On Strong NP-completeness of Rational Problems

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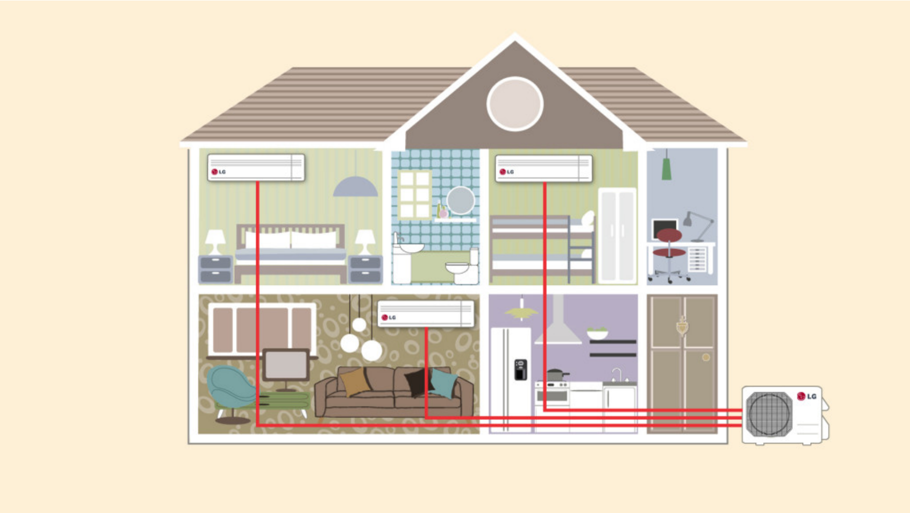
Motivation



What we found

> *A rather subtle point is the question of rational coefficients. Indeed, most textbooks get rid of this case, where some or all input values are non-integer, by the trivial statement that multiplying with a suitable factor, e.g. with the smallest common multiple of the denominators, if the values are given as fractions or by a suitable power of 10, transforms the data into integers. Clearly, this may transform even a problem of moderate size into a rather unpleasant problem with huge coefficients.*
>
> — Hans Kellerer, Ulrich Pferschy, and David Pisinger. *Knapsack problems*. Springer, 2004.

Definitions (1)

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Definition (Knapsack problems)

Assume there are n items whose non-negative rational weights and profits are given as a list $L = \{(w_1, v_1), \ldots, (w_n, v_n)\}$. Let the capacity be $W \in \mathbb{Q}_{\geq 0}$ and the profit threshold be $V \in \mathbb{Q}_{\geq 0}$.

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0-1 Knapsack: Is there a subset of L whose total weight does not exceed W and total profit is at least V?

Unbounded Knapsack: Is there a list of non-negative integers $(q_1, \ldots, q_n)$ such that

$$\sum_{i=1}^{n} q_i \cdot w_i \leq W \qquad \text{and} \qquad \sum_{i=1}^{n} q_i \cdot v_i \geq V?$$

(Intuitively, q_i denotes the number of times the i-th item in A is chosen.)

Definitions (2)

Definition (Subset Sum problems)

Assume we are given a list of n items with rational non-negative weights $A = \{w_1, \ldots, w_n\}$ and a target total weight $W \in \mathbb{Q}_{\geq 0}$.

0-1 Subset Sum: Does there exists a subset B of A such that the total weight of B is equal to W?

Unbounded Subset Sum: Does there exist a list of non-negative integer quantities $(q_1, \ldots, q_n)$ such that

$$\sum_{i=1}^{n} q_i \cdot w_i = W?$$

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Assume we are given a list of n items with non-negative rational weights $A = \{w_1, \ldots, w_n\}$.
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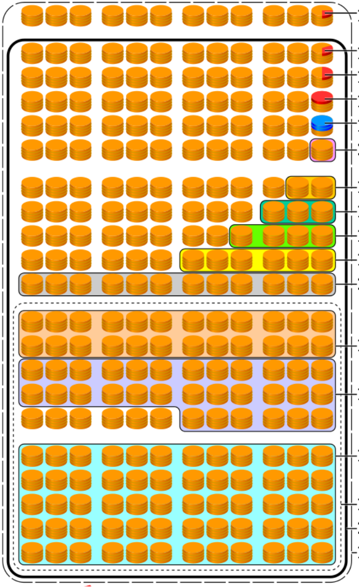
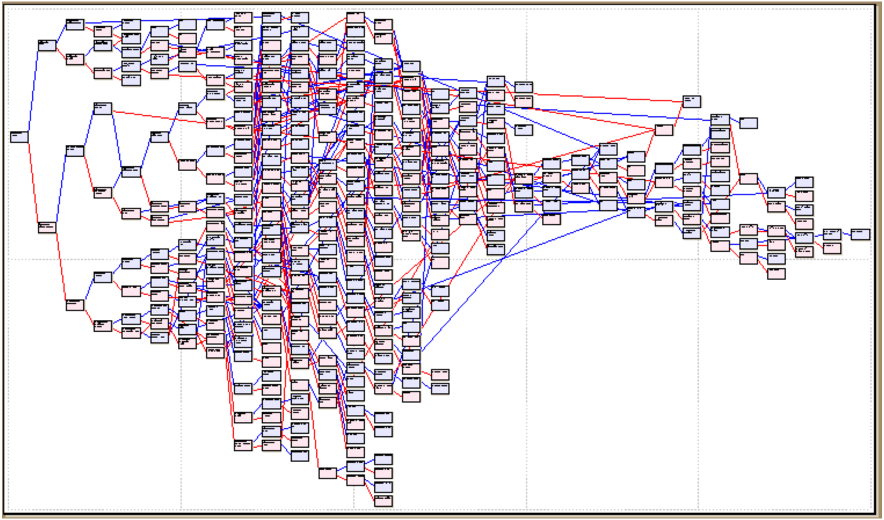
EXAMPLE PROBLEM

Money in 18th century England



1 quarter farthing	= 1/16 d.	= 1/192 shilling	= £1/3840
1 third farthing	= 1/12 d.	= 1/144 shilling	= £1/2880
1 half farthing	= 1/8 d.	= 1/96 shilling	= £1/1920
1 farthing	= 1/4 d.	= 1/48 shilling	= £1/960
1 halfpenny	= 1/2 d.	= 1/24 shilling	= £1/480
1 penny	**= 1 d.**	**= 1/12 shilling**	**= £1/240**
1 twopence	= 2 d.	= 1/6 shilling	= £1/120
1 threepence	= 3 d.	= 1/4 shilling	= £1/80
1 groat	= 4 d.	= 1/3 shilling	= £1/60
1 sixpence	= 6 d.	= 1/2 shilling	= £1/40
1 shilling	**= 12 d.**	**= 1/-**	**= £1/20**
1 florin	= 24 d.	= 2/-	= £1/10
1 half crown	= 30 d.	= 2/6	= £1/8
1 crown	= 60 d.	= 5/-	= £1/4
1 half sovereign	= 120 d.	= 10/-	= £1/2
1 sovereign	**= 240 d.**	**= 20 shillings**	**= £1**
1 guinea	= 252 d.	= 21 shillings	= £1.05

The Reductions



The Actual Reductions

One-in-Three-SAT for 3-CNF $\leq_m^p$ One-in-Three-SAT for 3-CNF$_{\leq 4}$ $\leq_m^p$ All-the-Same-SAT for 3-CNF$_{\leq 4}$ $\leq_m^p$ Unbounded Subset Sum $\leq_m^p$ Unbounded Knapsack

All-the-Same-SAT for 3-CNF$_{\leq 4}$ $\leq_m^p$ Partition

All-the-Same-SAT for 3-CNF$_{\leq 4}$ $\leq_m^p$ Subset Sum $\leq_m^p$ Knapsack

In the Pursuit of Satisfaction

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3-CNF$_{\leq 4}$ is the set of 3-CNF formulae that use each variable at most four times.

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Theorem

The ALL-THE-SAME-SAT *problem for 3-CNF$_{\leq 4}$ formulae is* NP*-complete.*

Prime Suspects (1)

Theorem (Rosser (1962))

$$\pi_i < i(\log i + \log\log i) \quad \textit{for} \quad i \geq 6$$

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Corollary

The total size of the first n prime numbers, when written down in unary, is $\mathcal{O}(n^2 \log n)$*. Furthermore, they can be computed in polynomial time.*

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Let $(a_0, a_1, \ldots, a_n)$ *and* $(b_0, b_1, \ldots, b_n)$ *be two lists of integers such that* $|a_i - b_i| < p_i$ *holds for all* $i = 1, \ldots, n$*.*

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We then have

$$a_0 + \frac{a_1}{p_1} + \ldots + \frac{a_n}{p_n} = b_0 + \frac{b_1}{p_1} + \ldots + \frac{b_n}{p_n}$$

if and only if

$$a_i = b_i \quad \text{for all} \quad i = 0, \ldots, n.$$

ALL-THE-SAME-SAT $\leq_m^p$ UNBOUNDED SUBSET SUM

Assume we are given a 3-CNF$_{\leq 4}$ formula

$$\phi = C_1 \wedge C_2 \wedge \ldots \wedge C_m$$

with m clauses $C_1, \ldots, C_m$ and n propositional variables $x_1, \ldots, x_n$,

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For a literal l, we write that $l \in C_j$ iff l is equal to a_j, b_j or c_j.
Let $p_i := \pi_{i+n+5}$ for all $i = 1, \ldots, n + m$.

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The weight of the item corresponding to the literal x_i is set to

$$1 + \frac{1}{p_i} - \frac{1}{p_{i \oplus_n 1}} + \sum_{\{j | x_i \in C_j\}} \left(\frac{1}{p_{n+j}} - \frac{1}{p_{n+j \oplus_m 1}} \right)$$

and corresponding to the literal $\neg x_i$ is set to

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Notice that the total weight of A is equal to

$$2n + \sum_{i=1}^{n} \left(\frac{2}{p_i} - \frac{2}{p_{i \oplus_n 1}} \right) + \sum_{j=1}^{m} \left(\frac{3}{p_{n+j}} - \frac{3}{p_{n+j \oplus_m 1}} \right)$$

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We claim that the target weight $W = n$ is achievable by picking items from A (each item possibly multiple times) iff ϕ is a positive instance of All-the-Same-SAT.

All-the-Same-SAT $\leq_m^p$ Unbounded Subset Sum

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- Notice that $T \leq W/(1 - \frac{5}{p_1}) < n/(1 - \frac{5}{n+5}) = n + 5$.

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The total weight of the selected items can be expressed as:

$$\sum_{i=1}^{n} t_i + \sum_{i=1}^{n} \frac{t_i - t_{i \ominus_n 1}}{p_i} + \sum_{j=1}^{m} \frac{t_{n+j} - t_{n+j \ominus_m 1}}{p_{n+j}} \qquad (\star)$$

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- Note that $|t_i - t_{i \ominus_n 1}| < n + 5$ for all $i = 1, \ldots, n$, and $|t_{n+j} - t_{n+j \ominus_m 1}| < n + 5$ for all $j = 1, \ldots, m$.

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- From the previously showed lemma this is equal to $W = n$ iff $\sum_{i=1}^n t_i = n$, and $t_1 = t_2 = \ldots = t_n$, and $t_{n+1} = t_{n+2} = \ldots = t_{n+m}$.

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($\Leftarrow$) Let ν be a valuation for which ϕ satisfies the All-the-Same-SAT condition.

- If $\nu(x_i) = \top$ then we set $q_i = 1$ and $q'_i = 0$.
- If $\nu(x_i) = \bot$ then we set $q_i = 0$ and $q'_i = 1$.

Let us define t_i-s as before.

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We now have $t_i = 1$ for all $i = 1, \ldots, n$ and $t_{n+1} = t_{n+2} = \ldots = t_{n+m}$, because the All-the-Same-SAT condition is satisfied by ν.

All-the-Same-SAT $\leq_m^p$ Unbounded Subset Sum

If $\sum_{i=1}^{n} t_i = n$, and $t_1 = t_2 = \ldots = t_n$, and $t_{n+1} = t_{n+2} = \ldots = t_{n+m}$, then we have:

- The first two imply that $t_i = 1$ for all $i = 1, \ldots, n$.
- The last one implies that in each clause exactly the same number of items corresponding to its literals is chosen.

($\Leftarrow$) Let ν be a valuation for which ϕ satisfies the All-the-Same-SAT condition.

- If $\nu(x_i) = \top$ then we set $q_i = 1$ and $q'_i = 0$.
- If $\nu(x_i) = \bot$ then we set $q_i = 0$ and $q'_i = 1$.

Let us define t_i-s as before.

We now have $t_i = 1$ for all $i = 1, \ldots, n$ and $t_{n+1} = t_{n+2} = \ldots = t_{n+m}$, because the All-the-Same-SAT condition is satisfied by ν.

From ($\star$) it follows that the total weight of these items is n.

The Other Reductions

We can simply repeat this proof to show.

Theorem

The PARTITION *problem with rational weights is strongly NP-complete.*

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The SUBSET SUM *problem with rational weights is strongly NP-complete.*

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Corollary

The 0-1 KNAPSACK *and* UNBOUNDED KNAPSACK *problems with rational weights are strongly NP-complete.*

Conclusions

- Subset Sum, Unbounded Subset Sum, Knapsack, Unbounded Knapsack, Partition are all strongly NP-hard with rational coefficients

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Thanks!