On Lower Bounds for Multiplicative Circuits and Linear Circuits in Noncommutative Domains

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CSR 2014

Outline

1. Lower Bounds for Multiplicative Circuits.
   1.1 lower bounds for Circuits over free monoids.
   1.2 lower bounds for Circuits over permutation groups.
2. Lower Bounds for Linear Circuits over Noncommutative Rings

Multiplicative Circuits

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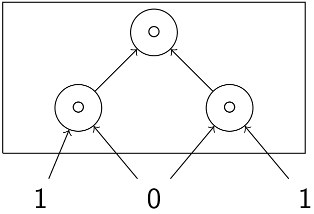

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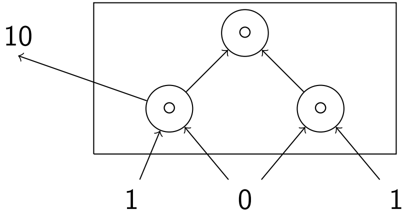

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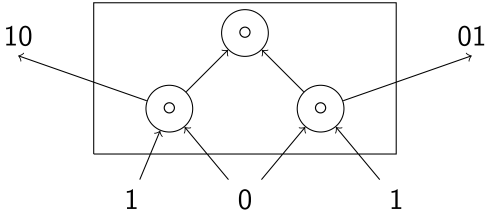

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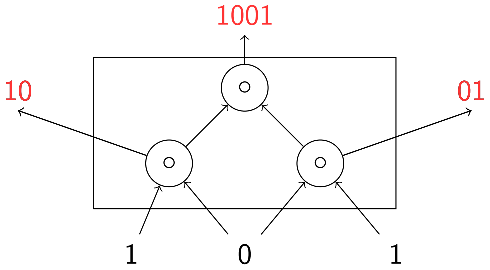

- fanin $= 2$

- size $=$ number of gates in the circuit

- multi-output circuit

Our Results

Theorem (Lower Bounds for Circuits over free Monoids)

Let $S = \{y_1, ..., y_n\} \subseteq \{0,1\}^n$ be the explicit set of n strings. Any concatenation circuit that takes $X = \{0,1\}$ as input and outputs $y_1, ..., y_n$ will require size $\Omega(\frac{n^2}{\log^2 n})$.

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Theorem (Lower Bounds for Circuits over permutation group)

Any composition circuit over the permutation group $(S_N, \cdot)$, *with domain size* $N = 2^{\frac{n^2}{\log^2 n}}$, *that takes as input* π_0, π_1 *and computes* $G_S = \{\pi_{y_i} | y_i \in S\} \subseteq S_N$ *as output is of size* $\Omega(\frac{n^2}{\log^2 n})$.

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Lower Bounds for Circuits over free Monoids, contd

Lemma

Let $s \in X^n$ be any string where $|X| \geq 2$, such that the number of distinct substrings of s of length ℓ is N. Then any concatenation circuit for s will require $\Omega(\frac{N}{\ell})$ gates.

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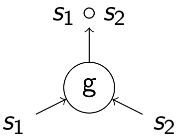

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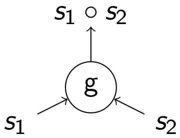

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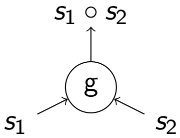

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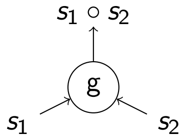

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- $\#$ of **new** substrings of length ℓ generated at any g is $\leq \ell - 1$ and this gives, $|C| = \Omega(\frac{N}{\ell})$. $\blacksquare$

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Let $S = \{y_1, ..., y_n\} \subseteq \{0,1\}^n$ be the explicit set of n strings defined above. Any concatenation circuit that takes $X = \{0,1\}$ as input and outputs S at its n output gates will require size $\Omega(\frac{n^2}{\log^2 n})$.

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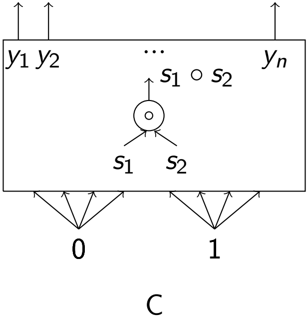
Proof

- Let C be a concatenation circuit computing S.
- Note that each $y_i \in S$ have $\Omega(\frac{n}{\log n})$ distinct substrings of length $O(\log n)$.
- In total $y_1, y_2, ..., y_n$ have $\Omega(\frac{n^2}{\log n})$ distinct substrings of length $O(\log n)$.

Lower Bounds for Circuits over free Monoids, contd



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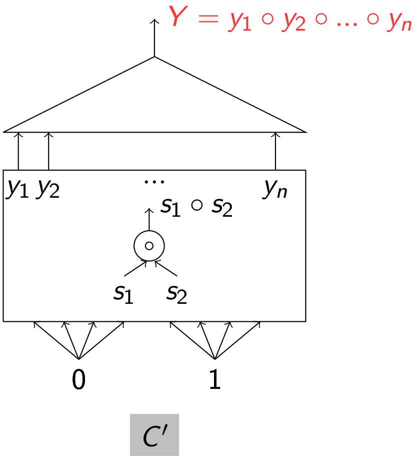

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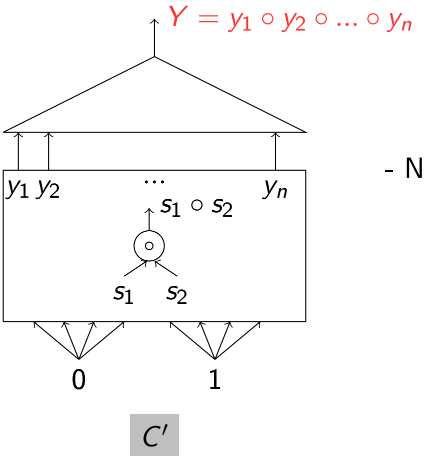

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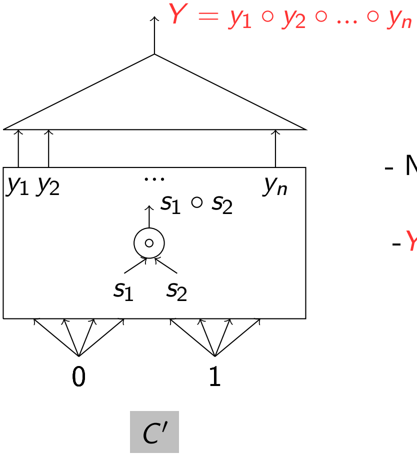

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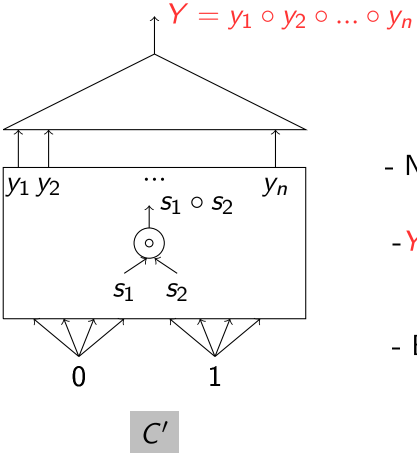

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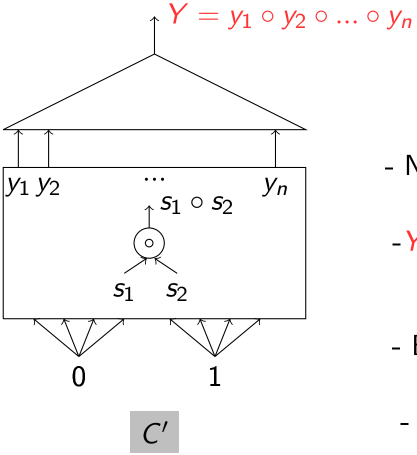

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2. Circuits over permutation groups

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Definition of generating elements: π_0 and π_1

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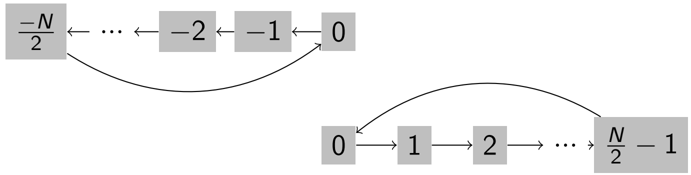
π_0

π_1



- Let $y = 101$. By π_y we mean the permutation $\pi_1\pi_0\pi_1$.

Definition: $G_S = \{\pi_{y_i} | y_i \in S\}$, where the set S be the explicit set of n strings defined before.

Goal: To compute G_S using composition circuits

Theorem

Any composition circuit over the permutation group $(S_N, \cdot)$, with domain size $N = 2^{\frac{n^2}{\log^2 n}}$, that takes as input π_0, π_1 and computes $G_S = \{\pi_{y_i} | y_i \in S\} \subseteq S_N$ as output is of size $\Omega(\frac{n^2}{\log^2 n})$.

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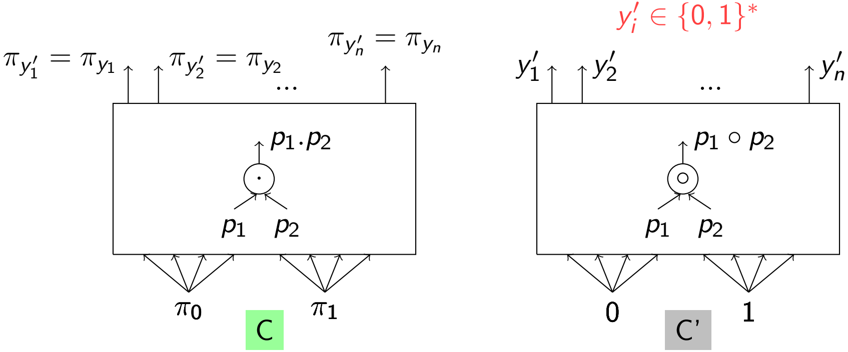

- Note that $|C| = |C'|$.

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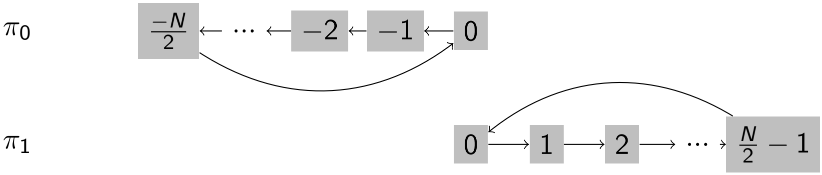
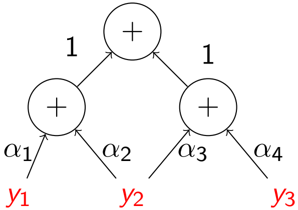


- Thus, u have at least $(\frac{N}{2} - 1)$ copies of π_1 and since fanin is 2, $|C| \geq \Omega(\log(\frac{N}{2} - 1))$, where $N = 2^{\frac{n^2}{\log^2 n}}$. $\blacksquare$

Linear Circuits over Rings

Example: Linear circuit over noncommutative ring R



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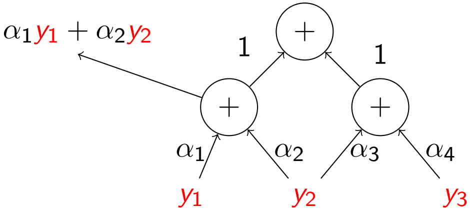

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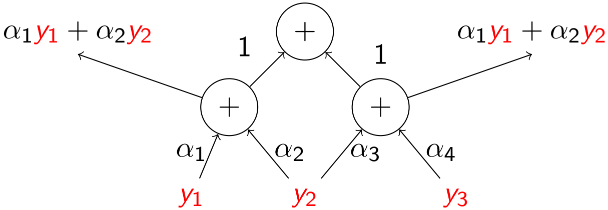

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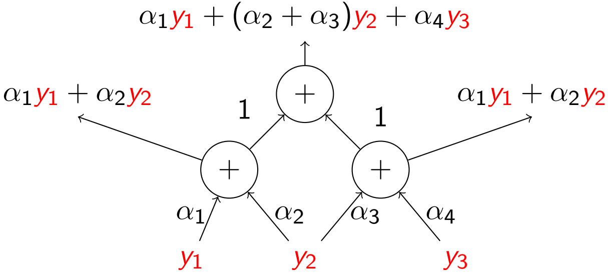

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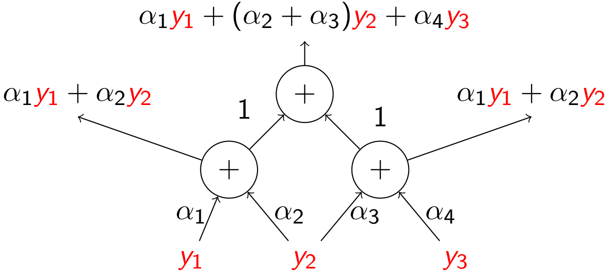

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Linear Circuits over Rings , contd

- When R is a field, we get the well-studied linear circuits model (see Satya Lokam's 2009 survey).
- No explicit superlinear size lower bounds are known for this model over fields (except for some special cases like the bounded coefficient model [Morgenstern'73])
- When the coefficients to come from a *noncommutative* ring $R = \mathbb{F}\langle x_0, x_1 \rangle$, we prove lower bounds for certain restricted linear circuits.

Linear Circuits over Rings, contd

Definition (Homogeneous polynomials)

A polynomial $P \in \mathbb{F}\langle x_0, x_1 \rangle$ is called homogeneous if degree of each monomial in P is the same.

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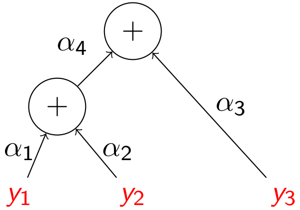
Homogeneous Linear Circuits over $\mathbb{F}\langle x_0, x_1 \rangle$



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- each gate g of the circuit computes a linear form $\sum_{i=1}^{n} \beta_i y_i$, where the $\beta_i \in \mathbb{F}\langle x_0, x_1 \rangle$ are all homogeneous polynomials of the same degree.

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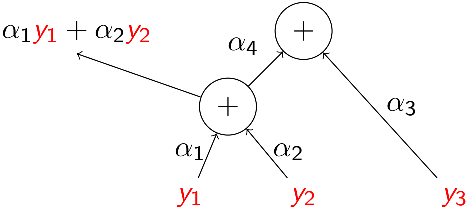

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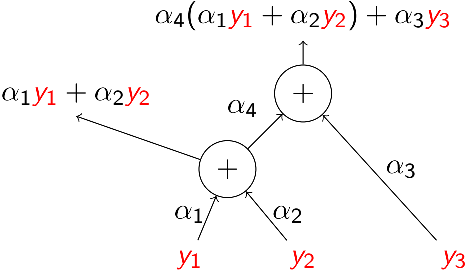

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- Our goal is to construct an explicit matrix $M \in \mathbb{F}^{n\times n}\langle x_0, x_1\rangle$ such that MY, where $Y = (y_1, y_2, \ldots, y_n)^T$ is a column vector of input variables, can not be computed by any homogeneous linear circuit C with size $O(n)$ and depth $O(\log n)$.

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Notation

Consider $n \times n$ matrices $\mathbb{F}^{n\times n}$ over field $\mathbb{F}$. The support of a matrix $A \in \mathbb{F}^{n\times n}$ is the set of locations $\mathrm{supp}(A) = \{(i,j) \mid A_{ij} \neq 0\}$.

Definition (Rigidity of a matrix)

Let $\mathbb{F}$ be any field. The rigidity of a matrix $A \in \mathbb{F}^{n \times n}$, denoted by $\mathcal{R}_r(A)$, is the smallest number t for which there are set of t positions $S \subseteq [n] \times [n]$ and a matrix E such that:

- $\mathrm{supp}(E) \subseteq S$
- rank of $A + E$ is upper bounded by r.

Linear Circuits over Rings, contd

Definition (Rigidity of a deck of matrices)

Let $\mathbb{F}$ be any field. The rigidity $\rho_r(D)$ of a deck of matrices $D = \{A_1, A_2, \ldots, A_N\} \subseteq \mathbb{F}^{n \times n}$ is the smallest number t for which there are set of t positions $S \subseteq [n] \times [n]$ and a deck of matrices $E = \{E_1, E_2, \ldots, E_N\}$ such that for all i:

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- Notice that for $N = 1$, this is the notion of rigid matrices.

Linear Circuits over Rings, contd

Definition (Construction of a Rigid deck)

Let $D = \{A_m \mid m \in \{x_0, x_1\}^{n^2}\}$ with matrices A_m indexed by string m of length n^2. The matrix A_m is defined as follows: $1 \leq i, j \leq n$

$$A_m[i,j] = \begin{cases} 1 & \text{if } m_{ni+j} = x_1 \\ 0 & \text{if } m_{ni+j} = x_0 \end{cases}$$

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Lemma

The above deck $D = \{A_m \mid m \in \{x_0, x_1\}^{n^2}\}$ is an explicit rigid deck for any field $\mathbb{F}$.

Linear Circuits over Rings, contd

- We now turn to the lower bound result for homogeneous linear circuits where the coefficient ring is $\mathbb{F}\langle x_0, x_1\rangle$.

- WANT: an explicit matrix $M \in \mathbb{F}^{n\times n}\langle x_0, x_1\rangle$ such that MY, where Y is a vector of input variables, can not be computed by any homogeneous linear circuits C with size $O(n)$ and depth $O(\log n)$.

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Definition (of matrix M)

We define an explicit $n \times n$ matrix M as $M = \sum_{m\in\{x_0,x_1\}^{n^2}} A_m m$, where $D = \{A_m \mid m \in \{x_0, x_1\}^{n^2}\}$ is the deck defined before

- each entry of matrix M can be expressed as

$$M_{ij} = (x_0 + x_1)^{(i-1)n+j-1} \cdot x_1 \cdot (x_0 + x_1)^{n^2-((i-1)n+j)}.$$

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Any homogeneous linear circuit C over the coefficient ring $\mathbb{F}\langle x_0, x_1 \rangle$ computing MY, for M defined before, requires either size $\omega(n)$ or depth $\omega(\log n)$.

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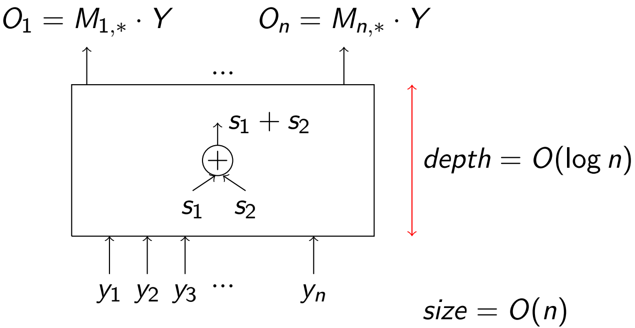

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- By Valiant's graph-theoretic argument, in the circuit C there is a set of gates V of cardinality $s = \frac{c_1 n}{\log \log n} = o(n)$ such that at least $n^2 - n^{1+\delta}$, for $\delta < 1$, input-output pairs have all their paths going through V.

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- Thus, we can write $M = B_1 B_2 + E$, where
  - $B_1 \in \mathbb{F}^{n \times s}\langle x_0, x_1 \rangle$ and
  - $B_2 \in \mathbb{F}^{s \times n}\langle x_0, x_1 \rangle$
  - $E \in \mathbb{F}^{n \times n}\langle x_0, x_1 \rangle$ and $|\mathrm{supp}(E)| \leq n^{1+\delta}$

- Write matrices M, E and B_1B_2 as a polynomial with matrix coefficients.

Example:

$$\begin{pmatrix} 6x_0 + x_1 & x_0 \\ 8x_1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 0 & 0 \end{pmatrix} x_0 + \begin{pmatrix} 1 & 0 \\ 8 & 0 \end{pmatrix} x_1$$

- $M = \sum_{m \in \{x_0, x_1\}^{n^2}} A_m \cdot m$, where $A_m \in \mathcal{A}$
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Summary:-

- We have shown lower bounds for
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Thank you.

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Lemma

The deck $\mathcal{A} = \{A_m \mid m \in \{x_0, x_1\}^{n^2}\}$ *is an explicit rigid deck for any field* $\mathbb{F}$.

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- Valiant showed that almost all $n \times n$ 0-1 matrices A over any field $\mathbb{F}$ have rigidity $\rho_r(A) = \Omega(\frac{(n-r)^2}{\log n})$ for target rank r.
- In particular, for $r = \epsilon \cdot n$, over any field $\mathbb{F}$, there is a 0-1 matrix R for which we have $\rho_r(R) \geq \frac{\delta \cdot n^2}{\log n}$ for some constant $\delta > 0$ depending on ϵ.

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- We claim that for the deck $\mathcal{A}$ we have $\rho_{\epsilon n}(\mathcal{A}) \geq \frac{\delta \cdot n^2}{\log n}$.

- To see this, $S \subseteq [n] \times [n]$ such that $|S| < \frac{\delta n^2}{\log n}$
- Let $E = \{E_m \in \mathbb{F}^{n \times n} | m \in \{x_0, x_1\}^{n^2}\}$ be any collection of matrices such that:
  - $\mathrm{supp}(E_m) \subseteq S$
  - Thus, we have for each m, $|\mathrm{supp}(E_m)| < \frac{\delta n^2}{\log n}$
- Since the deck $\mathcal{A}$ contains all 0-1 matrices, in particular $R \in \mathcal{A}$ and $R = A_m$ for some monomial m.
- From the rigidity of R we know that the rank of $R + E_m$ is at least ϵn.

This proves the claim and the lemma follows. ■

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