On vertex coloring without monochromatic triangles

Michał Karpiński, Krzysztof Piecuch

Institute of Computer Science
University of Wrocław
Poland

7 June, 2018

Outline

Definitions

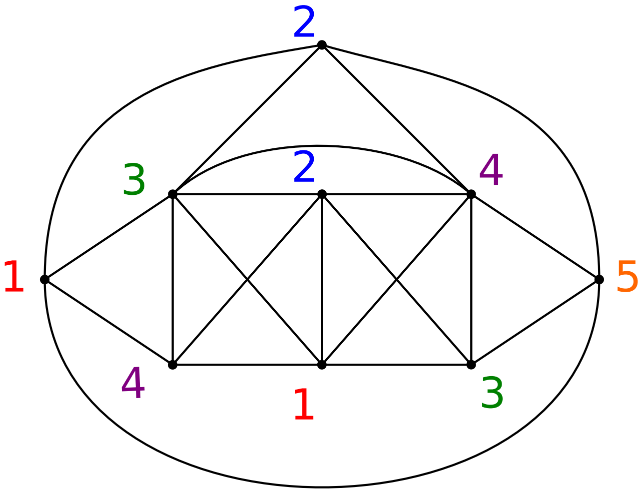
Definition

A *classic k-coloring* of a graph is a function $c : V \to \{1, \ldots, k\}$, such that there are no two adjacent vertices u and v, for which $c(u) = c(v)$.

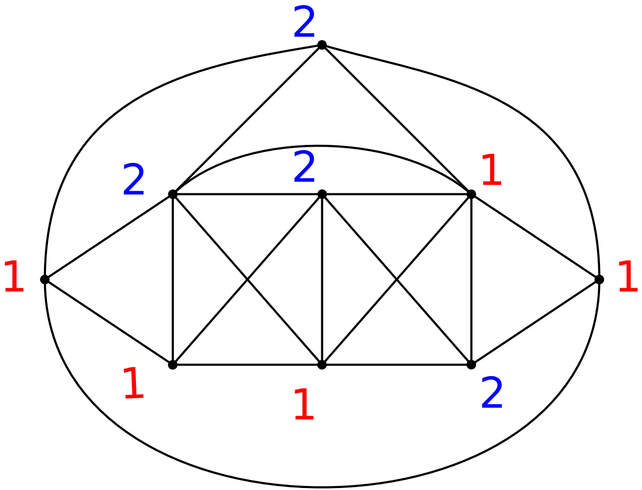
Definition

A *triangle-free k-coloring* of a graph is a function $c : V \to \{1, \ldots, k\}$, such that there are no three mutually adjacent vertices u, v and w, for which $c(u) = c(v) = c(w)$. If such vertices exist, then the induced subgraph (K_3) is called a *monochromatic triangle*.

Definitions



Definitions



Definitions

Definition

Given G, the smallest k for which there exists a classic k-coloring for G is called the *chromatic number* and is denoted as $\chi(G)$.

Definition

Given G, the smallest k for which there exists a triangle-free k-coloring for G we call the *triangle-free chromatic number* and we denote it as $\chi_3(G)$.

Definitions

TriangleFree-q-Coloring
Input: A finite, undirected, simple graph G.
Question: Is there a triangle-free q-coloring of G?

Definitions

Why this problem?

1. Coloring problems are fun!
2. Classic coloring is HARD!

Definitions

TriangleFree-q-Coloring is...

1. a special case of 3-uniform Hypergraph Coloring problem.
2. a special case of Bipartitioning without Subgraph H problem, where $H = K_3$ and $q = 2$.

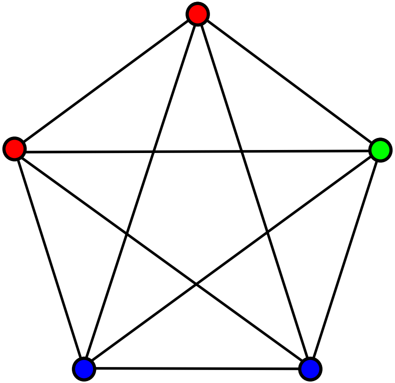
Graph-theoretical results

Theorem

For any graph G: $\left\lceil \frac{\omega(G)}{2} \right\rceil \leq \chi_3(G) \leq \left\lceil \frac{\chi(G)}{2} \right\rceil$.

Theorem

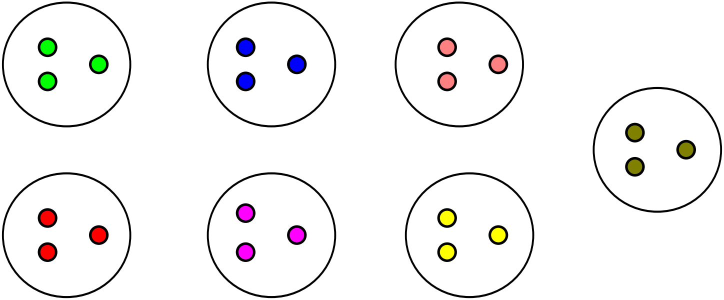
For any graph G: $\left\lceil \frac{\omega(G)}{2} \right\rceil \leq \chi_3(G) \leq \left\lceil \frac{\chi(G)}{2} \right\rceil$.



Graph-theoretical results

Theorem

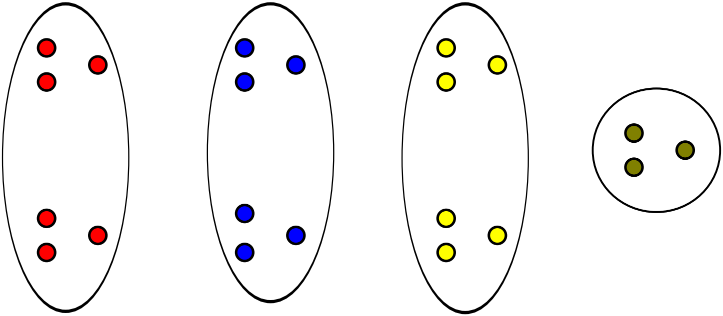
For any graph G*:* $\left\lceil \frac{\omega(G)}{2} \right\rceil \leq \chi_3(G) \leq \left\lceil \frac{\chi(G)}{2} \right\rceil.$



Graph-theoretical results

Theorem

For any graph G: $\left\lceil \frac{\omega(G)}{2} \right\rceil \leq \chi_3(G) \leq \left\lceil \frac{\chi(G)}{2} \right\rceil$.



Graph-theoretical results

Theorem

For any $k \geq 1$*, there exists a graph* G *for which* $\chi_3(G) = 1$ *and* $\chi(G) = k$*.*

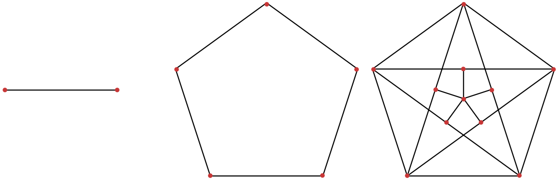
Graph-theoretical results

Theorem

For any $k \geq 1$, there exists a graph G for which $\chi_3(G) = 1$ and $\chi(G) = k$.

Proof.

Just take Mycielski graphs. □



Graph-theoretical results

Hypothesis: $\chi_3(G) \leq \left\lceil \frac{\omega(G)}{2} \right\rceil + c.$

Graph-theoretical results

~~Hypothesis: $\chi_3(G) \leq \left\lceil \frac{\omega(G)}{2} \right\rceil + c$.~~

Graph-theoretical results

~~Hypothesis: $\chi_3(G) \leq \left\lceil \frac{\omega(G)}{2} \right\rceil + c$.~~

Theorem

For every k, there exist a graph G where $\omega(G) \leq 3$, such that $\chi_3(G) > k$.

~~Hypothesis: $\chi_3(G) \leq \left\lceil \frac{\omega(G)}{2} \right\rceil + c$.~~

Theorem

For every k, there exist a graph G where $\omega(G) \leq 3$, such that $\chi_3(G) > k$.

Proof.

We know that for every k, t and g there exists t-uniform hypergraph with girth at least g that cannot be colored using only k colors (Erdos 1959). Take such H with $t = 3$ and $g = 4$. □

Graph-theoretical results

Theorem

Let $G = (V, E)$ be any graph where $|V| > 3$, where G is not a complete graph of odd number of vertices. Then $\chi_3(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil$.

Graph-theoretical results

Theorem

Let $G = (V, E)$ be any graph where $|V| > 3$, where G is not a complete graph of odd number of vertices. Then $\chi_3(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil$.

Proof.

1. If G is an odd cycle of length at least 5, then $\chi_3(G) = 1$ and $\left\lceil \frac{\Delta}{2} \right\rceil = 1$.
2. If G is a complete graph (a clique) of n (even) vertices, then $\chi_3(G) = \left\lceil \frac{n}{2} \right\rceil$ and $\Delta = n - 1$.
3. Otherwise use Brook's Theorem: $\chi_3(G) \leq \left\lceil \frac{\chi(G)}{2} \right\rceil \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil$.

□

Positive results

Definition

Let $\mathcal{G}$ be a class of graphs. We say that the triangle-free coloring problem is solvable in time $O(f(n,m), g(n,m))$ on $\mathcal{G}$, iff there exist algorithms $\mathcal{A}$ and $\mathcal{B}$, such that for every input $G \in \mathcal{G}$, (i) algorithm $\mathcal{A}$ outputs $\chi_3(G)$ in $O(f(n,m))$ time, and (ii) algorithm $\mathcal{B}$ outputs the triangle-free coloring that uses exactly $\chi_3(G)$ colors, in $O(g(n,m))$ time.

Positive results

Definition

Let $\mathcal{G}$ be a class of graphs. We say that the triangle-free coloring problem is solvable in time $O(f(n,m), g(n,m))$ on $\mathcal{G}$, iff there exist algorithms $\mathcal{A}$ and $\mathcal{B}$, such that for every input $G \in \mathcal{G}$, (i) algorithm $\mathcal{A}$ outputs $\chi_3(G)$ in $O(f(n,m))$ time, and (ii) algorithm $\mathcal{B}$ outputs the triangle-free coloring that uses exactly $\chi_3(G)$ colors, in $O(g(n,m))$ time.

Theorem

The triangle-free coloring problem is solvable:

- *in time $O(n, n^2)$ on planar graphs,*
- *in time $O(n, n)$ on: outerplanar graphs, chordal graphs, graphs with bounded maximum degree Δ, with $\Delta \leq 4$.*

Positive results

Observation

If $\chi(G) \leq 4$ (and therefore $\chi_3(G) \leq 2$) then:

1. $\chi_3(G) = 0$ iff G is empty,
2. $\chi_3(G) = 1$ iff G is triangle-free (K_3-free),
3. $\chi_3(G) = 2$ iff the above two cases do not hold.

Positive results

- planar graphs
  - check if G is triangle-free in $O(n)$ time (Papadimitriou and Yannakakis, 1981)
  - color graph with 4 colors in $O(n^2)$ time (Appel and Haken, 1989) and use SRS
- outerplanar graphs
  - check if G is triangle-free in $O(n)$ time (Papadimitriou and Yannakakis, 1981)
  - procude the classic $\chi(G)$-coloring in $O(n)$ time (Proskurowski and Sysło, 1986) and use SRS
- graphs with $\Delta \leq 4$
  - check if G is triangle-free in $O(n \cdot \Delta^2) = O(n)$
  - produce the classic Δ-coloring in $O(n)$ time (Skulrattanakulchai, 2006) and use SRS
- chordal graphs
  - $\chi(G) = \omega(G) \Rightarrow \chi_3(G) = \left\lceil \frac{\chi(G)}{2} \right\rceil = \left\lceil \frac{\omega(G)}{2} \right\rceil$
  - produce the classic $\chi(G)$-coloring in $O(n)$ time (Golumbic, 1980) and use SRS

Positive results

For planar graphs:

- finding $\chi(G)$ is NP-hard, even if G is 4-regular (Dailey, 1980)
- finding classic 4-coloring is done in $O(n^2)$, even if G is classically 3-colorable (Kawarabayashi, 2010)

Positive results

Theorem

The TRIANGLEFREE-q-COLORING *problem is* $\mathcal{FPT}$ *when parametrized by the vertex cover number.*

Positive results

Theorem

The TRIANGLEFREE-q-COLORING *problem is* $\mathcal{FPT}$ *when parametrized by the vertex cover number.*

Proof.

Let W be a minimum vertex cover of $G = (V, E)$, and let $|W| = k$. Then $I = V \setminus W$ is an independent set. Then:

1. find the triangle-free q-coloring of W by exhaustive search, then
2. color I by greedy strategy.

The total running time of the algorithm is $O(k^{\lceil k/2 \rceil + 1} n)$. □

Negative results

Previous work:

1. Vertex 2-coloring without monochromatic cycles of fixed size is NP-complete (Karpinski 2017; Farrugia 2004) $\Rightarrow$ TF-2-COL is NP-complete.
2. TriangleFreePolar-2-Coloring problem is NP-complete (Shitov 2017)

TriangleFreePolar-q-Coloring
Input: A finite, undirected, simple graph $G = (V, E)$ and a subset $S \subseteq E$.
Question: Is there a triangle-free q-coloring of G, such that no edge in S is monochromatic?

Negative results

Theorem

For any fixed $q \geq 2$, the TriangleFree-q-Coloring *problem is* $\mathcal{NP}$*-complete.*

Theorem

The TriangleFree-2-Coloring *problem is* $\mathcal{NP}$*-complete on* K_4*-free graphs.*

Theorem

The TriangleFreePolar-2-Coloring *problem is* $\mathcal{NP}$*-complete on graphs with maximum degree at most 3, but it is linear-time solvable on graphs with maximum degree at most 2.*

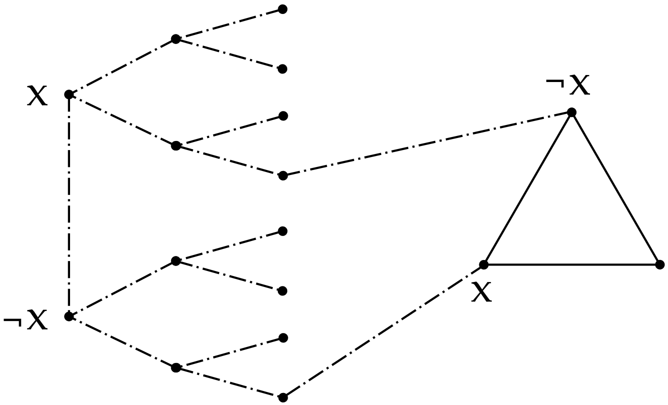
Negative results

NotAllEqual 3-Sat-4

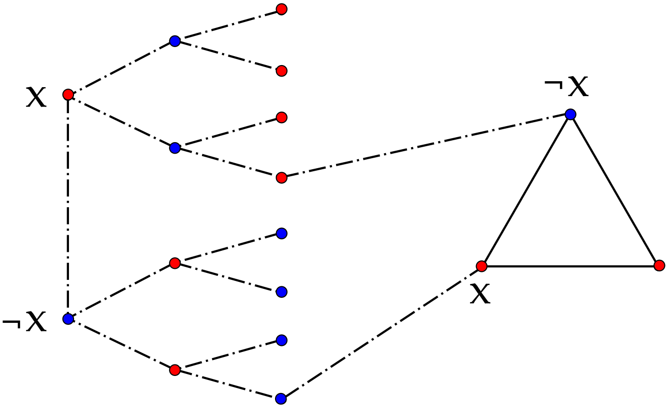
Input: Boolean formula ϕ in conjunctive normal form, where each clause consists of exactly three literals and every variable appears at most four times in the formula.

Question: Does ϕ have a nae-satisfying assignment, i.e., in each clause at least one literal is true and at least one literal is false?

Negative results



Negative results



What's next?

1. Sub-quadratic algorithm for finding a triangle-free coloring in planar graphs.
2. Algorithm that finds $\chi_3(G)$ in graphs with $\Delta(G) \geq 5$.
3. The smallest $\Delta(G)$, for which the TriangleFree-q-Coloring problem is $\mathcal{NP}$-hard.
4. Where the problems proved to be $\mathcal{NP}$-complete here reside in W-hierarchy?

What's next?

Generalize χ_3 to get the parameter χ_r, for any $r \geq 3$:

1. not allow monochromatic K_r
2. not allow monochromatic $C_{r'}$ (cycle of length r'), for any $3 \leq r' \leq r$.

The end

Thank You!