & & \vdots \\ \frac{1}{s_{n-1} - t_0} & \cdots & \frac{1}{s_{n-1} - t_{n-1}} \end{pmatrix}$

STRUCTURES AND PROPERTIES CAN BE EXTENDED TO MORE GENERAL CLASSES $\mathcal{T}$, $\mathcal{H}$, $\mathcal{V}_{\mathbf{s}}$, AND $\mathcal{C}_{\mathbf{s},\mathbf{t}}$

TOEPLITZ AND HANKEL MATRICES ARE USED IN SIGNAL AND IMAGE PROCESSING, ODE, PDE, INTEGRAL EQUATIONS, MARKOV CHAINS, QUEUING THEORY ETC.

WE COVER VANDERMONDE AND CAUCHY MATRICES WITH APPLICATIONS TO CLASSICAL POLYNOMIAL AND RATIONAL EVALUATION AND INTERPOLATION, SKIPPING THEIR OTHER IMPORTANT APPLICATIONS (STUDY OF POTENTIAL FIELDS, PARTICLE SIMULATION ETC.)

$\mathcal{T}$, $\mathcal{H}$, $\mathcal{V}$, AND $\mathcal{C}$ HAVE DISTINCT PROPERTIES

CAUCHY AND CAUCHY-LIKE MATRICES HAVE RATIONAL ENTRIES.

COLUMN INTERCHANGE KEEPS CAUCHY-LIKE STRUCTURE, BUT DESTROYS THE STRUCTURES OF $\mathcal{T}$, $\mathcal{H}$, $\mathcal{V}$.

$C_{\mathbf{as},\mathbf{at}} = \frac{1}{a} C_{\mathbf{s},\mathbf{t}}$ FOR ANY SCALAR $a \neq 0$ AND

$$C_{\mathbf{s},\mathbf{t}} = \left(\frac{1}{s_i - t_j}\right)_{i,j=0}^{n-1} = \left(\frac{1}{s_i - c - (t_j - c)}\right)_{i,j=0}^{n-1} = -C_{\mathbf{t},\mathbf{s}}^T,$$

BUT NOTHING SIMILAR IS KNOWN FOR $V_s = (s_i^j)_{i,j=0}^{n-1}$.

$\mathcal{T}$, $\mathcal{H}$, $\mathcal{V}$, AND $\mathcal{C}$ HAVE DISTINCT PROPERTIES, BUT ...

THE REVERSION (UNIT HANKEL) MATRIX

$$J = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \iddots & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

LET US REVERSE VECTORS AND MATRICES:

$$J \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = \begin{pmatrix} v_{n-1} \\ v_{n-2} \\ \vdots \\ v_0 \end{pmatrix}$$

$J^2 = I$, TJ IS H, JT IS H, HJ IS T, JH IS T,
$\mathcal{T}\mathcal{J}$ IS $\mathcal{H}$, $\mathcal{J}\mathcal{T}$ IS $\mathcal{H}$, $\mathcal{H}\mathcal{J}$ IS $\mathcal{T}$, $\mathcal{J}\mathcal{H}$ IS $\mathcal{T}$.

$$H^{-1} = J(HJ)^{-1},\ H\mathbf{b} = J(JH)\mathbf{b},\ H\mathbf{y} = \mathbf{b} \Longrightarrow JH\mathbf{y} = J\mathbf{b}.$$

$\mathcal{T}$, $\mathcal{H}$, $\mathcal{V}$, AND $\mathcal{C}$ HAVE DISTINCT PROPERTIES, BUT

$V_{\mathbf{s}}\mathcal{T}$ IS IN $\mathcal{V}_{\mathbf{s}}$, $V_{\mathbf{s}}^{T}\mathcal{V}_{\mathbf{s}}$ IS IN $\mathcal{H}$, $\mathcal{V}_{\mathbf{s}}V_{\mathbf{t}}^{-1}$ IS IN $\mathcal{C}_{\mathbf{s},\mathbf{t}}$, $\mathcal{C}_{\mathbf{s},\mathbf{t}}\mathcal{T}_{\mathbf{t}}$ IS IN $\mathcal{V}_{\mathbf{t}}$.

FAST INVERSION IN ONE OF THE CLASSES $\mathcal{T}$, $\mathcal{H}$, $\mathcal{V}$, OR $\mathcal{C}$, ENABLES FAST INVERSION IN ALL 4 CLASSES; THE SAME FOR PRODUCTS BY A VECTOR AND SOLVING A LINEAR SYSTEM OF EQUATIONS.

P89 (ISSAC), P90 (MATH OF COMP)

"HORNER'S" POLYNOMIAL EVALUATION

$$((\dots(p_n x + p_{n-1})x + \dots + p_2)x + p_1)x + p_0$$

USES n MULTIPLICATIONS AND n ADDITIONS

TO EVALUATE A POLYNOMIAL

$p_n x^n + \dots, p_2 x^2 + p_1 x + p_0$ AT A POINT $x = s$.

THIS IS OPTIMAL P63 (Prob. Kib), P66(UMN),

BUT NOT SO FOR EVALUATION AT n POINTS $s_0, s_1, \dots, s_{n-1}$

PROBLEM 1. MULTIPOINT POLYNOMIAL EVALUATION OR VANDERMONDE-BY-VECTOR MULTIPLICATION

PROBLEM 2. POLYNOMIAL INTERPOLATION OR A VANDERMONDE LINEAR SYSTEM OF EQUATIONS

$$\begin{pmatrix} 1 & s_0 & \cdots & s_0^{n-1} \\ 1 & s_1 & \cdots & s_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & s_{n-1} & \cdots & s_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{n-1} \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}$$

OR EQUIVALENTLY

$$v_i = p(s_i), \ \ i = 0, \ldots, n-1$$

FOR $\quad p(x) = p_0 + p_1 x + \cdots + p_{n-1} x^{n-1}$

PROBLEM 3. MULTIPOINT RATIONAL EVALUATION OR CAUCHY-BY-VECTOR MULTIPLICATION

PROBLEM 4. RATIONAL INTERPOLATION OR A CAUCHY LINEAR SYSTEM OF EQUATIONS

$$\begin{pmatrix} \frac{1}{s_0-t_0} & \cdots & \frac{1}{s_0-t_{n-1}} \\ \frac{1}{s_1-t_0} & \cdots & \frac{1}{s_1-t_{n-1}} \\ \vdots & & \vdots \\ \frac{1}{s_{n-1}-t_0} & \cdots & \frac{1}{s_{n-1}-t_{n-1}} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}$$

OR EQUIVALENTLY

$$v_i = \sum_{j=0}^{n-1} \frac{u_j}{s_i - t_j}, \;\; i = 0, \ldots, n-1.$$

THE FOUR PROBLEMS CAN BE SOLVED IN NEARLY LINEAR ARITHMETIC TIME, BY USING

$O(n \log^2(n)) = \tilde{O}(n)$ AR. OPS,

BUT THE KNOWN NUMERICAL ALGORITHMS

RUN IN QUADRATIC ARITHMETIC TIME, OF ORDER n^2.

OUR NUMERICAL ALGORITHMS USE

$O(n\log^2(n/\epsilon)) = \tilde{O}(n)$ AR. OPS FOR ϵ-EVALUATION,

AND

$O(n\log^3(n/\epsilon)) = \tilde{O}(n)$ FOR ϵ-INTERPOLATION.

$\log(n/\epsilon)$ is $O(\log n)$ if $\epsilon = 1/n^{O(1)}$.

EXTENSIONS:

VANDERMONDE-LIKE $\mathcal{V}_{\mathbf{s}}$,

CAUCHY-LIKE $\mathcal{C}_{\mathbf{s},\mathbf{t}}$,

TRANSPOSES $V_{\mathbf{s}}^T$ AND $\mathcal{V}_{\mathbf{s}}^T$.

NOTE THAT $\mathcal{C}_{\mathbf{s},\mathbf{t}}^T = -\mathcal{C}_{\mathbf{t},\mathbf{s}}$

1. CV MATRICES.

2. HSS (LOCALLY LOW-RANK) MATRICES. FAST COMPUTATIONS WITH THEM

3. CV MATRICES $\approx$ EXTENDED HSS MATRICES.

4. FAST NUMERICAL COMPUTATIONS WITH CV MATRICES.

5. EXTENSIONS AND CHALLENGES.

OUR BASIC TOOL IS

TRANSFORMATION OF MATRIX STRUCTURES

Table : VANDERMONDE AND CAUCHY MATRICES

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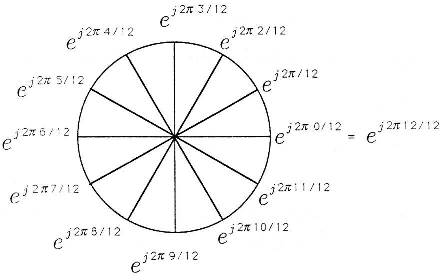
$V = V_{\mathbf{s}} = \left(s_i^j\right)_{i,j=0}^{n-1}$	$C = C_{\mathbf{s},\mathbf{t}} = \left(\frac{1}{s_i - t_j}\right)_{i,j=1}^{n}$
$\begin{pmatrix} 1 & s_1 & \cdots & s_1^{n-1} \\ 1 & s_2 & \cdots & s_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & s_n & \cdots & s_n^{n-1} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{s_1 - t_1} & \cdots & \frac{1}{s_1 - t_n} \\ \frac{1}{s_2 - t_1} & \cdots & \frac{1}{s_2 - t_n} \\ \vdots & & \vdots \\ \frac{1}{s_n - t_1} & \cdots & \frac{1}{s_n - t_n} \end{pmatrix}$

[MRT 05]: ANOTHER DIFFERENCE BETWEEN C AND V

MATRICES C CAN BE APPROXIMATED BY HSS MATRICES, WHICH HAVE LOCALLY LOW RANKS.

FAST MULTIPLOLE ALGORITHM (FMM) OPERATES WITH THEM FAST.
FMM IS ONE OF THE TEN MOST IMPORTANT ALGORITHMS OF XX CENTURY.

$\Longrightarrow$ TRANSFORM V INTO C; $\mathcal{V}$ INTO $\mathcal{C}$.



n-TH ROOTS OF UNITY: $1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$,

$\omega^n = 1$. $\omega = \exp(2\pi\sqrt{-1}/n)$ IS A PRIMITIVE *n*TH ROOT OF UNITY.

$$\Omega = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ \vdots & \vdots & & \vdots & \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{pmatrix} = (\omega^{ij})_{i,j=0}^{n-1}.$$

Ω IS THE MATRIX OF DISCRETE FOURIER TRANSFORM (DFT). IT IS A SPECIAL VANDERMONDE MATRIX.

FFT IS ALSO IN THE LIST OF TEN MOST IMPORTANT ALGORITHMS OF XX CENTURY.

IT MULTIPLIES Ω AND Ω^{-1} BY A VECTOR BY USING $1.5n\log_2(n)$ (RESP., $1.5n\log_2(n)+n$) AR. OPS,

VS. ORDER OF n^2, (RESP. n^3) FOR GENERAL $n \times n$ MATRIX.

FFT LINKS THE CLASSES $V_{\mathbf{s}}$ WITH $C_{\mathbf{s},\mathbf{t}(\Omega)}$ (CV MATRICES),

$\mathcal{V}_{\mathbf{s}}$ WITH $\mathcal{C}_{\mathbf{s},\mathbf{t}(\Omega)}$ (CV-LIKE MATRICES),

THAT IS, FFT LINKS VANDERMONDE AND CAUCHY MATRICES AND STRUCTURES.

$\mathbf{t}(\Omega) = (1, \omega, \omega^2, \ldots, \omega^{n-1})^T$ IS FILLED WITH ROOTS OF UNITY,
$\omega = \exp((2\pi/n)\sqrt{-1})$ IS A PRIMITIVE ROOT OF 1,
AND $\mathbf{s} = (s_i)_{i=0}^{n-1}$ WITH ANY (UNRESTRICTED) s_i.

$$V_{\mathbf{s}} = \frac{1}{\sqrt{n}} \operatorname{diag}\left(s_i^n - 1\right)_{i=0}^{N-1} C_{\mathbf{s},\mathbf{t}(\Omega)} \operatorname{diag}(\omega^j)_{j=0}^{n-1} \Omega.$$

WE CAN MULTIPLY A LOW-RANK MATRIX BY A VECTOR FAST.

AN 9×7 MATRIX M OF A RANK 2 HAS A GENERATOR OF LENGTH 2: $M = FG$, F IS 9×2 and G IS 2×7:

$$M = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \\ f_{41} & f_{42} \\ f_{51} & f_{52} \\ f_{61} & f_{62} \\ f_{71} & f_{72} \\ f_{81} & f_{82} \\ f_{91} & f_{92} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} & g_{17} \\ g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & 26 & g_{27} \end{pmatrix}.$$

NUMBER OF ENTRIES IS $(9+7)*2 < 9*7$,
GENERALLY $(m+n)\rho \ll mn$ IF $\rho \ll m$, $\rho \ll n$

NUMBER OF FLOPS IS $(2m-1)n$ FOR $M\mathbf{v}$;
IT IS $2(m+n)\rho - n - \rho$ FOR $FG\mathbf{v}$

WE CAN MULTIPLY A BANDED MATRIX BY A VECTOR FAST.

A (1,2)-BANDED MATRIX

$$M = \begin{pmatrix} \Sigma_0 & B_0 & C_0 & O & O & O & O & O \\ A_1 & \Sigma_1 & B_1 & C_1 & O & O & O & O \\ O & A_2 & \Sigma_2 & B_2 & C_2 & O & O & O \\ O & O & A_3 & \Sigma_3 & B_3 & C_3 & O & O \\ O & O & O & A_4 & \Sigma_4 & B_4 & C_4 & O \\ O & O & O & O & A_5 & \Sigma_5 & B_5 & C_5 \\ O & O & O & O & O & A_6 & \Sigma_6 & B_6 \\ O & O & O & O & O & O & A_7 & \Sigma_7 \end{pmatrix}.$$

(l, u)-BANDED MATRIX BY A VECTOR: $O((l+u)n)$ FLOPS.

(l, u)-BANDED LIN·SOLVE: $O((l+u)^2 n)$ FLOPS.

WE CAN MULTIPLY AN HSS AND AN EXTENDED HSS MATRIX BY A VECTOR FAST.

1. AN $n \times n$ HSS MATRIX HAS A BLOCK DIAGONAL WITH, SAY, $O(n \log(n))$ ENTRIES

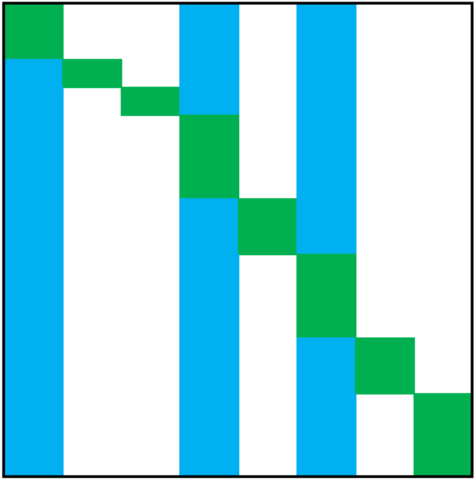
2. RANKS ρ OF ITS OFF-DIAGONAL BLOCKS ARE, SAY, $O(\log(n))$.
AND EACH OFF-DIAGONAL BLOCK HAS A GENERATOR (F, G) OF LENGTH $\leq \rho = O(\log(n))$.

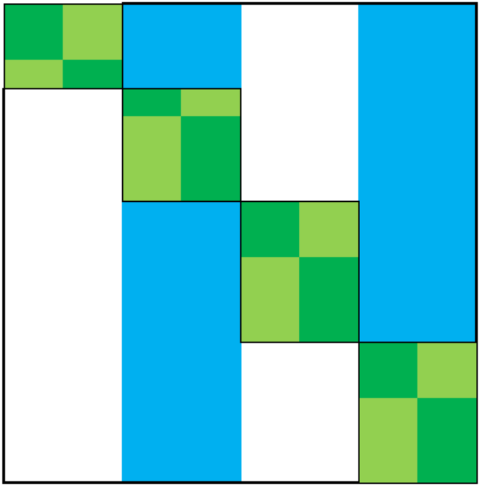
3. THE INVERSE IS AGAIN HSS MATRIX WITH THE SAME PROPERTIES.

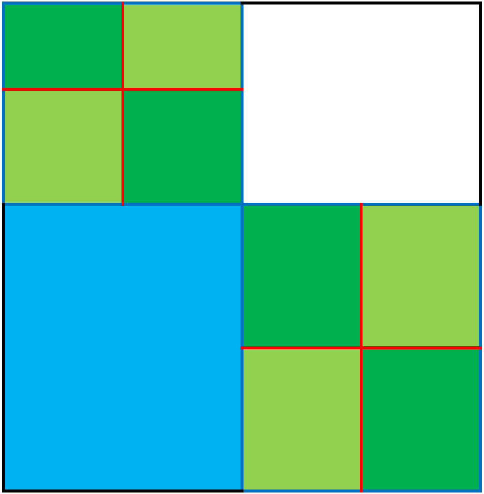
BANDED MATRICES AND THEIR INVERSES ARE A SPECIAL CASE OF HSS MATRICES.

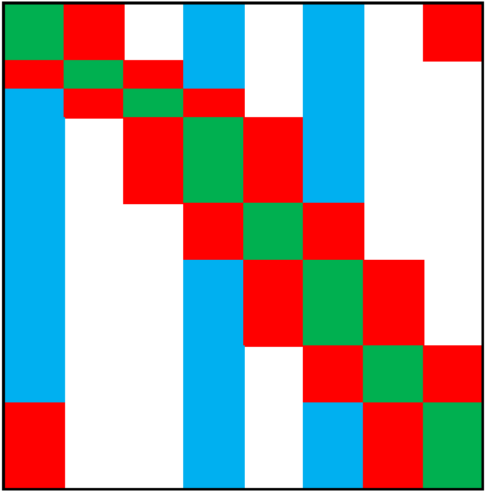
THE FMM EXTENDS THE COST BOUNDS ABOVE TO HSS MATRICES: USE SHORT GENERATORS FOR OFF-DIAGONAL BLOCKS

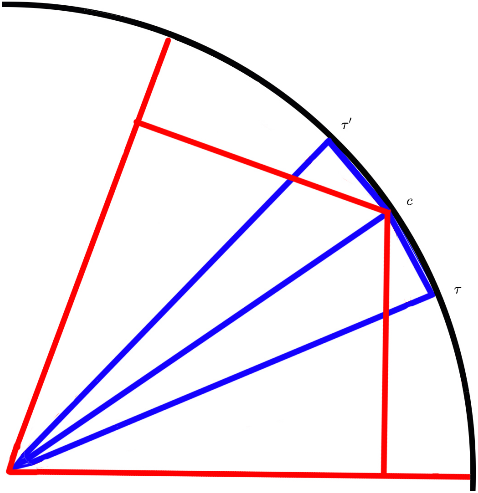
WE WORK WITH EXTENDED HSS MATRICES, AND EXTEND THE ALGORITHMS AND COST BOUNDS











τ'
c
τ

[R85], [MRT05], [CGS07]:
ASSUME $C = (\frac{1}{s_i - t_j})_{i,j=0}^{n-1}$ AND GLOBAL COMPLEX CENTER c
SUCH THAT $|q_{i,j}| = |\frac{t_j - c}{s_i - c}|$ ARE SMALL FOR ALL (i, j).

THEN C CAN BE APPROXIMATED BY A MATRIX M OF A LOW RANK.

LEMMA. WRITE $q = \frac{t-c}{s-c}$. LET $|q| \leq \theta < 1$. (THE PAIR s AND t IS (θ, c)-SEPARATED.) THEN

$$\frac{1}{s-t} = \frac{1}{s-c} \sum_{h=0}^{\rho-1} \frac{(t-c)^h}{(s-c)^h} + \frac{q_\rho}{s-c}, \quad |q_\rho| = \frac{|q|^\rho}{1-|q|} \leq \frac{\theta^\rho}{1-\theta}. \tag{0.1}$$

Proof.

$\frac{1}{s-t} = \frac{1}{s-c}\frac{1}{1-q}$, $\frac{1}{1-q} = \sum_{h=0}^{\infty} q^h$ (NEWMAN'S EXPANSION).

SO $\frac{1}{1-q} = \sum_{h=0}^{\rho-1} q^h + \sum_{h=\rho}^{\infty} q^h = \sum_{h=0}^{\rho-1} q^h + \frac{q^\rho}{1-q}$. □

Theorem

*[MRT05], [CGS07]. Assume two integers k and n such that $0 < k < n$ and a Cauchy matrix $C = (\frac{1}{s_i - t_j})_{i,j=0}^{n-1}$. Suppose all pairs (s_i, t_j) are (θ, c)-*separated *from one another for $0 < \theta < 1$ and a center c. Then $C = FG^T + \Delta$ where $\mathrm{rank}(\widehat{C}) \le k+1$, $F = (1/(s_i - c)^{h+1})_{i,h=0}^{n-1,k}$, $G^T = ((t_j - c)^h)_{j,h=0}^{n,k}$, and $|\Delta| \le \frac{\theta^k}{(1-\theta)\delta}$ for $\delta = \min_{i=0}^{n-1} |s_i - c|$. (One can compute the matrices F and G by using $2kn + 2n - 2$ ar. ops.)*

Proof.

Apply (0.1) for $s = s_i$, $t = t_j$ and all pairs $\{i, j\}$. □

GENERALLY A CAUCHY MATRIX $C = (\frac{1}{s_i - t_j})_{i,j=0}^{n-1}$ HAS NO SUCH GLOBAL CENTERS, BUT FOR ANY CV MATRIX C WE SEEK A SET OF LOCAL CENTERS THAT SUPPORT APPROXIMATION BY AN HSS MATRIX, WHICH STILL CAN BE MULTIPLIED BY A VECTOR AND INVERTED FAST.

WE SEEK A BLOCK DIAGONAL OF C WITH $O(n \log n)$ ENTRIES OVERALL SUCH THAT EVERY OFF-DIAGONAL BLOCK B THE RATIOS $|q_{i,j}| = |\frac{t_j - c(B)}{s_i - c(B)}|$ ARE SMALL FOR SOME CENTER $c = c(B)$ AND ALL PAIRS (s_i, t_j) IN THE BLOCK.

$C = C_{\mathbf{s},\mathbf{t}} = \left(\frac{1}{s_i - t_j}\right)_{i,j=1}^{n}$, $t_j = \omega^j$ and s_i UNRESTRICTED.

DEFINE LOCAL CENTERS AND DIAGONAL BLOCKS

1. REPRESENT ALL $s_i = |s_i| \exp(\phi_i \sqrt{-1})$ IN POLAR COORDINATES AND REENUMERATE THEM IN NONDECREASING ORDER OF ARGUMENTS ϕ_i. SIMILARLY DO FOR ALL $t_j = \exp(\psi_j \sqrt{-1})$.
2. PARTITION THE UNIT CIRCLE $z: \ |z| = 1$ INTO k ARCS $\mathcal{A}_q$ OF THE SAME LENGTH $2\pi/k$. CHOOSE LOCAL CENTERS AT THE MIDPOINTS.
3. DEFINE BLOCKS $C = (C_{pq})_{p,q=1}^{k}$, $C_{pq} = \left(\frac{1}{s_i - t_j}\right)_{\phi_i \in \mathcal{A}_p, \psi_j \in \mathcal{A}_q}$

WE HAVE APPROXIMATED A CV (CAUCHY-VANDERMONDE) MATRIX BY HSS MATRIX, WHOSE OFF-TRIDIAGONAL (ADMISSIBLE) BLOCKS HAVE SMALL RANKS

OUR PROGRESS:

WE USE $O(n\log^2(n/\epsilon)) = \tilde{O}(n)$ AR. OPS FOR ϵ-EVALUATION,

WE USE $O(n\log^3(n/\epsilon)) = \tilde{O}(n)$ FOR ϵ-INTERPOLATION.

$\log(n/\epsilon)$ is $O(\log n)$ if $\epsilon = 1/n^{O(1)}$.

MORE ADVANCED TECHNIQUES CAN PROBABLY SAVE A LOGARITHMIC FACTOR FOR MULTIPLICATION BY A VECTOR

WE CAN EXTEND THE RESULTS TO VANDERMONDE TRANSPOSES, TO MATRICES
WITH A STRUCTURE OF VANDERMONDE TYPE, AND

TO A LARGE CLASS OF MATRICES

WITH A STRUCTURE OF CAUCHY TYPE (REPRESENTING PROBLEMS 3 AND 4 OF RATIONAL EVALUATION AND INTERPOLATION)

1. VANDERMONDE AND CAUCHY LINKS

2. APPROXIMATION OF CAUCHY BY HSS MATRICES.

3. CV, CV^T MATRICES, AND EXTENDED HSS MATRICES.

4. FASTER VANDERMONDE AND CAUCHY COMPUTATIONS, EXTENSIONS

5. A CHALLENGE: NEW TRANSFORMATIONS OF MATRIX STRUCTURES WITH ALGORITHMIC APPLICATIONS BY EXTENDING [P 1990] (MATH. OF COMP.)

CAUCHY-LIKE LINEAR SYSTEM $\Longrightarrow$ CV-LIKE LINEAR SYSTEM + CV-BY-VECTOR

$$C_{\mathbf{s},\mathbf{t}}\mathbf{x} = \mathbf{b}.$$

WRITE $\mathbf{x} = C_{\mathbf{t},\mathbf{t}(\Omega)}\mathbf{y}$,

$\tilde{C} = C_{\mathbf{s},\mathbf{t}} C_{\mathbf{t},\mathbf{t}(\Omega)}\mathbf{y}$ IS CV-LIKE.

THEN $\tilde{C}\mathbf{y} = \mathbf{b}$.